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GROUP CONNECTIVITY, STRONGLY $\mathbb{Z}_m$-CONNECTIVITY, AND EDGE DISJOINT SPANNING TREES*

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Abstract. Let $\mathbb{Z}_m$ be the cyclic group of order $m \geq 3$. A graph G is $\mathbb{Z}_m$-connected if G has an orientation D such that for any mapping $b : V(G) \mapsto \mathbb{Z}_m$ with $\sum_{v\in V(G)} b(v) = 0$, there exists a mapping $f : E(G) \mapsto \mathbb{Z}_m - \{0\}$ satisfying $\sum_{e\in E^+_D(v)} f(e) - \sum_{e\in E^-_D(v)} f(e) = b(v)$ in $\mathbb{Z}_m$ for any $v \in V(G)$; and a graph G is strongly $\mathbb{Z}_m$-connected if, for any mapping $\theta : V(G) \to \mathbb{Z}_m$ with $\sum_{v\in V(G)} \theta(v) = |E(G)|$ in $\mathbb{Z}_m$, there is an orientation D such that $d^+_D(v) = \theta(v)$ in $\mathbb{Z}_m$ for each $v \in V(G)$. In this paper, we study the relation between $\mathbb{Z}_m$-connected graphs and strongly $\mathbb{Z}_m$-connected graphs and show that a graph G is $\mathbb{Z}_m$-connected if and only if $(m-2)G$ is strongly $\mathbb{Z}_m$-connected, where $(m-2)G$ is the graph obtained from G by replacing each edge in G with $m-2$ parallel edges. We also show that if G is $\mathbb{Z}_m$-connected, then $(m-2)G$ has $m-1$ edge disjoint spanning trees. Those results together with a result by Jaeger et al. [*J. Combin. Theory Ser. B*, 56 (1992), pp. 165–182] imply that every $\mathbb{Z}_3$-connected graph is A-connected for any abelian group A with $|A| \geq 4$. They are applied to determine the exact values of $ex(n, \mathbb{Z}_m)$ for all $m \geq 3$, where $ex(n, \mathbb{Z}_m)$ is the largest integer such that every simple graph on n vertices with at most $ex(n, \mathbb{Z}_m)$ edges is not $\mathbb{Z}_m$-connected, and to present characterizations of graphic and multigraphic sequences that have $\mathbb{Z}_m$-connected realizations.

Key words. nowhere-zero flow, modulo orientation, group connectivity, strongly $\mathbb{Z}_m$-connectivity, graphic sequence realization

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1. Introduction.

We consider finite graphs without loops but permitting multiple edges, and we follow [2, 31] for undefined terms and notation. Throughout this paper, $\mathbb{Z}$ denotes the additive group of the integers. For an integer $m \geq 2$, $\mathbb{Z}_m$ denotes the set of integers modulo m, as well as the (additive) cyclic group on m elements.

Let $D = D(G)$ be an orientation of a graph G. Following [2], (u, v) denotes an arc oriented from u to v, and $A(D)$ denotes the set of all arcs in D. For a vertex $v \in V(D)$, define

$$E^-_D(v) = \{(u, v) \in A(D)\}, E^+_D(v) = \{(v, u) \in A(D)\},$$
$$d^-_D(v) = |E^-_D(v)|, \text{ and } d^+_D(v) = |E^+_D(v)|.$$

The subscript D may be omitted when D is understood from the context.

Let A be an (additive) abelian group and G be a graph with an orientation $D = D(G)$. For subsets $X \subseteq E(G)$ and $A' \subseteq A$, define $F(X, A') = \{f | f : X \to A'\}$ to be the set of all mappings from X into A', and we use $F(G, A')$ for $F(E(G), A')$. To emphasize the orientation D, we often write a mapping $f \in F(X, A')$ as an ordered

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pair (D, f). When D is understood from the context, we simply use f for (D, f). If $f \in F(G, A)$, define $\partial f : V(G) \to A$, called the *boundary of* f, as follows:

$$\text{for any vertex } v \in V(G),\ \partial f(v) = \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e).$$

A function $b : V(G) \to A$ is an *A-zero-sum function* if $\sum_{v\in V(G)} b(v) = 0$, where 0 denotes the additive identity. The set of all A-zero-sum functions of G is denoted by $Z(G, A)$.

For any A-zero-sum function b of G, an (A', b)-flow is a mapping $f \in F(G, A')$ satisfying $\partial f = b$. When $b = 0$, an $(A - \{0\}, 0)$-flow is known as a nowhere-zero A-flow in the literature (see [25, 10, 31], among others). Following [11], if, for any $b \in Z(G, A)$, G always has an $(A - \{0\}, b)$-flow, then G is *A-connected.* The concept of strongly $\mathbb{Z}_{2s+1}$-connectedness was introduced in [17] (see also [15]). Motivated by the "θ-orientation" idea of Thomassen et al. (see [24, 18]), we will extend this notion to strongly $\mathbb{Z}_m$-connected graphs to include the case when m is even.

DEFINITION 1.1. *Let G be a graph, and let $\Theta(G, \mathbb{Z}_m) = \{\theta : V(G) \to \mathbb{Z}_m \mid \sum_{v\in V(G)} \theta(v) \equiv |E(G)| \pmod m\}$. A graph G is* strongly Z_m-connected *if, for any $\theta \in \Theta(G, \mathbb{Z}_m)$, there is an orientation D such that $d_D^+(v) \equiv \theta(v) \pmod m$ for every vertex $v \in V(G)$.*

Let $\langle A\rangle$ and $\langle \mathcal{S}\mathbb{Z}_m\rangle$ denote the family of all A-connected graphs and the family of all strongly $\mathbb{Z}_m$-connected graphs, respectively.

If a graph G has an orientation D such that for each vertex $v \in V(G)$, $d_D^+(v) - d_D^-(v) \equiv 0 \pmod m$, we say that G admits a *modulo m-orientation.* For $m = 2k + 1$, a modulo $(2k + 1)$-orientation of G can be also viewed as an orientation D such that $d_D^+(v) \equiv -kd_G(v)$ for each $v \in V(G)$; and G is strongly $\mathbb{Z}_{2k+1}$-connected can be equivalently defined as follows (see Proposition 1.3):

for any $b \in Z(G, \mathbb{Z}_{2k+1})$, there exists an orientation D of G such that $d_D^+(v) - d_D^-(v) \equiv b(v) \pmod{2k+1}$ for every $v \in V(G)$.

The strongly $\mathbb{Z}_{2k+1}$-connected graphs are also known as contractible configurations for modulo $(2k + 1)$-orientations (see [15, 17]).

It is known that a connected graph G has a modulo $2k$-orientation if and only if G is eulerian. Since $d_D^+(v) - d_D^-(v) = 2d_D^+(v) - d(v) \equiv d(v) \pmod 2$, every possible $\mathbb{Z}_{2k}$ boundary β must satisfy that $\beta(v) \equiv d(v) \pmod 2$ for every $v \in V(G)$. This motivates us to introduce the following definition.

DEFINITION 1.2. *Let $\Phi(G, \mathbb{Z}_{2k})$ be the collection of all functions $\beta : V(G) \to \mathbb{Z}$ satisfying that $0 \le \beta(v) \le 2k - 1$ and $\beta(v) \equiv d(v) \pmod 2$ for every $v \in V(G)$, and that $\sum_{v\in V(G)} \beta(v) \equiv 0 \pmod{2k}$. A graph G is* uniformly Z_{2k}-connected *if, for any $\beta \in \Phi(G, \mathbb{Z}_{2k})$, there is an orientation D such that for every vertex $v \in V(G)$, $d_D^+(v) - d_D^-(v) \equiv \beta(v) \pmod{2k}$. Let $\langle \mathcal{U}\mathbb{Z}_{2k}\rangle$ be the family of all uniformly Z_{2k}-connected graphs.*

In [24], Thomassen commented that an argument of Anton Kotzig implies that G is strongly $\mathbb{Z}_2$-connected if and only if G is connected. The following relations are observed in Wu's dissertation.

