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MOD $(2p+1)$-ORIENTATION ON BIPARTITE GRAPHS AND COMPLEMENTARY GRAPHS*

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Abstract. A mod $(2p+1)$-orientation D is an orientation of G such that $d^+_D(v) - d^-_D(v) \equiv 0 \pmod{2p+1}$ for any vertex $v \in V(G)$. Jaeger conjectured that every $4p$-edge-connected graph has a mod $(2p+1)$-orientation. A graph G is strongly $\mathbb{Z}_{2p+1}$-connected if for every mapping $b : V(G) \mapsto \mathbb{Z}_{2p+1}$ with $\sum_{v\in V(G)} b(v) = 0$, there exists an orientation D of G such that $d^+_D(v) - d^-_D(v) = b(v)$ in $\mathbb{Z}_{2p+1}$ for any $v \in V(G)$. A strongly $\mathbb{Z}_{2p+1}$-connected graph admits a mod $(2p+1)$-orientation, and it is a contractible configuration for mod $(2p+1)$-orientation. We prove Jaeger's module orientation conjecture is equivalent to its restriction to bipartite simple graphs and investigate strongly $\mathbb{Z}_{2p+1}$-connectedness of certain bipartite graphs, particularly for $p = 2$. We also show that if G is a simple graph with $|V(G)| \geq N(p) = 1152p^4$ and $\min\{\delta(G), \delta(G^c)\} \geq 4p$, then either G or G^c is strongly $\mathbb{Z}_{2p+1}$-connected. When $p = 2$, the value of $N(2)$ can be reduced to $N(2) = 80$.

Key words. modulo orientations, nowhere-zero flows, group connectivity, complementary graphs

AMS subject classifications. 05C15, 05C21, 05C40

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1. Introduction.

We consider finite and loopless graphs and follow [2] for undefined terms and notation. Let $\mathbb{Z}$ denote the set of integers. For $k \in \mathbb{Z}$ with $k > 1$, $\mathbb{Z}_k$ denotes the set of all integers modulo k, as well as the (additive) cyclic group of order k. For a graph G, $\kappa'(G)$ and $\delta(G)$ denote the edge-connectivity and the minimum degree, respectively. If G is a simple graph, then G^c denotes the complement of G. For vertex subsets $U, W \subseteq V(G)$, let $[U, W]_G = \{uw \in E(G) | u \in U, w \in W\}$. When $U = \{u\}$ or $W = \{w\}$, we use $[u, W]_G$ or $[U, w]_G$ for $[U, W]_G$, respectively. As in [2], we define $\partial_G(X) = [X, V(G) - X]_G$. The subscript G may be omitted when G is understood from the context.

Let $D = D(G)$ denote an orientation of G. For each $v \in V(G)$, define $E^+_D(v)$ to be the set of all edges directed out from v and $E^-_D(v)$ to be the set of all edges directed into v. Following [2], we use $d^+_D(v) = |E^+_D(v)|$ and $d^-_D(v) = |E^-_D(v)|$ to denote the out-degree and the in-degree of v under the orientation D, respectively. If a graph G has an orientation D such that $d^+_D(v) - d^-_D(v) \equiv 0 \pmod{k}$ for every vertex $v \in V(G)$, then we say that G admits a *modulo k-orientation*, or a mod k-orientation for short. Let $\mathcal{M}_k$ denote the family of all graphs admitting a mod k-orientation. As a connected graph G has a modulo $2k$-orientation if and only if G is Eulerian, we focus on the case where $k = 2p + 1$ is odd in this paper.

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Let A be an (additive) abelian group with identity 0, and $A^* = A - \{0\}$. Assume that G has an orientation $D(G)$. Following Jaeger et al. [10], we define $F(G, A) = \{f|f : E(G) \to A\}$ and $F^*(G, A) = \{f|f : E(G) \to A^*\}$. For a function $f : E(G) \to A$, define $\partial f : V(G) \to A$ by

$$\partial f(v) = \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e),$$

where "$\sum$" refers to the addition in A.

A mapping $b : V(G) \to A$ is an *A-valued zero sum function* on G if $\sum_{v\in V(G)} b(v) = 0$. The set of all A-valued zero sum functions on G is denoted by $Z(G, A)$. For a mapping $b \in Z(G, A)$, a function $f \in F^*(G, A)$ is a nowhere-zero (A, b)-flow if $\partial f(v) = b(v)$ for each vertex $v \in V(G)$. A graph G is *A-connected* if for any $b \in Z(G, A)$, G has a nowhere-zero (A, b)-flow. For a positive integer k, the nowhere-zero $(\mathbb{Z}, 0)$-flow with $|f(e)| < k$ for each edge $e \in E(G)$ is known as *nowhere-zero k-flow*. Tutte [22] showed that the existence of nowhere-zero k-flow is equivalent to the existence of nowhere-zero $(\mathbb{Z}_k, 0)$-flow. The concept of strongly $\mathbb{Z}_{2p+1}$-connectedness was introduced in [14] (see also [13]).

