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Adjacent vertex distinguishing total coloring of graphs with maximum degree 4

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ABSTRACT

Let k be a positive integer. An adjacent vertex distinguishing (for short, AVD) total k-coloring ϕ of a graph G is a proper total k-coloring of G such that no pair of adjacent vertices have the same set of colors, where the set of colors at a vertex v is $\{\phi(v)\} \cup \{\phi(e) : e \text{ is incident to } v\}$. Zhang et al. conjectured in 2005 that every graph with maximum degree Δ has an AVD total $(\Delta + 3)$-coloring. Recently, Papaioannou and Raftopoulou confirmed the conjecture for 4-regular graphs. In this paper, by applying the Combinatorial Nullstellensatz, we verify the conjecture for all graphs with maximum degree 4.

1. Introduction

All graphs considered in this paper are simple and undirected. We follow the standard notation and terminology as can be found in [6]. Let $G = (V(G), E(G))$ be a graph and $T(G) = V(G) \cup E(G)$. For a vertex $x \in V(G)$, we use $N_G(v)$ and $E_G(v)$ to denote the set of vertices adjacent to v and the set of edges incident to v, respectively. An ℓ-vertex or ℓ^--vertex of G is a vertex of degree ℓ or at most ℓ, respectively. Let $V_\ell(G)$ and $V_{\ell^-}(G)$ be the sets of ℓ-vertices and ℓ^--vertices, respectively, in G. We also use V_ℓ and V_{ℓ^-} for short if the graph G is understood in context. The maximum degree of G is denoted by $\Delta(G)$.

Let k be a positive integer and $[k] = \{1, 2, \ldots, k\}$. A mapping $\phi : T(G) \to [k]$ is a proper total k-coloring if, for any two adjacent or incident elements $z_1, z_2 \in T(G)$, it is $\phi(z_1) \neq \phi(z_2)$. Let $C_\phi(v) = \{\phi(v)\} \cup \{\phi(e) : e \in E_G(v)\}$ and $m_\phi(v) = \phi(v) + \sum_{e \in E_G(v)} \phi(e)$ for any vertex $v \in V(G)$. A proper total k-coloring ϕ of G is adjacent vertex distinguishing (for short, AVD) if $C_\phi(u) \neq C_\phi(v)$ whenever $uv \in E(G)$. The AVD total chromatic number $\chi_a^t(G)$ is the smallest integer k such that G has an AVD total k-coloring.

The AVD total coloring is related to vertex-distinguishing edge coloring which requires that every pair of vertices receives different the sets of colors. The vertex-distinguishing edge coloring was introduced by Burris and Schelp [7], and independently by Černý et al. [9] (under the notion of observability). This type of coloring has been well studied over the last decade (see, for example, [2–5]). It was later extended to require only adjacent vertices to be distinguished by Zhang et al. [14], which was in turn extended to total coloring [13].

Zhang et al. [13] determined $\chi_a^t(G)$ for some basic graphs such as complete graphs and complete bipartite graphs and made the following conjecture.

Conjecture 1.1 ([13]). *For any graph G, $\chi_a^t(G) \leq \Delta(G) + 3$.*

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Chen [8] and Wang [12], independently, confirmed Conjecture 1.1 for graphs with maximum degree 3. Later, Hulgan [10] presented a concise proof on this result. Recently Papaioannou and Raftopoulou [11] verified Conjecture 1.1 for 4-regular graphs.

Theorem 1.2 (*[11]*). *For any 4-regular graph G,* $\chi_a^t(G) \le 7$.

The aim of this paper is to extend Theorem 1.2 from 4-regular graphs to graphs with maximum degree 4. We prove the following result.

Theorem 1.3. *For any graph G with maximum degree 4,* $\chi_a^t(G) \le 7$.

We use a polynomial method based on the Combinatorial Nullstellensatz due to Alon [1]. In fact, in Section 3, we prove a stronger result as follows.

Theorem 1.4. *Every graph with maximum degree 4 has a proper total 7-coloring satisfying:*

(i) *For any two adjacent 4-vertices u and v,* $C_\phi(u) \ne C_\phi(v)$;
(ii) *For any two adjacent* 3^-*-vertices u and v,* $m_\phi(u) \ne m_\phi(v)$.