PROPOSITION 1.3 (see Wu [27]). *Let $k \ge 3$ be an integer. Then each of the following holds:*

(i) $\langle \mathcal{S}\mathbb{Z}_k\rangle = \langle \mathcal{U}\mathbb{Z}_{2k}\rangle$.

(ii) $G \in \langle \mathcal{S}\mathbb{Z}_{2k+1}\rangle$ *if and only if for any* $b \in Z(G, \mathbb{Z}_{2k+1})$, *there exists an orientation* D *of* G *such that* $d^+_D(v) - d^-_D(v) \equiv b(v) \pmod{2k+1}$ *for each* $v \in V(G)$.

In fact, for a given mapping $\theta \in \Theta(G, \mathbb{Z}_k)$, the orientation D of G with $d^+_D(v) \equiv \theta(v) \pmod{k}$ for each $v \in V(G)$ is precisely an orientation such that $d^+_D(v) - d^-_D(v) \equiv \beta(v) \pmod{2k}$ with $\beta(v) \equiv 2\theta(v) - d(v) \pmod{2k}$ for each $v \in V(G)$, where $\beta \in \Phi(G, \mathbb{Z}_{2k})$. Similarly, an orientation D of G with $d^+_D(v) \equiv \theta(v) \pmod{2k+1}$ is an orientation such that $d^+_D(v) - d^-_D(v) \equiv b(v) \pmod{2k+1}$ with $b(v) = 2\theta(v) - d(v)$ for each $v \in V(G)$, where $b \in Z(G, \mathbb{Z}_{2k+1})$. Hence any possible elements $\beta \in \Phi(G, \mathbb{Z}_{2k})$ or $b \in Z(G, \mathbb{Z}_{2k+1})$ can be realized by an orientation D via carefully choosing a mapping $\theta \in \Theta(G, \mathbb{Z}_m)$, and vice versa. So Proposition 1.3 follows from these arguments.

Jaeger et al. [11] prove the following result concerning the group connectivity and edge disjoint spanning trees.

THEOREM 1.4 (see [11, Theorem 3.1]). *Let* G *be a graph with two edge disjoint spanning trees. Then* G *is* A*-connected for any abelian group with* $|A| \geq 4$.

Let $t \geq 1$ be an integer and G be a graph. Define tG to be the graph obtained from G by replacing each edge of G with t parallel edges.

Motivated by Theorem 1.4 and Proposition 1.3, we prove the following result.

THEOREM 1.5. *Let* $m \geq 3$ *be an integer, and let* G *be a graph. Each of the following holds:*

(i) $G \in \langle \mathbb{Z}_m \rangle$ *if and only if* $(m-2)G \in \langle \mathcal{S}\mathbb{Z}_m \rangle$.

(ii) *If* G *is* $\mathbb{Z}_m$*-connected, then* $(m-2)G$ *has* $m-1$ *edge disjoint spanning trees.*

Jaeger et al. [11] pointed out that there exists a $\mathbb{Z}_5$-connected graph which is not $\mathbb{Z}_6$-connected. Nevertheless, Theorem 1.5(ii) with $m = 3$ together with Theorem 1.4 implies the following theorem.

THEOREM 1.6. *Every* $\mathbb{Z}_3$*-connected graph is* A*-connected for any abelian group* A *with* $|A| \geq 4$.

This paper is organized as follows. In section 2, we present a couple of other interesting applications of Theorem 1.5, including Theorem 2.8, which characterizes degree sequences with $\mathbb{Z}_k$-connected realizations and whose proof will be postponed to the last section. Section 3 is devoted to the proof of Theorem 1.5.

2. Other applications of Theorem 1.5.

2.1. The size of non-*A*-connected graphs. In [19], motivated by an open problem (Problem 7.21 of [16]), Luo, Xu, and Yu define $ex(n, A)$ for any integer n and any finite abelian group A: the largest integer k such that every simple graph on n vertices with at most k edges is not A-connected, and they prove the following.

THEOREM 2.1 (see [19, Theorems 2, 3, and 4]). *Let* A *be an abelian group with* $|A| = k \geq 4$, *and let* $n \geq k$ *be an integer.*

(i) *If* $n \geq 6$, *then* $\frac{3n}{2} \leq ex(n, \mathbb{Z}_3) \leq 2n - 3$.

(ii) $ex(n, A) \leq \lceil \frac{(n-1)(k-1)}{k-2} \rceil - 1$.

They conjecture that the upper bound is the exact value of $ex(n, A)$.

CONJECTURE 2.2 (see [19]). *If* $n \geq |A| \geq 4$ *or if* $n \geq 6$ *and* $A = \mathbb{Z}_3$, *then* $ex(n, A) = \lceil \frac{(n-1)(|A|-1)}{|A|-2} \rceil - 1$.

Wu et al. [28] verify Conjecture 2.2 for some finite cyclic groups.

THEOREM 2.3 (see [28, Theorem 1.5]).

(i) *If k is odd, $n \geq k \geq 4$ or if $n \geq 6$ and $k = 3$, then* $ex(n, \mathbb{Z}_k) = \lceil \frac{(n-1)(k-1)}{k-2} \rceil - 1$.

(ii) *If $n \geq 4$, then* $ex(n, \mathbb{Z}_4) = \lceil \frac{3n-3}{2} \rceil - 1$.

As a direct consequence of Theorem 1.5(ii), we prove that Conjecture 2.2 holds for all finite cyclic groups.

THEOREM 2.4. $ex(n, \mathbb{Z}_k) = \lceil \frac{(k-1)(n-1)}{k-2} \rceil - 1$ *for $n \geq k \geq 4$ or for $k = 3$ and $n \geq 6$.*

Proof. By Theorem 1.5(ii), if G is $\mathbb{Z}_k$-connected, then $|E(G)| \geq \lceil \frac{(k-1)(|V(G)|-1)}{k-2} \rceil$. Thus $ex(n, \mathbb{Z}_k) \geq \lceil \frac{(k-1)(n-1)}{k-2} \rceil - 1$. Hence the theorem follows from Theorem 2.1. ◻

2.2. Graphic degree sequences with $\mathbb{Z}_k$-connected realizations. A sequence of n nonnegative integers is graphic (multigraphic, respectively) if it is the degree sequence of a simple graph (a multigraph, respectively) G, where G is called a realization of the sequence. It has been extensively studied whether a degree sequence has a realization with certain properties. A noticeable application (see [21]) of graph realization with 4-flows has been found in the design of critical partial Latin squares which leads to the proof of the so-called simultaneous edge-coloring conjecture by Keedwell [12] and Cameron [3]. All graphic sequences which have realizations admitting a nowhere-zero 3-flow or 4-flow are characterized in [20, 21], respectively.

Wu et al. [28] present a characterization of graphic sequences with $\mathbb{Z}_4$-connected realizations which was conjectured by Luo, Xu, and Yu in [19].

THEOREM 2.5 (see [28, Theorem 1.5]). *Let $d = (d_1, d_2, \ldots, d_n)$ be a graphic sequence. Then d has a $\mathbb{Z}_4$-connected realization if and only if $\sum_{i=1}^n d_i \geq 3n - 3$ and* $\min\{d_1, d_2, \ldots, d_n\} \geq 2$.