Definition 1.1. *Let G be a graph, and let $Z(G, \mathbb{Z}_{2p+1}) = \{b : V(G) \to \mathbb{Z}_{2p+1} \mid \sum_{v\in V(G)} b(v) \equiv 0 \pmod{2p+1}\}$. A graph G is* strongly $\mathbb{Z}_{2p+1}$-connected *if, for every $b \in Z(G, \mathbb{Z}_{2p+1})$, there is an orientation D such that for every vertex $v \in V(G)$, $d_D^+(G) - d_D^-(G) \equiv b(v) \pmod{2p+1}$.*

It is routine to see that strongly $\mathbb{Z}_3$-connectedness and $\mathbb{Z}_3$-connectedness are the same.

Tutte and Jaeger proposed the following conjectures concerning mod $(2p + 1)$-orientations. A conjecture on strongly $\mathbb{Z}_{2p+1}$-connected graphs has also been proposed.

Conjecture 1.2.
(i) (*Tutte* [22]) *Every 4-edge-connected graph has a nowhere-zero 3-flow.*
(ii) (*Jaeger et al.* [10]) *Every 5-edge-connected graph is $\mathbb{Z}_3$-connected.*
(iii) (*Jaeger* [8]) *Every $4p$-edge-connected graph has a mod $(2p + 1)$-orientation.*
(iv) (Lai [13]) *Every $(4p + 1)$-edge-connected graph is strongly $\mathbb{Z}_{2p+1}$-connected.*

By a result of Kochol [11], Conjecture 1.2(i) is equivalent to its restriction to 5-edge-connected graphs. Thus, Conjecture 1.2(ii) implies Conjecture 1.2(i). For $p = 1$, Conjecture 1.2(iii) is Conjecture 1.2(i).

It is well known that the $p = 2$ case of Conjecture 1.2(iii), if true, would imply Tutte's 5-flow conjecture. It is also known that Conjecture 1.2(iv), if true for $p = 2$, would imply the Jaeger et al. conjecture that every 3-edge-connected graph is $\mathbb{Z}_5$-connected (see [17]). Thus, the $p = 2$ case of Conjecture 1.2(iii) and (iv) deserve special attention. These conjectures remain open by far to the best of our knowledge. The best known results so far have been recently obtained by Thomassen [21], Wu [23], and Lovász et al. [20].

Theorem 1.3 (Thomassen [21]). *Every 8-edge-connected graph is $\mathbb{Z}_3$-connected.*

Theorem 1.4 (Lovász et al. [20], Wu [23]). *Let $p > 0$ be an integer. Every $6p$-edge-connected graph is strongly $\mathbb{Z}_{2p+1}$-connected.*

In this paper, we show that each of Conjecture 1.2(i)–(iv) is equivalent to its restriction to bipartite simple graphs. This motivates us to investigate the strongly $\mathbb{Z}_{2p+1}$-connectedness of some complete bipartite graphs. The investigation leads us to find a Ramsey type theorem on strongly $\mathbb{Z}_{2p+1}$-connectedness. In [7], Hou et al. proved the following.

THEOREM 1.5 (Hou et al. [7]). *Let G be a simple graph with $|V(G)| \geq 44$. If $\min\{\delta(G), \delta(G^c)\} \geq 4$, then either G or G^c is strongly $\mathbb{Z}_3$-connected.*

We extend Theorem 1.5 from $p = 1$ to all integer $p > 0$, stated as Theorem 1.6 below. As it is well known that $\kappa'(G) \leq \delta(G)$ for any graph G, Theorem 1.6 remains valid if the condition $\min\{\delta(G), \delta(G^c)\} \geq 4p$ is replaced by $\min\{\kappa'(G), \kappa'(G^c)\} \geq 4p$. Thus in some sense, Theorem 1.6 supports Conjecture 1.2.

THEOREM 1.6. *Let G be a simple graph with $|V(G)| \geq N(p) = 1152p^4$. If $\min\{\delta(G), \delta(G^c)\} \geq 4p$, then either G or G^c is strongly $\mathbb{Z}_{2p+1}$-connected.*

While we make minimum efforts to improve the bound $N(p)$ in the general case, we will show that when $p = 2$, the value of $N(2)$ can be reduced to $N(2) = 80$.