Remark. If $m_\phi(u) \ne m_\phi(v)$, then the sets of colors must be different. Also, in the definition of AVD total coloring it requires that any two adjacent vertices have different color sets. Theorem 1.4 does not cover a 4-vertex that is adjacent to a 3^--vertex. Of course, in a proper total coloring, two adjacent vertices of different degrees also have different color sets.

2. A polynomial associated with AVD total coloring

Let G be a graph with maximum degree 4 and H be an induced subgraph in $G[V_{3^-}]$. An *H-partial AVD total 7-coloring* of G is a mapping $\phi : T(G) - T(H) \to [7]$, satisfying the following two conditions:

(a) For any two adjacent or incident elements $z_1, z_2 \in T(G) - T(H)$, $\phi(z_1) \ne \phi(z_2)$;
(b) For $uv \in E(G-H)$, $C_\phi(u) \ne C_\phi(v)$ if $d_G(u) = d_G(v) = 4$, and $m_\phi(u) \ne m_\phi(v)$ if $d_G(u) \le 3$ and $d_G(v) \le 3$.

H is called *reducible* if every H-partial AVD total 7-coloring can be extended to a proper total coloring of G satisfying the conditions of Theorem 1.4. We will use a polynomial method to prove Theorem 1.4. For this, we need the following theorem, known as the Combinatorial Nullstellensatz due to Alon [1].

Theorem 2.1 (*[1]*). *Let* $\mathbb{F}$ *be an arbitrary field and* $P \in \mathbb{F}[x_1, \ldots, x_n]$ *with degree* $deg(P) = \sum_{j=1}^{n} i_j$*, where each* i_j *is a nonnegative integer. If the coefficient of the monomial* $x_1^{i_1} \ldots x_n^{i_n}$ *in P is nonzero, and if* $S_1, \ldots, S_n$ *are subsets of* $\mathbb{F}$ *with* $|S_j| > i_j$*, then there are* $s_1 \in S_1, \ldots, s_n \in S_n$ *such that* $P(s_1, \ldots, s_n) \ne 0$.

Let H be an induced subgraph of $G[V_{3^-}]$. Denote $V(H) = \{v_1, \ldots, v_h\}$ and $E(H) = \{e_1, \ldots, e_k\}$. Each element $z \in T(H) = V(H) \cup E(H)$ is associated with a variable x_z. Let D be an arbitrary orientation of H and ϕ be an H-partial AVD total 7-coloring of G. For each vertex $v \in V(H)$, $N_D^+(v)$ is the set of arcs with v as the initial vertex. For each vertex $v \in V(H)$, let $\mu_H(v) = x_v + \sum_{e \in E_H(v)} x_e$ and

$$
\begin{aligned}
\mathcal{P}_{D,\phi}(H; v) = & \prod_{u \in N_G(v) \setminus V(H)} \left((x_v - \phi(u))(x_v - \phi(uv)) \prod_{e \in E_H(v)} (x_e - \phi(uv)) \right) \\
& \cdot \prod_{u \in (V_{3^-} \cap N_G(v)) \setminus V(H)} \left(\mu_H(v) + \sum_{e \in E_G(v) \setminus E_H(v)} \phi(e) - \phi(u) - \sum_{e \in E_G(u)} \phi(e) \right) \\
& \cdot \prod_{u \in N_D^+(v)} (x_v - x_u) \left(\mu_H(v) + \sum_{e \in E_G(v) \setminus E_H(v)} \phi(e) - \mu_H(u) - \sum_{e \in E_G(u) \setminus E_H(u)} \phi(e) \right) \\
& \cdot \prod_{e \in E_H(v)} (x_v - x_e) \prod_{\substack{e_i, e_j \in E_H(v) \\ i<j}} (x_{e_i} - x_{e_j}).
\end{aligned}
$$

Remark. In $\mathcal{P}_{D,\phi}(H; v)$, the first product assures that v and every edge $e \in E_H(v)$ would have different colors than its incident elements in $T(G) - T(H)$; while the last two products (together with some parts of the third product) assure that v and every edge $e \in E_H(v)$ would have different color than its incident elements in $T(H)$. Moreover, the second and third products guarantee $m_\phi(u) \ne m_\phi(v)$ for any vertex $u \in V_{3^-} \cap N_G(v)$.