Sufficient conditions for A-connected realization problems have been proved by Luo, Xu, and Yu in [19] for $|A| = 4$, and by Yin, Luo, and Guo [30] for $|A| \geq 5$.

THEOREM 2.6. *Let $d = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with* $\min\{d_1, d_2, \ldots, d_n\} \geq 2$ *and A be an abelian group with $|A| \geq 4$. Suppose* $\sum_{i=1}^n d_i \geq 2\lceil \frac{(|A|-1)(n-1)}{|A|-2} \rceil$.

(i) *(see* [19]*) If $|A| = 4$, then d has an A-connected realization.*

(ii) *(see* [30]*) If $|A| \geq 5$, then d has an A-connected realization.*

Very recently, Dai and Yin [6] presented a characterization of graphic sequences with a $\mathbb{Z}_3$-connected realization. If a sequence d consists of the terms $d_1, \ldots, d_t$ having multiplicities $m_1, \ldots, m_t$, we may write $d = (d_1^{m_1}, \ldots, d_t^{m_t})$ for convenience. For $n \geq 5$, let

$$S_1(n) = \{((n-1)^2, 3^{n-k-2}, 2^k) : 0 \leq k \leq n-4 \text{ and } k \equiv n \pmod 2\}$$

and

$$S_2(n) = \{(d_1, d_2, d_3, d_4, 2^{n-4}) : n-1 \geq d_1 \geq d_2 \geq d_3 \geq d_4 \geq 3 \text{ and } d_1+d_2+d_3+d_4 = 2n+4\}.$$

Denote

$$R(n) = \begin{cases} S_1(n) \cup S_2(n) & \text{if } n \text{ is odd;} \\ S_1(n) \cup S_2(n) \cup \{(n-1, 3^{n-1})\} & \text{if } n \text{ is even.} \end{cases}$$

THEOREM 2.7 (see [6]). *Let $n \geq 5$, and let $d = (d_1, d_2, \ldots, d_n)$ be a nonincreasing graphic sequence with $d_n \geq 2$. Then d has a $\mathbb{Z}_3$-connected realization if and only if both $\sum_{i=1}^n d_i \geq 4n - 4$ and $d \notin R(n)$.*

In this paper, by applying our main result (Theorem 1.5), we present a characterization of graphic and multigraphic sequences that have $\mathbb{Z}_k$-connected realizations for all $k \geq 4$ and $k \geq 3$, respectively.

THEOREM 2.8. *Let k be an integer. Each of the following holds:*

(i) *For $k \geq 4$, a graphic sequence $d = (d_1, d_2, \ldots, d_n)$ has a $\mathbb{Z}_k$-connected realization if and only if both* $\min\{d_1, d_2, \ldots, d_n\} \geq 2$ *and* $\sum_{i=1}^n d_i \geq 2\lceil \frac{(k-1)(n-1)}{k-2} \rceil$.

(ii) *For $k \geq 3$, a multigraphic sequence $d = (d_1, d_2, \ldots, d_n)$ has a $\mathbb{Z}_k$-connected realization if and only if both* $\min\{d_1, d_2, \ldots, d_n\} \geq 2$ *and* $\sum_{i=1}^n d_i \geq 2\lceil \frac{(k-1)(n-1)}{k-2} \rceil$.

3. Proof of Theorem 1.5.

The main goal of this section is to prove Theorem 1.5, showing a characterization of $\mathbb{Z}_m$-connectedness of a graph G in terms of the strongly $\mathbb{Z}_m$-connectedness of $(m-2)G$, as well as a conclusion on the lower bound of the strength (as defined in [4]) for $\mathbb{Z}_m$-connected graphs. Throughout this section, for each edge $e = uv \in E(G)$, we always let $[e]$ denote the set of $m-2$ parallel edges joining u and v in $(m-2)G$. We assume that if e_1 and e_2 are two distinct edges in $E(G)$ (possibly e_1 and e_2 are parallel edges in G), then $[e_1] \cap [e_2] = \emptyset$ in $(m-2)G$.

We shall prove Theorem 1.5(i) differently when m has different parities. Applying Proposition 1.3(ii), we first show Proposition 3.1 below when $m = 2k+1$ is an odd integer.

PROPOSITION 3.1. *Let $k > 0$ be an integer, and let G be a graph. Then $G \in \langle \mathbb{Z}_{2k+1} \rangle$ if and only if $(2k-1)G \in \langle \mathcal{S}\mathbb{Z}_{2k+1} \rangle$.*

Proof. By Proposition 1.3(ii), it is sufficient to show $G \in \langle \mathbb{Z}_{2k+1} \rangle$ if and only if for any $b \in Z((2k-1)G, \mathbb{Z}_{2k+1})$, there exists an orientation D of $(2k-1)G$ such that $d^+_D(v) - d^-_D(v) \equiv b(v) \pmod{2k+1}$ for each $v \in V((2k-1)G)$.

We denote $\mathbb{Z}_{2k+1} = \{0, 1, \ldots, 2k\}$ and $\mathbb{Z}^*_{2k+1} = \{1, \ldots, 2k\}$, and take the convention of regarding $\mathbb{Z}_{2k+1}$ and $\mathbb{Z}^*_{2k+1}$ as subsets of integers with the arithmetic operations taken modulo $2k+1$.

Suppose $G \in \langle \mathbb{Z}_{2k+1} \rangle$. Let $b \in Z(G, \mathbb{Z}_{2k+1})$. Since $G \in \langle \mathbb{Z}_{2k+1} \rangle$, there exist an orientation $D = D(G)$ and a mapping $f \in F^*(G, \mathbb{Z}_{2k+1})$ such that $\partial f = b$. For each $e = (u, v)$ of $D(G)$ with integral value $f(e)$, let

$$t(e) = \begin{cases} \frac{1}{2}(f(e) + 2k - 1) & \text{if } f(e) \text{ is odd;} \\ \frac{1}{2}f(e) + 2k & \text{if } f(e) \text{ is even and } f(e) < 0; \\ \frac{1}{2}f(e) - 1 & \text{if } f(e) \text{ is even and } f(e) > 0. \end{cases} \tag{1}$$

Since $0 < |f(e)| \leq 2k$ by (1), we have $0 \leq t(e) \leq 2k-1$ for any $e \in E(G)$. We shall give $(2k-1)G$ an orientation D' as follows. For each $e = (u, v)$ of $D(G)$, orient $t(e)$ edges in $[e]$ from u to v, and the rest of the $2k-1-t(e)$ edges in $[e]$ from v to u. Under the orientation D' of $(2k-1)G$, for any vertex $v \in V((2k-1)G)$,

$$\begin{aligned} &d^+_{D'}(v) - d^-_{D'}(v) \\ &= \left[\sum_{e \in E^+_{D(G)}(v)} t(e) + \sum_{e \in E^-_{D(G)}(v)} (2k-1-t(e)) \right] - \left[\sum_{e \in E^+_{D(G)}(v)} (2k-1-t(e)) + \sum_{e \in E^-_{D(G)}(v)} t(e) \right] \\ &= \sum_{e \in E^+_{D(G)}(v)} [t(e) - (2k-1-t(e))] - \sum_{e \in E^-_{D(G)}(v)} [t(e) - (2k-1-t(e))] \\ &= \sum_{e \in E^+_{D(G)}(v)} [2t(e) - 2k + 1] - \sum_{e \in E^-_{D(G)}(v)} [2t(e) - 2k + 1]. \end{aligned} \tag{2}$$

Since $4k \equiv 2k-1 \equiv -2 \pmod{2k+1}$, it follows from (1) and (2) that

$$d^+_{D'}(v) - d^-_{D'}(v) \equiv \sum_{e \in E^+_{D(G)}(v)} f(e) - \sum_{e \in E^-_{D(G)}(v)} f(e) \equiv b(v) \pmod{2k+1}.$$

Therefore $(2k-1)G \in \langle \mathcal{S}\mathbb{Z}_{2k+1} \rangle$ by Proposition 1.3(ii).