THEOREM 1.7. *Let G be a simple graph with $|V(G)| \geq 80$. If $\min\{\delta(G), \delta(G^c)\} \geq 8$, then either G or G^c is strongly $\mathbb{Z}_5$-connected.*

In the next section, we will present the mechanisms that will be needed in the proof of our main theorem, including two of our key tools, stated as Lemmas 2.3 and 2.4, whose proofs are postponed to the last section. The equivalence of Conjecture 1.2 (i)–(iv) to its restriction to bipartite simple graphs will also be shown in the next section. In section 3, we will prove Theorems 1.6 and 1.7 assuming the validity of Lemmas 2.3 and 2.4.

2. Preliminaries.

We display some elementary properties on contractible configurations and boundary functions related to strongly $\mathbb{Z}_{2p+1}$-connectedness of graphs.

2.1. Contractible configurations.

For graphs G and H, we use $H \subseteq G$ to mean that H is a subgraph of G. For an edge set $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. If H is a subgraph of G, then we use G/H for $G/E(H)$. Following the notation in [3] and [4], define

$$\mathcal{M}^o_{2p+1} = \{H : \text{for any graph } G \text{ with } H \subseteq G,\ G \in \mathcal{M}_{2p+1} \Longleftrightarrow G/H \in \mathcal{M}_{2p+1}\}.$$

Liang et al. proved that a graph H is in $\mathcal{M}^o_{2p+1}$ if and only if H is strongly $\mathbb{Z}_{2p+1}$-connected.

THEOREM 2.1 (Liang et al. [19]; see also [18]). *For any integer $p > 0$, $\mathcal{M}^o_{2p+1}$ consists of precisely all strongly $\mathbb{Z}_{2p+1}$-connected graphs.*

By Theorem 2.1, we will use $\mathcal{M}^o_{2p+1}$ to denote the set of all strongly $\mathbb{Z}_{2p+1}$-connected graphs in the following.

LEMMA 2.2 ([13], [14], and [18]). *Let G be a graph and let $m, p > 0$ be integers. Each of the following holds:*

(i) *If $G \in \mathcal{M}^o_{2p+1}$ and $e \in E(G)$, then $G/e \in \mathcal{M}^o_{2p+1}$.*
(ii) *If $H \subseteq G$, and if both $H \in \mathcal{M}^o_{2p+1}$ and $G/H \in \mathcal{M}^o_{2p+1}$, then $G \in \mathcal{M}^o_{2p+1}$.*
(iii) *Let mK_2 denote the loopless graph with two vertices and m parallel edges. Then mK_2 is strongly $\mathbb{Z}_{2p+1}$-connected if and only if $m \geq 2p$.*
(iv) *The complete graph K_n is strongly $\mathbb{Z}_{2p+1}$-connected if and only if $n = 1$ or $n \geq 4p + 1$.*

A graph H is a $\mathcal{M}^o_{2p+1}$-graph if $H \in \mathcal{M}^o_{2p+1}$. By definition, K_1 is an $\mathcal{M}^o_{2p+1}$-graph. Thus for any graph G, every vertex lies in a maximal $\mathcal{M}^o_{2p+1}$-subgraph of G.

Let $H_1, H_2, \ldots, H_c$ denote the collection of all maximal $\mathcal{M}^o_{2p+1}$-subgraph of G. Then $G' = G/(\cup_{i=1}^c E(H_i))$ is the $\mathcal{M}^o_{2p+1}$-*reduction* of G. It follows that a graph G is strongly $\mathbb{Z}_{2p+1}$-connected if and only if its $\mathcal{M}^o_{2p+1}$-reduction is K_1, a singleton. A graph G is $\mathcal{M}^o_{2p+1}$-*reduced* if G does not have any nontrivial subgraph in $\mathcal{M}^o_{2p+1}$. By definition, the $\mathcal{M}^o_{2p+1}$-reduction of a graph is always $\mathcal{M}^o_{2p+1}$-reduced. Since contraction may bring in new parallel edges, even when G is a simple graph, its $\mathcal{M}^o_{2p+1}$-reduction may have multiple edges.

The study of complementary strongly $\mathbb{Z}_{2p+1}$-connected graphs needs the following lemmas. Lemma 2.4 indicates that when $p = 2$, Lemma 2.3 can be improved.

LEMMA 2.3. *Let $p > 0$ be an integer. Then $K_{4p,16p^2}$ is strongly $\mathbb{Z}_{2p+1}$-connected.*

LEMMA 2.4. *The complete bipartite graph $K_{8,8}$ is strongly $\mathbb{Z}_5$-connected.*

The proofs of the two lemmas above will be presented in the last section. Example 2.5 below shows that Lemma 2.4 is sharp in some sense.

Example 2.5. The graph $K_{7,7}$ has a mod 5-orientation but $K_{7,7} \notin \mathcal{M}^o_5$.