By the Combinatorial Nullstellensatz (Theorem 2.1), we need to find the existence of certain monomials of degree $deg(P)$ with nonzero coefficient in the expansion of $\mathcal{P}_{D,\phi}(H;v)$. Since $\phi(x)$ is a constant for each element $x \in T(G) - T(H)$, one can drop each constant term from $\mathcal{P}_{D,\phi}(H;v)$ to only consider its homogeneous part which can be expressed as follows.

$$\widetilde{\mathcal{P}}_D(H;v) = \left(x_v^2 \prod_{e\in E_H(v)} x_e\right)^{|N_G(v)\setminus N_H(v)|} (\mu_H(v))^{|(V_{3^-}\cap N_G(v))\setminus V(H)|} \cdot \prod_{u\in N_D^+(v)} (x_v - x_u)(\mu_H(v) - \mu_H(u)) \prod_{e\in E_H(v)} (x_v - x_e) \prod_{\substack{e_i,e_j\in E_H(v)\\ i<j}} (x_{e_i} - x_{e_j}).$$

Note that $\prod_{v\in V(H)} \mathcal{P}_{D,\phi}(H;v)$ and $\prod_{v\in V(H)} \widetilde{\mathcal{P}}_D(H;v)$ are independent of the orientation D. Thus we define the following two polynomials.

$$\mathcal{P}_\phi(H) = \mathcal{P}_\phi(x_{v_1}, x_{v_2}, \ldots, x_{v_h}, x_{e_1}, x_{e_2}, \ldots, x_{e_k}) = \prod_{v\in V(H)} \mathcal{P}_{D,\phi}(H;v),$$

$$\widetilde{\mathcal{P}}(H) = \widetilde{\mathcal{P}}(x_{v_1}, x_{v_2}, \ldots, x_{v_h}, x_{e_1}, x_{e_2}, \ldots, x_{e_k}) = \prod_{v\in V(H)} \widetilde{\mathcal{P}}_D(H;v).$$

Lemma 2.2. *Let G be a graph with maximum degree 4 and let H be an induced subgraph of $G[V_{3^-}]$ such that there exists an H-partial AVD total 7-coloring of G. If $\widetilde{\mathcal{P}}(H)$ has a monomial $\prod_{z\in T(H)} x_z^{j_z}$ with nonzero coefficient such that $j_z \le 6$ for each $z \in T(H)$ and $deg(\widetilde{\mathcal{P}}(H)) = \sum_{z\in T(H)} j_z$, then H is reducible.*

Proof. Let $\prod_{z\in T(H)} x_z^{j_z}$ be a monomial of $\widetilde{\mathcal{P}}(H)$ with nonzero coefficient such that $deg(\widetilde{\mathcal{P}}(H)) = \sum_{z\in T(H)} j_z$ and $j_z \le 6$ for each $z \in T(H)$. Let ϕ be an H-partial AVD total 7-coloring of G. Then $\prod_{z\in T(H)} x_z^{j_z}$ is also a monomial of $\mathcal{P}_\phi(H)$ such that $\sum_{z\in T(H)} j_z = deg(\mathcal{P}_\phi(H))$ and its coefficient in $\mathcal{P}_\phi(H)$ is the same as its coefficient in $\widetilde{\mathcal{P}}(H)$. Thus by Theorem 2.1, for each $z \in T(H)$, there is an integer $a_z \in [7]$ such that

$$\mathcal{P}_\phi(a_{v_1}, a_{v_2}, \ldots, a_{v_h}, a_{e_1}, a_{e_2}, \ldots, a_{e_k}) \neq 0.$$

Hence ϕ can be extended to a coloring of G satisfying the conditions of Theorem 1.4 by coloring each $z \in T(H)$ with a_z. By definition, H is reducible. ■

The following lemma is needed in the proof of the main result and its proof is straightforward and thus omitted.

Lemma 2.3. *Let $\mathbb{F}$ be an arbitrary field and P, Q, R be three polynomials in $\mathbb{F}[x_1, \ldots, x_n]$ with $P = Q \cdot R$. If the coefficient of the monomial $x_1^{i_1} \ldots x_n^{i_n}$ in P is nonzero and $deg(P) = \sum_{j=1}^n i_j$, then there are nonnegative integers $i'_j \le i_j$ for each $j \in [n]$ such that $deg(Q) = \sum_{j=1}^n i'_j$ and the coefficient of the monomial $x_1^{i'_1} \ldots x_n^{i'_n}$ in Q is also nonzero.*

3. Graphs with $\Delta = 4$

In this section, we will prove Theorem 1.4. Let us start with the special case of graphs in which V_{3^-} is an independent set of vertices. Note that if V_{3^-} is an independent set of vertices, then any AVD total 7-coloring of G satisfies the two conditions of Theorem 1.4.