Conversely, assume $(2k-1)G \in \langle \mathcal{S}\mathbb{Z}_{2k+1} \rangle$. Let $b \in Z(G, \mathbb{Z}_{2k+1})$. By Proposition 1.3(ii), $(2k-1)G$ has an orientation D' such that for any vertex v, $d^+_{D'}(v) - d^-_{D'}(v) \equiv b(v) \pmod{2k+1}$. Let $D = D(G)$ be an orientation of G. For each $e = (u, v)$ in $D(G)$, let $t(e)$ be the number of edges in $[e]$ oriented from u to v under the orientation D'. Define $f(e) = 2t(e) - (2k-1)$ as integers. Since $f(e)$ is odd, $f(e) \neq 0$. Since $0 \leq t(e) < 2k$, it follows that $-(2k-1) \leq f(e) \leq 2k-1$, and so $f \in F^*(G, \mathbb{Z}_{2k+1})$. By (2) and by the definition of f, we conclude that $\partial f(v) = b(v)$ for every $v \in V(G)$. Hence $G \in \langle \mathbb{Z}_{2k+1} \rangle$. This proves Proposition 3.1. ∎

Next, we are to prove Theorem 1.5(i) when m is even. By Proposition 1.3(i), it suffices to show that, when $m = k$ is even, $G \in \langle \mathbb{Z}_k \rangle$ if and only if $(k-2)G \in \langle \mathcal{U}\mathbb{Z}_{2k} \rangle$. To justify this, we need the following technical lemma. Throughout the rest of this section, we adopt the convention of viewing $\mathbb{Z}_{2k} = \{0, 1, 2, \ldots, 2k-1\}$ as a subset of integers, with the arithmetic operations taken modulo $2k$. Similarly, we view $\mathbb{Z}_k = \{0, 1, 2, \ldots, k-1\}$ as a subset of integers, with its arithmetic operations taken modulo k.

LEMMA 3.2. *Let G be a graph and $k \geq 1$ be an integer. Let $D' = D'(G)$ be an orientation of G. The following are equivalent:*

(i) $G \in \langle \mathbb{Z}_{2k} \rangle$.

(ii) *For any $b \in Z(G, \mathbb{Z}_{2k})$, there exists a mapping $f_1 \in F(G, \mathbb{Z}_{2k} - \{k\})$ such that $\partial f_1 = b$ in $\mathbb{Z}_{2k}$.*

(iii) *For any $b \in Z(G, \mathbb{Z}_{2k})$, there exist an orientation D of G and $f \in F(G, \{0, 1, \ldots, k-1\})$ such that $\partial f = b$ in $\mathbb{Z}_{2k}$ under orientation D.*

Proof. Throughout the proof of this lemma, the mappings b_0 and f_0 are defined as follows. Let $b_0 : V(G) \mapsto \{0, k\} \subseteq \mathbb{Z}_{2k}$ be a mapping such that for any $v \in V(G)$, $b_0(v) = 0$ if $d_G(v)$ is even, and $b_0(v) = k$ if $d_G(v)$ is odd. Since the number of odd degree vertices in any graph is even, it follows that $b_0 \in Z(G, \mathbb{Z}_{2k})$. Let $f_0 = F(G, \{k\})$ be the constant function. Then under any orientation of G, by the definitions of b_0 and f_0, we have $\partial f_0 = b_0$.

Assume that (i) holds. For any $b \in Z(G, \mathbb{Z}_{2k})$, let $b_2 = b - b_0 \in Z(G, \mathbb{Z}_{2k})$. Since $G \in \langle \mathbb{Z}_{2k} \rangle$, there exists $f_2 \in F(G, Z_{2k} - \{0\})$ such that $\partial f_2 = b_2 = b - b_0$. Let $f_1 = f_2 + f_0$. Then $\partial f_1 = \partial f_2 + \partial f_0 = b$ and for every $e \in E(G)$, $f_1(e) = f_2(e) + f_0(e) = f_2(e) + k \neq k$. Thus (ii) holds.

Assume that (ii) holds. Then for any $b \in Z(G, \mathbb{Z}_{2k})$, there exists a mapping $f_1 \in F(G, \mathbb{Z}_{2k} - \{k\})$ such that $\partial f_1 = b$ in $\mathbb{Z}_{2k}$. We define a new mapping f and a new orientation D as follows. For each edge $e \in E(G)$, if $0 \leq f_1(e) \leq k-1$, then define $f(e) = f_1(e)$ and the orientation of e in D is the same as in D'; if $k+1 \leq f_1(e) \leq 2k-1$, then define $f(e) = 2k - f_1(e)$ and oriented e in D by reversing the orientation of e in D'. Since $f_1 \in F(G, \mathbb{Z}_{2k} - \{k\})$ and $\partial f_1 = b$, we have $f \in F(G, \{0, 1, \ldots, k-1\})$ and, under the orientation D, $\partial f = b$ in $\mathbb{Z}_{2k}$. Thus (iii) holds.

Assume that (iii) holds. Let $b \in Z(G, \mathbb{Z}_{2k})$. Then $b - b_0 \in Z(G, \mathbb{Z}_{2k})$. By (iii), there exist an orientation D of G and $f_1 \in F(G, \{0, 1, \ldots, k-1\})$ such that $\partial f_1 = b - b_0$. Let $f = f_1 + f_0$. Then as $f_1 \in F(G, \{0, 1, \ldots, k-1\})$, we have $f \in F^*(G, Z_{2k})$. Moreover, $\partial f = \partial f_1 + \partial f_0 = (b - b_0) + b_0 = b$. Hence $G \in \langle \mathbb{Z}_{2k} \rangle$ by definition. ∎

PROPOSITION 3.3. *Let $k > 0$ be an even integer. The following are equivalent:*
(i) $G \in \langle \mathbb{Z}_k \rangle$.
(ii) $(k-2)G \in \langle \mathcal{U}\mathbb{Z}_{2k} \rangle$.

Proof. Let $G \in \langle \mathbb{Z}_k \rangle$, and let $\beta \in \Phi((k-2)G, \mathbb{Z}_{2k})$. For any $v \in V(G)$, since $k-2$ is even, $\beta(v)$ is even and thus $\beta(v) = 2j_v$ for some integer j_v with $0 \le j_v \le k-1$. Define $b(v) = j_v$ for each $v \in V(G)$. By Definition 1.2, there exists an integer t such that $\sum_{v\in V(G)} \beta(v) = 2tk$, and so $\sum_{v\in V(G)} b(v) = tk \equiv 0 \pmod{k}$. Hence $b \in Z(G, \mathbb{Z}_k)$. Since $G \in \langle \mathbb{Z}_k \rangle$, by Lemma 3.2 there exist an orientation $D = D(G)$ of G and $f \in F(G, \{0, 1, \dots, k/2-1\})$ such that $\partial f = b$. Thus $f(e) \ne k/2$ for any $e \in E(G)$. We will construct an orientation D' of $(k-2)G$ as follows. For any edge $e = uv \in E(G)$, if $(u, v) \in A(D)$, orient $f(e) + k/2 - 1$ edges in $[e]$ from u to v and the other $k/2 - 1 - f(e)$ edges in $[e]$ from v to u. Thus, under orientation D' of $(k-2)G$,

$$\begin{aligned} d^+_{D'}(w) - d^-_{D'}(w) &= \sum_{e\in E^+_D(w)} \left[\left(f(e) + \frac{k}{2} - 1\right) - \left(\frac{k}{2} - 1 - f(e)\right)\right] \\ &\quad - \sum_{e\in E^-_D(w)} \left[\left(f(e) + \frac{k}{2} - 1\right) - \left(\frac{k}{2} - 1 - f(e)\right)\right] \\ &= \sum_{e\in E^+_D(w)} 2f(e) - \sum_{e\in E^-_D(w)} 2f(e) \\ &\equiv 2\partial f(w) \equiv 2b(w) \equiv \beta(w) \pmod{2k}. \end{aligned}$$

Therefore $(k-2)G \in \langle \mathcal{U}\mathbb{Z}_{2k} \rangle$.