Let C be a Hamiltonian cycle of $K_{7,7}$. Then $K_{7,7} - E(C)$ is a 5-regular bipartite graph, and so it has a mod 5-orientation. The mod 5-orientation of $K_{7,7,} - E(C)$ together with a strong orientation of C yields a mod 5-orientation of $K_{7,7}$. To see that $K_{7,7} \notin \mathcal{M}^o_5$, fix $x_0 \in V(K_{7,7})$ and define, for any $x \in V(K_{7,7}) - \{x_0\}$, $b(x) = 1$ and $b(x_0) = 2$. It is routine to verify that $b \in Z(K_{7,7}, \mathbb{Z}_5)$ and that there is no orientation satisfying b by a similar arguments of Proposition 2.9 below.

2.2. Boundary functions. Our boundary functions are motivated by the following.

LEMMA 2.6 (Hakimi [6]). *Let G be a graph and $\ell : V(G) \mapsto \mathbb{Z}$ be a function such that $\sum_{v\in V(G)} \ell(v) = 0$ and $\ell(v) \equiv d_G(v) \pmod 2$ $\forall v \in V(G)$. Then the following are equivalent:*

(i) *G has an orientation D such that $d^+_D(v) - d^-_D(v) = \ell(v)$ $\forall v \in V(G)$.*

(ii) *$|\sum_{v\in S} \ell(v)| \le |\partial_G(S)|$ $\forall S \subset V(G)$.*

DEFINITION 2.7. *Let $p > 0$ be an integer, let G be a graph, and let $b \in Z(G, \mathbb{Z}_{2p+1})$.*

(i) *An orientation $D = D(G)$ satisfies b if for any $v \in V(G)$, $d^+_D(v) - d^-_D(v) = b(v)$ in $\mathbb{Z}_{2p+1}$.*

(ii) *Define $L(b)$ to be the collection of all mappings $\ell : V(G) \to \mathbb{Z}$ satisfying each of the following:*

(ii-a)

$$\ell(v) \equiv \begin{cases} b(v) \pmod{2p+1} \\ d_G(v) \pmod 2 \end{cases} \quad \forall v \in V(G),$$

(ii-b) $\sum_{v\in V(G)} \ell(v) = 0$, *and*

(ii-c) $\max\{\ell(v) : v \in V(G)\} - \min\{\ell(v) : v \in V(G)\} \le 4p + 2$.

(iii) *For an $\ell \in L(b)$, an orientation D realizes ℓ if for any $v \in V(G)$, $d^+_D(v) - d^-_D(v) = \ell(v)$.*

LEMMA 2.8 (Proposition 3.1 and Lemma 3.2 in [15]). *Let G be a graph and $b \in Z(G, \mathbb{Z}_{2p+1})$. Then each of the following holds:*

(i) $L(b) \ne \emptyset$.

(ii) *If there is no orientation satisfying b, then for any $\ell \in L(b)$, there is no orientation realizing ℓ.*

Jaeger et al. [10] constructed the first 4-edge-connected graph which is not $\mathbb{Z}_3$-connected. Other infinite families of 4-edge-connected graphs that are also not $\mathbb{Z}_3$-connected but with additional properties are found in [13] and [16], among others. The boundary functions can be utilized to extend the construction of Jaeger et al. to build $4p$-edge-connected nonstrongly $\mathbb{Z}_{2p+1}$-connected graphs.

Proposition 2.9. *For any integer $p > 0$, there exist $4p$-edge-connected non-strongly $\mathbb{Z}_{2p+1}$-connected graphs.*