Lemma 3.1. *Let G be a connected graph with maximum degree 4. If V_{3^-} is an independent set of vertices in G, then G has an AVD total 7-coloring.*

Proof. For $i \in \{1, 2, 3, 4\}$, let G_i be a copy of G, and use $v_i \in V(G_i)$ to denote the copy of $v \in V(G)$ in G_i. Let $G' = \cup_{i=1}^4 G_i$. Then we augment G' to a new graph G'' by adding some new edges to G' such that for each $v \in V(G)$, the induced subgraph $G''[\{v_1, v_2, v_3, v_4\}]$ is $(4 - d_G(v))$-regular in G''. It is obvious that G'' is a 4-regular graph. By Theorem 1.2, G'' has an AVD total 7-coloring ψ. Since V_{3^-} is an independent set of G, ψ also induces an AVD total 7-coloring of G_1 (and thus of G) satisfying the two conditions of Theorem 1.4. ■

Outline of the proof of Theorem 1.4. The proof is based on a minimal counterexample. By the minimality of G, we first show that no connected induced subgraph in $G[V_{3^-}]$ with at least two vertices is reducible. Applying this claim repeatedly, we show that $G[V_{3^-}]$ contains certain configurations. Finally we prove that such configurations are indeed reducible, which contradicts to the first claim.

Proof of Theorem 1.4. Suppose that Theorem 1.4 does not hold. Let G be a counterexample to Theorem 1.4 such that $|E(G)|$ is minimum. Obviously, G is connected and V_{3^-} is not an independent set of G by Lemma 3.1.

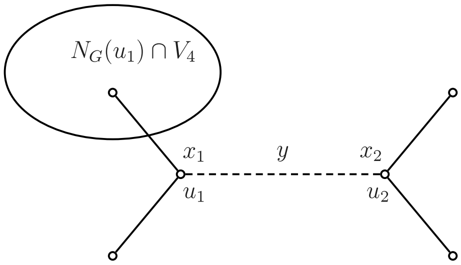

Fig. 1. H and the corresponding variables in Claim 3.3.

Claim 3.1. *Let H be a connected subgraph induced by at least two vertices of V_{3^-}. Then G has an H-partial AVD total 7-coloring and thus H is not reducible.*

Proof. Since H is connected and has at least two vertices, it is $|E(H)| \geq 1$. By the minimality of G, $G - E(H)$ has a proper total 7-coloring ψ satisfying

(i) For any two adjacent 4-vertices u and v, $C_\psi(u) \neq C_\psi(v)$;
(ii) For any two adjacent 3^--vertices u and v, $m_\psi(u) \neq m_\psi(v)$.

Uncoloring every vertex in $V(H)$, we obtain a proper total 7-coloring ϕ on $T(G) - T(H)$. It is easy to see that ϕ is an H-partial AVD total 7-coloring of G. Since G is a counterexample, ϕ cannot be extended to an AVD total 7-coloring of G satisfying the conditions of Theorem 1.4 and thus H is not reducible. ■

The following claim is a direct corollary of Claim 3.1 and Lemma 2.2.

Claim 3.2. *Let H be an induced subgraph of $G[V_{3^-}]$ with $|E(H)| \neq 0$. Then $\widetilde{\mathcal{P}}(H)$ has no monomials $\prod_{z\in T(H)} x_z^{i_z}$ with nonzero coefficient such that $\sum_{z\in T(H)} i_z = deg(\widetilde{\mathcal{P}}(H))$ and $i_z \leq 6$ for each $z \in T(H)$.*

Claim 3.3. *For any edge u_1u_2 in $G[V_{3^-}]$, we have the following two assertions.*
(1) $d_G(u_1) = d_G(u_2) = 3$ and thus V_2 is an independent set.
(2) At least one of u_1 and u_2 is adjacent to three 3-vertices.