Conversely, assume $(k-2)G \in \langle \mathcal{U}\mathbb{Z}_{2k} \rangle$. Let $b \in Z(G, \mathbb{Z}_k)$ and $\beta = 2b$. Since $\mathbb{Z}_k = \{0, 1, \dots, k-1\}$ is a subset of integers, we have $0 \le \beta(w) \le 2k-2$ for any $w \in V(G)$. Since $k-2$ is even, for any $w \in V(G)$, $\beta(w) = 2b(w) \equiv 0 \equiv d_{(k-2)G}(w) \pmod{2}$. As $\sum_{w\in V(G)} b(w) \equiv 0 \pmod{k}$, there exists an integer t with $\sum_{w\in V(G)} b(w) = tk$, and so $\sum_{w\in V(G)} \beta(w) = 2\sum_{w\in V(G)} b(w) = 2tk \equiv 0 \pmod{2k}$. Hence $\beta \in \Phi((k-2)G, \mathbb{Z}_{2k})$. Since $(k-2)G \in \langle \mathcal{U}\mathbb{Z}_{2k} \rangle$, there exists an orientation D' of $(k-2)G$ such that $d^+_{D'}(v) - d^-_{D'}(v) \equiv 2b(v) \pmod{2k}$. Thus $d^+_{D'}(v) - d^-_{D'}(v)$ is an even integer, and it follows that for every $v \in V((k-2)G)$,

$$\frac{1}{2}(d^+_{D'}(v) - d^-_{D'}(v)) \equiv b(v) \pmod{k}. \tag{3}$$

For each edge $e = uv \in E(G)$, let $t(e)$ be the number of edges in $[e]$ oriented from u to v in D'. We are to construct an orientation D of G by orienting e from u to v if and only if $t(e) \ge \frac{k}{2} - 1$. For each $e = (u, v) \in A(D)$, define $f(e) = t(e) - \frac{k}{2} + 1$. Then $0 \le f(e) \le \frac{k}{2} - 1$, and hence $f \in F(G, \{0, 1, \dots, k/2-1\})$. Moreover, for any

$w \in V(G)$,

(4)

$$\begin{aligned}\partial f(w) &= \sum_{e\in E_D^+(w)} f(e) - \sum_{e\in E_D^-(w)} f(e)\\ &= \sum_{e\in E_D^+(w)} \left(t(e) - \frac{k}{2} + 1\right) - \sum_{e'\in E_D^-(w)} \left(t(e') - \frac{k}{2} + 1\right)\\ &= \frac{1}{2}\left[\sum_{e\in E_D^+(w)} (t(e) - (k-2-t(e))) + \sum_{e'\in E_D^-(w)} ((k-2-t(e')) - t(e'))\right]\\ &= \frac{1}{2}\left[\sum_{e\in E_D^+(w)} t(e) + \sum_{e'\in E_D^-(w)} (k-2-t(e'))\right.\\ &\qquad \left. - \left(\sum_{e\in E_D^+(w)} (k-2-t(e)) + \sum_{e'\in E_D^-(w)} t(e')\right)\right].\end{aligned}$$

Under the orientation D' of $(k-2)G$, we have

$$d^+_{D'}(w) = \sum_{e\in E_D^+(w)} t(e) + \sum_{e'\in E_D^-(w)} (k-2-t(e')), \tag{5}$$

$$d^-_{D'}(w) = \sum_{e\in E_D^+(w)} (k-2-t(e)) + \sum_{e'\in E_D^-(w)} t(e'). \tag{6}$$

Combining (5), (6), (4), and (3), we have for any $w \in V(G)$,

$$\partial f(w) = \frac{1}{2}(d^+_{D'}(w) - d^-_{D'}(w)) \equiv b(w) \pmod{k}.$$

Since $f \in F(G, \{0, 1, \ldots, k/2-1\})$, we conclude that $G \in \langle \mathbb{Z}_k \rangle$ by Lemma 3.2. □

Theorem 1.5(i) now follows from Propositions 3.1 and 3.3. We will prove Theorem 1.5(ii).

For an edge set $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. If H is a subgraph of G, then we use G/H for $G/E(H)$. As in [2], $c(G)$ denotes the number of components of G. Let H be a graph with $|E(H)| > 1$. Following [4, 9], define $g(H) = \frac{|E(H)|}{|V(H)|-c(H)}$. The *strength* of G, as defined in [4, 5], is $\eta(G) = \min\{g(G/X) | X \subseteq E(G)$ with $V(G[X]) \neq V(G)\}$. Let $\tau(G)$ be the maximum number of edge disjoint spanning trees in G. A fundamental theorem proved independently by Nash-Williams and Tutte implies the following.

THEOREM 3.4 (see Nash-Williams [22] and Tutte [26]). *For a connected graph G, $\tau(G) = \lfloor \eta(G) \rfloor$.*

By the definitions of τ and $\langle \mathcal{S}\mathbb{Z}_k \rangle$, we note that $\tau(K_1) = \infty$ and $K_1 \in \langle \mathcal{S}\mathbb{Z}_k \rangle$ and make the following observation.

OBSERVATION 3.5. *Let $k \geq 1$ be an integer, and let $\mathcal{T}_k = \{G | \tau(G) \geq k\}$. If $\mathcal{F} = \langle \mathcal{S}\mathbb{Z}_k \rangle$ or if $\mathcal{F} = \mathcal{T}_k$, then each of the following holds:*

(i) *If* $G \in \mathcal{F}$ *and if* $e \in E(G)$*, then* $G/e \in \mathcal{F}$.

(ii) *If* H *is a subgraph of* G *and if both* H *and* G/H *are in* $\mathcal{F}$*, then* $G \in \mathcal{F}$.

Proof. The proof is routine when $\mathcal{F} = \mathcal{T}_k$, as k edge disjoint spanning trees of G can be found by combining the k edge disjoint spanning trees of G/H and of H. For the proofs of Observation 3.5(i)–(ii) when $\mathcal{F} = \langle \mathcal{S}\mathbb{Z}_k \rangle$, it is more convenient to apply Proposition 1.3(i) with $\langle \mathcal{S}\mathbb{Z}_k \rangle = \langle \mathcal{U}\mathbb{Z}_{2k} \rangle$.