Proof. For $i \in \{1, \ldots, 2p+1\}$, let G^i be a copy of K_{4p} with vertex set $V(G^i) = \{v_1^i, \ldots, v_{4p}^i\}$ such that if $1 \le i < j \le 2p+1$, then $V(G^i) \cap V(G^j) = \emptyset$. Define $W_t = \{v_j^t v_{2p+j}^{t+1} : 1 \le j \le 2p\}$ for each $t \in \mathbb{Z}_{2p+1}$. Obtain a graph $G = G(p)$ from $\cup_{i=1}^{2p+1} G^i$ by adding the new edges $\cup_{t\in\mathbb{Z}_{2p+1}} W_t$. If $X \subseteq E(G)$ is an edge cut of G, then either for some $i \in \mathbb{Z}_{2p+1}$, $X \cap E(G^i) \ne \emptyset$, whence $X \cap (\cup_{t\in\mathbb{Z}_{2p+1}} W_t) \ne \emptyset$ and so $|X| \ge \kappa'(K_{4p}) + 1 \ge 4p$, or $X \subseteq \cup_{t\in\mathbb{Z}_{2p+1}} W_t$, whence X contains at least two of the W_t's and so $|X| \ge 2|W_t| = 4p$. Thus $\kappa'(G) \ge 4p = \Delta(G) = \delta(G)$. To show that $G \notin \mathcal{M}_{2p+1}^o$, we argue by contradiction and assume that $G \in \mathcal{M}_{2p+1}^o$. Set $b(v) = 4p$ for each $v \in V(G)$. As $|V(G)| = 4p(2p+1)$, we have $\sum_{v\in V(G)} b(v) \equiv 0 \pmod{2p+1}$, and so $b \in Z(G, \mathbb{Z}_{2p+1})$. Since G is strongly $\mathbb{Z}_{2p+1}$-connected, there is an orientation D satisfying b. Denote $\ell(v) = d_D^+(v) - d_D^-(v)\ \forall v \in V(G)$. Since $b = 4p$, $\ell(v) = d_D^+(v) - d_D^-(v) \equiv 4p \pmod{2p+1}$, and G is $4p$-regular, we have $\ell(v) \in \{-2, 4p\}$ for each $v \in V(G)$. By checking Definition 2.7(ii-a)–(ii-c), $\ell \in L(b)$, and thus D realizes ℓ. For $i \in \{-2, 4p\}$, let $N_i = \{v \in V(G) : \ell(v) = i\}$. If $x, y \in N_{4p}$ with $x \ne y$, then by Definition 2.7(ii), $d_D^+(x) = d_D^+(y) = 4p$. Since G is $4p$-regular, $xy \notin E(G)$. It follows that $|N_{4p} \cap V(G^i)| \le 1$ for any $i \in \mathbb{Z}_{2p+1}$, and so $|N_{4p}| \le 2p+1$. By Definition 2.7(ii), we have $0 = \sum_{v\in V(G)} \ell(v) = 4p|N_{4p}| - 2|N_{-2}| = 4p|N_{4p}| - 2(4p(2p+1) - |N_{4p}|) \le (2-4p)(2p+1) < 0$, a contradiction. This contradiction shows that there is no orientation satisfying b, and so G is not strongly $\mathbb{Z}_{2p+1}$-connected. ∎

2.3. An equivalent version of Jaeger's module orientation conjecture. The main results of this subsection are the following.

Theorem 2.10. *Let $p > 0$ be an integer. The following statements are equivalent:*

(i) *Every $(4p+1)$-edge-connected graph is strongly $\mathbb{Z}_{2p+1}$-connected.*
(ii) *Every $(4p+1)$-edge-connected bipartite simple graph is strongly $\mathbb{Z}_{2p+1}$-connected.*

Proof. (i) $\Rightarrow$ (ii) is straightforward. To prove (ii) $\Rightarrow$ (i), we let G be a $(4p+1)$-edge-connected graph and let $m \ge 4p+1$ be an integer. For each edge $e = uv \in E(G)$, subdivide each edge $e = uv$ with a middle vertex x_e, and attach a graph $\Gamma_e \cong K_{m,m}$ with a distinguished edge x_1y_1 by identifying the edge ux_e with x_1y_1 (see Figure 1). After we have performed this operation on each edge of G, we obtained a simple bipartite graph $\Gamma(G)$. The construction of $\Gamma(G)$ indicates that $\kappa'(\Gamma(G)) \ge 4p+1$. By (ii), $\Gamma(G)$ is strongly $\mathbb{Z}_{2p+1}$-connected. Since

$$\Gamma(G)/(\cup_{e\in E(G)} E(\Gamma_e)) \cong G,$$

it follows by Lemma 2.2(i) that G is also strongly $\mathbb{Z}_{2p+1}$-connected. ∎

By the definition of mod k-orientation, if $G \in \mathcal{M}_k$ and $e \in E(G)$, then $G/e \in \mathcal{M}_k$. Thus with similar arguments, we also have the following.

Theorem 2.11. *Let $p > 0$ be an integer. The following statements are equivalent:*

(i) *Every $4p$-edge-connected graph has a mod $(2p+1)$-orientation.*
(ii) *Every $4p$-edge-connected bipartite simple graph has a mod $(2p+1)$-orientation.*