Proof. For $i = 1, 2$, let $d_i = d_G(u_i) - 1$ and $k_i = |N_G(u_i) \cap V_4|$. Since $u_1u_2 \in E(G[V_{3^-}])$, we have $0 \leq k_i \leq d_i \leq 2$. Let H be the induced subgraph of G with vertex set $V(H) = \{u_1, u_2\}$ and edge set $\{u_1u_2\}$, i.e. $H = u_1u_2$. For convenience denote $x_i = x_{u_i}$ for $i = 1, 2$ and $y = x_{u_1u_2}$. Then by the definition of $\widetilde{\mathcal{P}}(H)$, we have

$$\widetilde{\mathcal{P}}(H) = (x_1^2 y)^{d_1}(x_1 + y)^{d_1-k_1}(x_1 - x_2)(x_1 + y - x_2 - y)(x_1 - y) \cdot (x_2^2 y)^{d_2}(x_2 + y)^{d_2-k_2}(x_2 - y).$$

(1) We first prove that $d_G(u_1) = d_G(u_2) = 3$. Suppose to the contrary that, without loss of generality, $d_G(u_1) \leq 2$ and $d_G(u_2) \leq 3$. Then $0 \leq d_1 \leq 1$ and $0 \leq d_2 \leq 2$. It is easy to check that $deg(\widetilde{\mathcal{P}}(H)) = 4d_1 + 4d_2 - k_1 - k_2 + 4$ and that the coefficient of the monomial $x_1^{3d_1-k_1+3} x_2^{2d_2} y^{d_1+2d_2-k_2+1}$ of $\widetilde{\mathcal{P}}(H)$ is -1. We have that $3d_1 - k_1 + 3 \leq 6$, $2d_2 \leq 4$, $d_1 + 2d_2 - k_2 + 1 \leq 6$, and $3d_1 - k_1 + 3 + 2d_2 + d_1 + 2d_2 - k_2 + 1 = deg(\widetilde{\mathcal{P}}(H))$, a contradiction to Claim 3.2. This proves $d_G(u_1) = d_G(u_2) = 3$.

Note that the above assertion implies that V_{2^-} is an independent set of $G[V_{3^-}]$. Otherwise, we would have an edge u_1u_2 in $G[V_{3^-}]$ with $\min\{d_G(u_1), d_G(u_2)\} \leq 2$, a contradiction to the fact that $d_G(u_1) = d_G(u_2) = 3$.

(2) We now prove that at least one of u_1 and u_2 is adjacent to three 3-vertices (See Fig. 1). Suppose to the contrary that both u and v are adjacent to at most two 3-vertices. Then $k_1 \geq 1$ and $k_2 \geq 1$ since $d_G(u) = d_G(v) = 3$ and V_{2^-} is an independent set of $G[V_{3^-}]$ by the first part of the claim. Let $P = \widetilde{\mathcal{P}}(H) \cdot (x_1 + y)^{k_1-1}(x_2 + y)^{k_2-1}$. Since $d_1 = d_2 = 2$,

$$\begin{aligned} P &= \widetilde{\mathcal{P}}(H) \cdot (x_1 + y)^{k_1-1}(x_2 + y)^{k_2-1} \\ &= (x_1^2 y)^2 (x_2^2 y)^2 (x_1 + y)(x_2 + y)(x_1 - y)(x_2 - y)(x_1 - x_2)^2 \\ &= -2x_1^6 x_2^6 y^6 - x_1^4 x_2^4 y^6 (x_1^4 - 2x_1^3 x_2 - 2x_1 x_2^3 + x_2^4) + x_1^4 x_2^4 y^4 (x_1^2 x_2^2 + y^4)(x_1 - x_2)^2. \end{aligned}$$

Since $deg(P) = 18$ and the monomial $x_1^6 x_2^6 y^6$ of P has coefficient -2, by Lemma 2.3, $\widetilde{\mathcal{P}}(H)$ has a monomial $x_1^{i_1} x_2^{i_2} y^{i_3}$ with nonzero coefficient such that $i_1 + i_2 + i_3 = deg(\widetilde{\mathcal{P}}(H))$ and $i_j \leq 6$ for $j = 1, 2, 3$, a contradiction to Claim 3.2. ■

Claim 3.4. *If $u_1u_2u_3u_1$ is a triangle in $G[V_3]$, then each u_i is adjacent to three 3-vertices.*

Proof. By Claim 3.3, at most one vertex in the triangle is adjacent to a 4-vertex. Suppose to the contrary (without loss of generality) that u_1 is adjacent to one 4-vertex, while both u_2 and u_3 are adjacent to three 3-vertices. Let H be the triangle

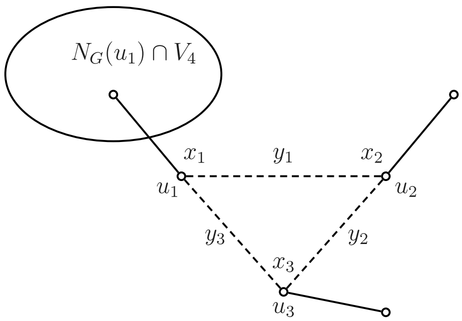

Fig. 2. H and the corresponding variables in Claim 3.4.