Assume that $G \in \langle \mathcal{U}\mathbb{Z}_{2k} \rangle$. Let $e = uv \in E(G)$, and denote w to be the contracted vertex corresponding to e in G/e. For any $\beta \in \Phi(G/e, \mathbb{Z}_{2k})$, define a mapping $\beta' : V(G) \mapsto \mathbb{Z}_{2k}$ to be $\beta'(x) = \beta(x)$ for any $x \in V(G) - \{u, v\}$, $\beta'(u) = d(u)$ and $\beta'(v) = \beta(w) - d(u)$ in $\mathbb{Z}_{2k}$. Then $\beta'(v) \equiv \beta(w) - d(u) \equiv (d(v) + d(u)) - d(u) \equiv d(v) \pmod 2$. Moreover, we have $\sum_{x\in V(G)} \beta'(x) \equiv \sum_{x\in V(G/e)} \beta(x) \equiv 0 \pmod{2k}$. Hence $\beta' \in \Phi(G, \mathbb{Z}_{2k})$. As $G \in \langle \mathcal{U}\mathbb{Z}_{2k} \rangle$, there exists an orientation D' of G such that $d^+_{D'}(x) - d^-_{D'}(x) \equiv \beta'(x) \pmod{2k}$ for any $x \in V(G)$. By contracting the edge e, this results in an orientation D of G/e such that for any $x \in V(G/e)$, $d^+_D(x) - d^-_D(x) \equiv \beta(x) \pmod{2k}$. Hence $G/e \in \langle \mathcal{U}\mathbb{Z}_{2k} \rangle$ by definition. This proves Observation 3.5(i) for $\mathcal{F} = \langle \mathcal{S}\mathbb{Z}_k \rangle$. The proof of Observation 3.5(ii) for $\mathcal{F} = \langle \mathcal{S}\mathbb{Z}_k \rangle$ is similar and thus omitted. ∎

A graph is nontrivial if it contains at least one nonloop edge. The next lemma follows from the arguments of Nash-Williams in [23]. A detailed proof can be found in Theorem 2.4 of [29].

LEMMA 3.6. *Let* G *be a nontrivial graph, and let* $k > 0$ *be a integer. If* $g(G) \geq k$*, then* G *has a nontrivial subgraph* H *with* $\tau(H) \geq k$.

The following theorem is a special case of Theorem 4 in [4].

THEOREM 3.7 (see Catlin et al. [4]). *Let* s, t *be integers with* $s \geq t > 0$ *and* H *be a nontrivial graph; then* $\eta(H) \geq \frac{s}{t}$ *if and only if* $\tau(tH) \geq s$.

By Theorem 3.7, Theorem 1.5(ii) is equivalent to the following.

THEOREM 3.8. *Let* $k \geq 3$ *be an integer and* G *be a* $\mathbb{Z}_k$*-connected graph on* $n \geq 2$ *vertices. Each of the following holds:*

(i) $\eta(G) \geq \frac{k-1}{k-2}$.

(ii) *In particular,* $|E(G)| \geq \lceil \frac{(k-1)(n-1)}{k-2} \rceil$.

By the definition of η, Theorem 3.8(ii) follows from Theorem 3.8(i). By Theorems 3.4 and 3.7, Theorem 3.8 follows from Theorem 1.5(i) and Proposition 3.9 below.

PROPOSITION 3.9. *Let* $k \geq 1$ *be an integer. If* $G \in \langle \mathcal{S}\mathbb{Z}_k \rangle$*, then* $\tau(G) \geq k - 1$.

Proof. Since all graphs in $\langle \mathcal{S}\mathbb{Z}_k \rangle$ are connected, we may assume $k \geq 2$. Let G be a counterexample with $|V(G)| + |E(G)|$ minimized and with $n = |V(G)| > 1$. We first claim that G has no nontrivial subgraph $H \in \mathcal{T}_{k-1}$. Otherwise, let $H \in \mathcal{T}_{k-1}$ be a nontrivial subgraph of G. Then by Observation 3.5(i), $G/H \in \langle \mathcal{S}\mathbb{Z}_k \rangle$. By the minimality of G, $G/H \in \mathcal{T}_{k-1}$. Hence by Observation 3.5(ii), $G \in \mathcal{T}_{k-1}$, a contradiction to the assumption that G is a counterexample. Therefore G does not have a nontrivial subgraph in $\mathcal{T}_{k-1}$. By Lemma 3.6, we have $g(G) < k - 1$. Thus $|E(G)| < (k - 1)(n - 1)$ since G is connected.

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Set $\theta(v_i) = k - 1$ if $1 \leq i \leq n - 1$, and $\theta(v_n) = |E(G)| - \sum_{i=1}^{n-1} \theta(v_i)$. Then $\theta \in \Theta(G, \mathbb{Z}_k)$. Since $G \in \langle \mathcal{S}\mathbb{Z}_k \rangle$, there exists an orientation D of G such that for any $v \in V(G)$, $d^+_D(v) \equiv \theta(v) \pmod k$. Thus for each $1 \leq i \leq n - 1$, $d_G(v_i) \geq d^+_D(v_i) \geq k - 1$, and so $|E(G)| \geq \sum_{i=1}^{n-1} d^+(v_i) \geq (k - 1)(n - 1)$, a contradiction to the fact that $|E(G)| < (k-1)(n-1)$. This completes the proof of the proposition. ∎

4. Proof of Theorem 2.8. We will prove Theorem 2.8 in this section. By Theorem 3.8, if G is a $\mathbb{Z}_k$-connected realization of $d = (d_1, d_2, \ldots, d_n)$, then $\sum_{i=1}^n d_i = 2|E(G)| \geq 2\lceil \frac{(k-1)(n-1)}{k-2} \rceil$. Hence, together with Theorem 2.6, Theorem 2.8(i) follows. It remains to prove Theorem 2.8(ii). The following former results and newly developed lemmas will be needed in our arguments.

THEOREM 4.1 (see Hakimi [8]). *If $d = (d_1, d_2, \ldots, d_n)$ is a nonincreasing integral sequence with $n \geq 2$ and $d_n \geq 0$, then d is a multigraphic sequence if and only if $\sum_{i=1}^n d_i$ is even and $d_1 \leq d_2 + \cdots + d_n$.*

THEOREM 4.2 (see Boesch and Harary [1]). *Let $d = (d_1, \ldots, d_n)$ be a nonincreasing integral sequence with $n \geq 2$ and $d_n \geq 0$. Let j be an integer with $2 \leq j \leq n$ such that $d_j \geq 1$. Then the sequence $(d_1, d_2, \ldots, d_n)$ is multigraphic if and only if the sequence $(d_1 - 1, d_2, \ldots, d_{j-1}, d_j - 1, d_{j+1}, \ldots, d_n)$ is multigraphic.*

LEMMA 4.3 (see [13, Proposition 3.2 and Lemma 3.3] and [14, Lemma 2.1]). *Let $k \geq 3$ be an integer, G be a graph, and H be a subgraph of G.*

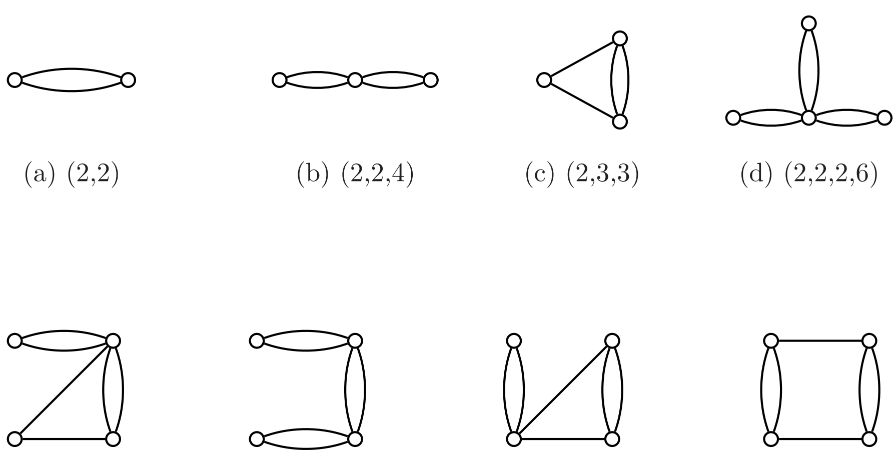
(i) *If $H \in \langle \mathbb{Z}_k \rangle$ and $G/H \in \langle \mathbb{Z}_k \rangle$, then $G \in \langle \mathbb{Z}_k \rangle$.*