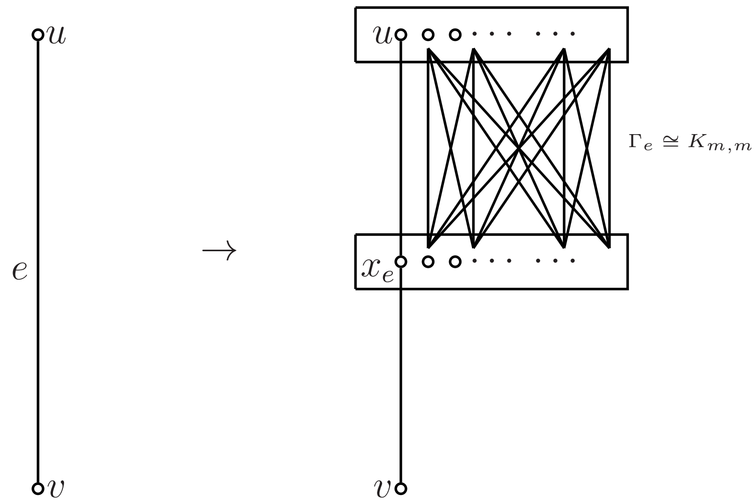

FIG. 1. *The edge transaction.*

3. Strongly $\mathbb{Z}_{2p+1}$-connectedness on complementary graphs. Throughout this section, $p > 0$ denotes an integer. We shall assume the validity of Lemmas 2.4 and 2.3 to prove Theorems 1.6 and 1.7. We start displaying some tools that will be needed in our arguments. For a graph G, define $\overline{\kappa}'(G) = \max\{\kappa'(H) : H \subseteq G \text{ with } |E(H)| > 0\}$. The lemma below follows from Theorem 1.4.

LEMMA 3.1. *Let G' be the $\mathcal{M}^o_{2p+1}$-reduction of a connected graph G such that $G' \neq K_1$. Then $\overline{\kappa}'(G') \leq 6p - 1$.*

LEMMA 3.2 (Gu et al. [5]). *Let G be a graph with order n and let $k > 0$ be an integer. If $\overline{\kappa}'(G) \leq k$, then $|E(G)| \leq (n-1)k$.*

LEMMA 3.3. *Let G be a simple graph with order $n > 24p$ and $|E(G)| \geq \frac{n(n-1)}{4}$. Then G contains a strongly $\mathbb{Z}_{2p+1}$-connected subgraph H with $|V(H)| > \sqrt{n/2 - 12p}$.*

Proof. Let $H_1, H_2, \ldots, H_c$ denote the collection of all maximal strongly $\mathbb{Z}_{2p+1}$-connected subgraphs of G. Then $G' = G/(\cup_{i=1}^{c} E(H_i))$ is the $\mathcal{M}^o_{2p+1}$-reduction of G. Let $m = \max_{1 \leq i \leq c}\{|V(H_i)|\}$. By Lemmas 3.1 and 3.2, we have $|E(G')| \leq (6p-1)(c-1)$, and so

$$\begin{aligned}\frac{cm^2}{2} + (6p-1)c &\geq \sum_{i=1}^{c} \frac{|V(H_i)|(|V(H_i)|-1))}{2} + (6p-1)(c-1) \\ &\geq \sum_{i=1}^{c} |E(H_i)| + |E(G')| = |E(G)| \geq \frac{n(n-1)}{4}.\end{aligned}$$

Since $c \leq n$, we conclude that $m > \sqrt{n/2 - 12p}$. ∎

LEMMA 3.4 (lifting). *Let G be a graph, let $P = v_1 v_2 \ldots v_t$ be a path of G, and let $G_{[P, v_1 v_t]}$ be the graph obtained from G by deleting $E(P)$ and adding a new edge $e = v_1 v_t$. If $G_{[P, v_1 v_t]}$ is strongly $\mathbb{Z}_{2p+1}$-connected, then G is strongly $\mathbb{Z}_{2p+1}$-connected.*

Proof. For any $b \in Z(G_{[P, v_1 v_t]}, \mathbb{Z}_{2p+1})$, there exists an orientation D' satisfying b. Subdivide $D'(e)$ with internal vertices $v'_2, \ldots, v'_{t-1}$, and then identity v'_i with v_i for $2 \leq i \leq t-1$. This yields an orientation D of G satisfying b. ∎

3.1. Proof of Theorem 1.6. Let $n = |V(G)|$, and for a vertex $v \in V(G)$, let $N_G(v)$ denote the set of all vertices adjacent to v in G. Arguing by contradiction, we assume that

(1) $\quad G$ is a counterexample to Theorem 1.6 with $n = |V(G)|$ minimized,

and let $\mathcal{X} = \{X \subset V : G[X] \in \mathcal{M}^o_{2p+1}$ or $G^c[X] \in \mathcal{M}^o_{2p+1}\}$. Choose $X \in \mathcal{X}$ with $|X|$ maximized, and let $Y = V(G) - X$. Since $\max\{|E(G)|, |E(G^c)|\} \geq \frac{|E(K_n)|}{2} = \frac{1}{4}n(n-1)$, by Lemma 3.3 and as $n \geq 1152p^4$, we have $|X| \geq \sqrt{n/2 - 12p} \geq 24p^2 - 4p$. By (1), $|X| < n$, and so $Y \neq \emptyset$. By switching G and G^c if necessary, we may assume $H_0 = G[X] \in \mathcal{M}^o_{2p+1}$. By Lemma 2.2(i)–(iii), we have

(2) $$\text{for any } y \in Y,\ |[X, y]_G| \leq 2p - 1.$$

Claim A. $|Y| \geq 4p$.