$u_1u_2u_3u_1$. For convenience denote $x_i = x_{u_i}$ for $i = 1, 2, 3$, $y_1 = x_{u_1u_2}$, $y_2 = x_{u_2u_3}$, and $y_3 = x_{u_3u_1}$ (See Fig. 2). Then we have

$$\begin{aligned}\widetilde{\mathcal{P}}(H) = &(x_1^2y_1y_3)(x_1 - y_1)(x_1 - y_3)(y_1 - y_3)(x_1 - x_2)(x_1 + y_3 - x_2 - y_2)\\ &\cdot(x_2^2y_1y_2)(x_2 + y_1 + y_2)(x_2 - y_1)(x_2 - y_2)(y_1 - y_2)(x_2 - x_3)(x_2 + y_1 - x_3 - y_3)\\ &\cdot(x_3^2y_2y_3)(x_3 + y_2 + y_3)(x_3 - y_2)(x_3 - y_3)(y_2 - y_3)(x_3 - x_1)(x_3 + y_2 - x_1 - y_1).\end{aligned}$$

One can check that the coefficient of the monomial $x_1^5x_2^5x_3^5y_1^6y_2^6y_3^2$ in $\widetilde{\mathcal{P}}(H)$ is -1 and $deg(\widetilde{\mathcal{P}}(H)) = 29$, a contradiction to Claim 3.2. ■

The final step. Let G_1 be a connected component of $G[V_3]$ with $|V(G_1)|$ maximum, and let $u_1 \in V(G_1)$ with $d_{G_1}(u_1)$ minimum. Then $|V(G_1)| \geq 2$ since V_3 is not independent in G by Lemma 3.1 and V_{2^-} is an independent set of $G[V_{3^-}]$ by Claim 3.3(1). Moreover, $d_{G_1}(u_1) \leq 2$ since $\Delta(G) = 4$ and G is connected. Pick an edge $u_1u_2 \in E(G_1)$. By Claim 3.3, u_2 is adjacent to three 3-vertices, and so let $u_3 \in N_{G_1}(u_2) \setminus \{u_1\} \subseteq V_3$.

Let $H = G[\{u_1, u_2, u_3\}]$, $k_1 = |N_G(u_1) \cap V_4|$ and $k_3 = |N_G(u_3) \cap V_4|$. Then $1 \leq k_1 \leq 2$ and $0 \leq k_3 \leq 2$. By Claim 3.4, H can not be a triangle and thus H is an induced path $u_1u_2u_3$ in G_1. For convenience denote $x_i = x_{u_i}$ for $i = 1, 2, 3$, $y_1 = x_{u_1u_2}$, and $y_2 = x_{u_2u_3}$. Then we have

$$\begin{aligned}\widetilde{\mathcal{P}}(H) = &(x_1^2y_1)^2(x_1 + y_1)^{2-k_1}(x_1 - y_1)(x_1 - x_2)(x_1 - x_2 - y_2)\\ &\cdot(x_2^2y_1y_2)(x_2 + y_1 + y_2)(x_2 - y_1)(x_2 - y_2)(y_1 - y_2)(x_2 - x_3)(x_2 + y_1 - x_3)\\ &\cdot(x_3^2y_2)^2(x_3 + y_2)^{2-k_3}(x_3 - y_2).\end{aligned}$$

Let $P = \widetilde{\mathcal{P}}(H)\cdot(x_1 + y_1)^{k_1-1}(x_3 + y_2)^{k_3}$. One can check that $deg(P) = 29$ and the coefficient of the monomial $x_1^6x_2^6x_3^6y_1^5y_2^6$ in P is 3. Therefore, by Lemma 2.3, $\widetilde{\mathcal{P}}(H)$ has a monomial $x_1^{i_1}x_2^{i_2}x_3^{i_3}y_1^{i_4}y_2^{i_5}$ with nonzero coefficient such that $i_j \leq 6$ for each $j = 1, \ldots, 5$ and $i_1 + \cdots + i_5 = deg(\widetilde{\mathcal{P}}(H))$, a contradiction to Claim 3.2. This contradiction completes the proof of Theorem 1.4.

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