(ii) *(see also [11]) A cycle of length n is in $\langle \mathbb{Z}_k \rangle$ if and only if $n \leq k-1$.*

(iii) *If G is connected and every edge lies in a cycle of length at most $k-1$, then G is $\mathbb{Z}_k$-connected.*

LEMMA 4.4. *Let $d = (d_1, d_2, \ldots, d_n)$ be a nonincreasing multigraphic sequence with $d_n \geq 2$ and $\sum_{i=1}^n d_i = 4n - 4$. Then d has a $\mathbb{Z}_3$-connected realization.*

Proof. We argue by induction on n. If $2 \leq n \leq 4$, then all the graphs whose degree sequences satisfy the hypothesis of Lemma 4.4 are depicted in Figure 1 below.



(a) (2,2) (b) (2,2,4) (c) (2,3,3) (d) (2,2,2,6)

(e) (2,2,3,5) (f) (2,2,4,4) (g) (2,3,3,4) (h) (3,3,3,3)

FIG. 1. *Multigraphic degree sequences and their $\mathbb{Z}_3$-connected realization when $2 \leq n \leq 4$.*

It follows from Lemma 4.3 that every graph in Figure 1 is $\mathbb{Z}_3$-connected, and so Lemma 4.4 holds if $2 \leq n \leq 4$. We now assume that $n \geq 5$ and that the lemma holds for smaller values of n. Since $d_n \geq 2$ and $\sum_{i=1}^{n} d_i = 4n-4$, we conclude that $2 \leq d_n \leq 3$.

Case 1. $d_n = 2$.

Since $n \geq 5$ and $\sum_{i=1}^{n} d_i = 4n-4$, we have $d_1 \geq 4$. Let $d' = (d_1-2, d_2, \dots, d_{n-1})$. If d' is a multigraphic sequence, d' has a $\mathbb{Z}_3$-connected realization H such that $v_1 \in V(H)$ has degree $d_1 - 2$ in H by induction hypothesis. Construct a new graph G from H by adding a new vertex $v_n \notin V(H)$ and two new parallel edges joining v_1 and v_n. Then G is a realization of d. Moreover, since $H \in \langle \mathbb{Z}_3 \rangle$ and since G/H is a cycle of length 2, by Lemma 4.3, G is a $\mathbb{Z}_3$-connected realization of d. Hence we assume that d' is not multigraphic. By Theorem 4.1, we must have $d_2 > d_1 - 2 + \sum_{i=3}^{n-1} d_i$. Since d is a nonincreasing sequence and $n \geq 5$, we have $d_2 > d_1 - 2 + 2 + 2 = d_1 + 2 \geq d_2 + 2$, a contradiction. This shows that d must have a $\mathbb{Z}_3$-connected realization in Case 1.

Case 2. $d_n = 3$.

By $\sum_{i=1}^{n} d_i = 4n-4$, we have $d_{n-3} = d_{n-2} = d_{n-1} = d_n = 3$ and $4 \leq d_1 \leq n-1$. Thus $\max\{d_1-2, d_2\} \leq n-1 \leq 3n-11 \leq d_1 - 2 + d_2 - \max\{d_1-2, d_2\} + \sum_{i=3}^{n-2} d_i$. Let $d^* = (d_1 - 2, d_2, \dots, d_{n-2})$. Then by Theorem 4.1, d^* is multigraphic. By induction, d^* has a $\mathbb{Z}_3$-connected realization H with a vertex $v_1 \in V(H)$ having degree $d_1 - 2$ in H. Construct a new graph G from H by adding two new vertices $v_{n-1}, v_n \notin V(H)$, two new parallel edges joining v_{n-1} and v_n, and two new edges v_1v_{n-1}, v_1v_n. By Lemma 4.3, G is a $\mathbb{Z}_3$-connected realization of d since $H \in \langle \mathbb{Z}_3 \rangle$ and $G/H \in \langle \mathbb{Z}_3 \rangle$. This proves the lemma. ∎

A graph G is *supereulerian* if G contains a spanning eulerian subgraph.

LEMMA 4.5 (see [7, Theorem 1.6]). *Let $d = (d_1, d_2, \dots, d_n)$ be a nonincreasing multigraphic sequence. Then d has a supereulerian realization if and only if either $n = 1$ and $d_1 = 0$, or $n \geq 2$ and $d_n \geq 2$.*

LEMMA 4.6. *Let $n \geq 2$ and $k \geq 4$. Let $d = (d_1, d_2, \dots, d_n)$ be a nonincreasing integral sequence with $d_n \geq 2$. Each of the following holds:*

(i) *If d is multigraphic with $n \leq k$, and $\sum_{i=1}^{n} d_i \geq 2\lceil \frac{(k-1)(n-1)}{k-2} \rceil$, then d has a $\mathbb{Z}_k$-connected realization.*

(ii) *If $\sum_{i=1}^{n} d_i = 2\lceil \frac{(k-1)(n-1)}{k-2} \rceil$, then d is multigraphic and has a $\mathbb{Z}_k$-connected realization.*

Proof. (i) Assume $n \leq k$. By Lemma 4.5, d has a supereulerian realization G. Thus G is 2-edge-connected with $|V(G)| \leq k$. If $n \leq k-1$, then every edge lies in a cycle of length at most $k-1$. Thus by Lemma 4.3(iii), $G \in \langle \mathbb{Z}_k \rangle$. Now assume $n = k$. If G contains a cycle of length 2, say C, then G/C remains 2-edge-connected and has $k-1$ vertices. Thus $G/C \in \langle \mathbb{Z}_k \rangle$. By Lemma 4.3(ii) and (i), $G \in \langle \mathbb{Z}_k \rangle$. Hence we may further assume that G is simple. Since $\sum_{i=1}^{n} d_i \geq 2\lceil \frac{(k-1)(n-1)}{k-2} \rceil$, G is not a cycle. It follows that every edge of G is in a cycle of length at most $k-1$. By Lemma 4.3(iii), $G \in \langle \mathbb{Z}_k \rangle$. This proves (i).

(ii) Since $d_n \geq 2$ and $\sum_{i=1}^{n} d_i = 2\lceil \frac{(k-1)(n-1)}{k-2} \rceil$, we have $d_1 = 2\lceil \frac{(k-1)(n-1)}{k-2} \rceil - \sum_{i=2}^{n} d_i \leq 2\lceil \frac{(k-1)(n-1)}{k-2} \rceil - 2(n-1) \leq 2(n-1) \leq \sum_{i=2}^{n} d_i$. Thus by Theorem 4.1, d is multigraphic.

Now we show that d has a $\mathbb{Z}_k$-connected realization. By (i), we may assume $n \geq k+1$.