As $\delta(G) \geq 4p$ and by (2), we have $|Y| \geq |(N_G(y) - X) \cup \{y\}| \geq 4p - (2p-1) + 1 = 2p + 2$. Let $G_0 = G/H_0$, and let u_0 be the vertex in G_0 onto which H_0 is contracted. We shall show that if $|Y| \leq 4p - 1$, then $G_0 \in \mathcal{M}^o_{2p+1}$, and so by Lemma 2.2(ii), $G \in \mathcal{M}^o_{2p+1}$. This contradicts (1) and so Claim A holds.

Assume that $|Y| \leq 4p - 1$. For any vertex $u \in Y$, we have

(3) $$|[u_0, u]_{G_0}| \geq 4p - d_{G[Y]}(u) \geq |Y| + 1 - d_{G[Y]}(u).$$

Let t denote the number of different unordered pairs of distinct vertices in Y that are not adjacent in G_0, and let $\{u_1, u_2\}, \ldots, \{u_{2t-1}, u_{2t}\}$ be all such pairs. Note that different u_i's may represent the same vertex of G_0. Let $P_i = u_{2i-1}u_0u_{2i}$ denote a path of length two in G_0 for each $1 \leq i \leq t$. For each fixed $u \in Y$, there are t_u pairs of such pairs $\{u, u'_j\}$, where $t_u = |Y - (N_{G[Y]}(u) \cup \{u\})| = |Y| - 1 - d_{G[Y]}(u)$ and $Y - (N_{G[Y]}(u) \cup \{u\}) = \{w_1, w_2, \ldots, w_{t_u}\}$. By (3), these paths $P_1, \ldots, P_t$ can be so chosen that $E(P_i) \cap E(P_j) = \emptyset$ for any $1 \leq i < j \leq t$. Obtain a graph G_1 by lifting $P_1, \ldots, P_t$. Then, we have $G_1[Y]$ is isomorphic to the complete graph $K_{|Y|}$. By the definition of G_1, for each $u \in Y$, $[u_0, u]_{G_1} = [u_0, u]_{G_0} - \{u_0 w_j : w_j \in Y - (N_{G[Y]}(u) \cup \{u\})\}$, and so by (3),

(4) $$|[u_0, u]_{G_1}| \geq 4p - d_{G[Y]}(u) - t_u = 4p - |Y| + 1 \text{ for each } u \in Y.$$

As $G_1 - u_0$ contains a complete spanning subgraph isomorphic to $K_{|Y|}$, it follows by (4) that $K_{4p+1}/K_{4p+1-|Y|}$ is a spanning subgraph of G_1. By Lemma 2.2(iv), $K_{4p+1} \in \mathcal{M}^o_{2p+1}$, and so by Lemma 2.2(i), $G_1 \in \mathcal{M}^o_{2p+1}$. Therefore, $G_0 \in \mathcal{M}^o_{2p+1}$ by Lemma 3.4. Hence the claim follows.

By Claim A, Y contains a subset Y_1 with $|Y_1| = 4p$. Let $X_1 = \{x \in X : [x, Y_1]_G = \emptyset\}$. Thus $[X_1, Y_1]_{G^c}$ is isomorphic to $K_{4p,|X_1|}$. As $|X| \geq 24p^2 - 4p$ and by (2), we have $|X_1| \geq |X| - |Y_1|(2p-1) \geq 16p^2$, and so by Lemma 2.3, $[X_1, Y_1]_{G^c} \in \mathcal{M}^o_{2p+1}$. Let $X_2 = \{x \in X : |[x, Y_1]_{G^c}| \geq 2p\}$. As $|[x, Y_1]_{G^c}| = 4p > 2p$ for any $x \in X_1$, we have $X_1 \subset X_2$.

Claim B. $|X_2| \geq |X| - (4p - 3)$.

By contradiction, we assume that $|X_2| \leq |X| - (4p-2)$. Then $|[X, Y_1]_{G^c}| = |X| \cdot |Y_1| - |[X, Y_1]_G| \leq |X| \cdot |Y_1| - (4p-2)(2p+1)$, and so $|[X, Y_1]_G| \geq (4p-2)(2p+1) > 4p(2p-1)$, contrary to (2). This proves Claim B.