We first prove the following statement:

$$d_n = \cdots = d_{n-(k-3)} = 2 \text{ and } d_1 \geq 3. \tag{7}$$

Since $nd_1 \geq \sum_{i=1}^n d_i = 2\lceil \frac{(k-1)(n-1)}{k-2} \rceil \geq 2n-2+\frac{2(n-1)}{k-2} > 2n$, we have $d_1 \geq 3$. Suppose to the contrary that $d_{n-(k-3)} \geq 3$. Then, since $n \geq k+1$,

$$\sum_{i=1}^n d_i \geq 2\left\lceil \frac{3(n-(k-3))+2(k-3)}{2} \right\rceil = 2\left\lceil \frac{(k-1)(n-1)}{k-2} + \frac{(k-4)n}{2(k-2)} + \frac{k-1}{k-2} - \frac{k-3}{2} \right\rceil$$
$$\geq 2\left\lceil \frac{(k-1)(n-1)}{k-2} + \frac{4k-12}{2(k-2)} \right\rceil \geq 2\left\lceil \frac{(k-1)(n-1)}{k-2} + 1 \right\rceil > 2\left\lceil \frac{(k-1)(n-1)}{k-2} \right\rceil,$$

a contradiction to the assumption that $\sum_{i=1}^n d_i = 2\lceil \frac{(k-1)(n-1)}{k-2} \rceil$. Since d is nonincreasing, $d_n = \cdots = d_{n-(k-3)} = 2$. This proves (7).

We argue by induction on n to prove Lemma 4.6(ii). Assume that $n \geq k+1 \geq 5$ and Lemma 4.6 holds for smaller values of n.

Case 1. $d_2 \geq 3$.

Let $d' = (d_1-1, d_2-1, d_3, \ldots, d_{n-(k-2)})$. By (7), we have

$$\sum_{i=1}^{n-(k-2)} d'_i = 2\left\lceil \frac{(k-1)(n-1)}{k-2} \right\rceil - 2 - 2(k-2) = 2\left\lceil \frac{(k-1)(n-(k-2)-1)}{k-2} \right\rceil.$$

Since $\min_{i \leq n-(k-2)}\{d'_i\} \geq \min\{d_2-1, d_{n-(k-2)}\} \geq 2$, by induction, d' is a multigraphic sequence with a $\mathbb{Z}_k$-connected realization G' with $V(G') = \{v_1, v_2, \ldots, v_{n-(k-2)}\}$ such that $d_{G'}(v_i) = d'_i$ for each $i = 1, 2, \ldots, n-k+2$.

Construct a new graph G from G' by adding a new path $P = v_{n-(k-3)} \cdots v_n$ with $V(P) \cap V(G') = \emptyset$, and two new edges $v_nv_1, v_{n-(k-3)}v_2$. Then G is a realization of d. Moreover, since $G' \in \langle \mathbb{Z}_k \rangle$ and G/G' is a cycle of length $k-1$, it follows from Lemma 4.3 that G is a $\mathbb{Z}_k$-connected realization of d.

Case 2. $d_2 = 2$.

Since $d_2 = \cdots = d_n = 2$ and $\sum_{i=1}^n d_i$ must be even, we have $d_1 \geq 4$ by (7). Let $d^* = (d_1-2, d_2, \ldots, d_{n-(k-2)})$. Since

$$\sum_{i=1}^{n-(k-2)} d'_i = 2\left\lceil \frac{(k-1)(n-(k-2)-1)}{k-2} \right\rceil$$

and $\min_{i \leq n-(k-2)}\{d^*_i\} \geq \min\{d_1-2, d_{n-(k-2)}\} \geq 2$, by induction, d^* is a multigraphic sequence and has a $\mathbb{Z}_k$-connected realization G^*. Denote $V(G^*) = \{v_1, v_2, \ldots, v_{n-(k-2)}\}$, where $d_{G^*}(v_1) = d_1-2$ and $d_{G^*}(v_i) = d_i$ for each $i = 2, \ldots, n-(k-2)$. Construct a new graph G from G^* by adding a new path $P = v_{n-(k-3)} \cdots v_n$ with $V(P) \cap V(G^*) = \emptyset$, and two new edges $v_nv_1, v_{n-(k-3)}v_1$. Then G is a realization of d. Moreover, since $G^* \in \langle \mathbb{Z}_k \rangle$ and G/G^* is a cycle of length $k-1$, by Lemma 4.3, G is a $\mathbb{Z}_k$-connected realization of d. This completes the proof of Lemma 4.6. ∎

Proof of Theorem 2.8(ii). Let $m = \sum_{i=1}^n d_i$. Since d is multigraphic, $m \equiv 0 \pmod 2$. We argue by induction on m. By Lemmas 4.4 and 4.6, Theorem 2.8(ii) holds if $m = 2\lceil \frac{(k-1)(n-1)}{k-2} \rceil$. Assume that $m \geq 2\lceil \frac{(k-1)(n-1)}{k-2} \rceil + 2$, and that Theorem 2.8(ii) holds for smaller values of m.

If $d_2 \geq 3$, then by Theorem 4.2, $d' = (d_1-1, d_2-1, d_3, \ldots, d_n)$ is multigraphic. By induction, d' has a $\mathbb{Z}_k$-connected realization G'. Denote $V(G') = \{v_1, v_2, \ldots, v_n\}$ such that $d_{G'}(v_1) = d_1-1$, $d_{G'}(v_2) = d_2-1$, and $d_{G'}(v_i) = d_i$ for each $i = 3, \ldots, n$. Let $G = G' + v_1v_2$. Since G' is $\mathbb{Z}_k$-connected, G is a $\mathbb{Z}_k$-connected realization of d.

If $d_2 = 2$, then $d_2 = d_3 = \cdots = d_n = 2$. Thus $d = (d_1, \ldots, d_n) = (m-2(n-1), 2, \ldots, 2)$. Let $t = \frac{1}{2}d_1 = \frac{m-2(n-1)}{2}$. Since $m \equiv 0 \pmod 2$, t is an integer.

By Theorem 4.1, $d_1 \le \sum_{i=2}^n d_i = 2(n-1)$. Thus $n-1 \ge \frac{1}{2}d_1 = t$. Since $m \ge 2\lceil \frac{(k-1)(n-1)}{k-2} \rceil + 2$, we have

$$\begin{aligned} (8) \qquad 2t(k-1) &= (m-2(n-1))(k-1) = m(k-1) - 2(n-1)(k-1) \\ &= m + [(k-2)m - 2(n-1)(k-1)] > m = 2t + 2(n-1). \end{aligned}$$

By (8), there exist t integers $k-1 \ge s_1 \ge s_2 \ge \cdots \ge s_t \ge 2$ such that $\sum_{i=1}^t s_i = \frac{m}{2}$. Let $C_1, C_2, \ldots, C_t$ be a set of disjoint cycles such that C_i has length s_i. For each cycle C_i, we designate a vertex $u_i \in V(C_i)$. Construct a graph G by identifying $u_1, u_2, \ldots, u_t$ into a single vertex labeled as v_1. Then v_1 has degree $2t$ in G, and $|V(G)| = \sum_{i=1}^t (|V(C_i)| - 1) + 1 = \sum_{i=1}^t s_i - t + 1 = \frac{m}{2} - t + 1 = n$. Label the other vertices in $V(G) - \{v_1\}$ by $v_2, v_3, \ldots, v_n$, respectively. Then for each $j \ge 2$, $d_G(v_j) = 2$, so G is a realization of $d = (d_1, \ldots, d_n) = (m - 2(n-1), 2, \ldots, 2)$. Since $s_i \le k-1$, every edge in G lies in a cycle of length at most $k-1$, Therefore by Lemma 4.3, G is $\mathbb{Z}_k$-connected. This completes the proof of Theorem 2.8(ii). ◻

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