By the definition of X_2, every edge in $[X_2, Y_1]_{G^c}/[X_1, Y_1]_{G^c}$ lies in a $(2p)K_2$. By Lemma 2.2(iii), $[X_2, Y_1]_{G^c}/[X_1, Y_1]_{G^c} \in \mathcal{M}^o_{2p+1}$. As $[X, Y_1]_{G^c} \in \mathcal{M}^o_{2p+1}$ and by Lemma 2.2(ii), $[X_2, Y_1]_{G^c} \in \mathcal{M}^o_{2p+1}$. Thus $X_2 \cup Y_1 \in \mathcal{X}$ and $|X_2 \cup Y_1| \geq |X| - (4p - 3) + |Y_1| > |X|$, contrary to the choice of X. This proves the theorem.

3.2. Proof of Theorem 1.7. While we make no effort to reduce the bound of $N(p)$ in Theorem 1.6, we in this section will assume the validity of Lemma 2.4 to prove Theorem 1.7 to show that $N(2)$ can be as small as 80 here. From the proof of Theorem 1.7, one can see that if we can prove $K_{4p+1,4p+1}$ is strongly $\mathbb{Z}_{2p+1}$-connected, then the bound on $N(p)$ in Theorem 1.6 may be significantly reduced. Let $K_{m,n}^{-3}$ denote any graph obtained from $K_{m,n}$ by deleting arbitrarily three edges. The following lemma, a consequence of Lemma 2.4, will be useful in our arguments.

LEMMA 3.5. *$K_{9,9}^{-3}$ is strongly $\mathbb{Z}_5$-connected. Thus for integers $m, n \geq 9$, $K_{m,n}^{-3}$ is strongly $\mathbb{Z}_5$-connected.*

Proof. Let G denote a $K_{9,9}^{-3}$ and E^c be the edge set of G^c.

Case 1. G^c is isomorphic to $K_{1,3}$, or P_4 (a path on 4 vertices), or $P_2 \cup P_3$.

By symmetry, if $G^c \cong K_{1,3}$, then we may assume $E^c = \{x_1y_1, x_1y_2, x_1y_3\}$; if $G^c \cong P_4$, then we assume $E^c = \{x_1y_1, x_1y_2, x_2y_1\}$; and if $G^c \cong P_2 \cup P_3$, then we assume $E^c = \{x_1y_2, x_1y_3, y_1x_2\}$. In any case, let $H = G[(X - \{x_1\}) \cup (Y - \{y_1\})] \cong K_{8,8}$. It follows from Lemma 2.2(iii) that $G/H = K_{9,9}^{-3}/K_{8,8} \in \mathcal{M}_5^o$, and so by Lemmas 2.4 and 2.2(ii), $G \in \mathcal{M}_5^o$.

Case 2. G^c is a matching.

By symmetry, we assume $E^c = \{x_1y_1, x_2y_2, x_3y_3\}$. Let G' be a graph obtained from G by deleting the edges y_1x_2, x_2y_3, y_3x_1 and adding edge x_1y_1. Hence $G'[(X - \{x_2\} \cup (Y - \{y_3\})] \cong K_{8,8}$. By Lemma 2.2(iii), $G'/K_{8,8} \in \mathcal{M}_5^o$, and so by Lemmas 2.4 and 2.2(ii), $G' \in \mathcal{M}_5^o$. By Lemma 3.4, $G \in \mathcal{M}_5^o$.

If $K_{9,9}^{-3} \in \mathcal{M}_5^o$ and if $m, n \geq 9$, then by Lemma 2.2(ii)–(iii), $K_{m,n}^{-3}$ is strongly $\mathbb{Z}_5$-connected. This completes the proof of the lemma. ∎

Proof of Theorem 1.7. We argue by contradiction and assume that

$$G \text{ is a counterexample with } |V(G)| \text{ minimized.} \tag{5}$$

Define $\mathcal{X} = \{X \subset V : G[X] \in \mathcal{M}_5^o \text{ or } G^c[X] \in \mathcal{M}_5^o\}$. Choose $X \in \mathcal{X}$ such that $|X|$ is maximized, and let $Y = V(G) - X$. As $|V(G)| \geq 80$, we have $\max\{|E(G)|, |E(G^c)|\} \geq \frac{1}{4}|V(G)|(|V(G)| - 1) \geq 20(|V(G)| - 1) > 12(|V(G)| - 1)$, and so $|X| > 1$ by Lemmas 3.1 and 3.2. By (5), we have $1 < |X| < |V(G)|$, and so $Y \neq \emptyset$. By symmetry between G and G^c, we may assume $H_0 = G[X] \in \mathcal{M}_5^o$. Hence by Lemma 2.2(iii),

$$\text{for any } y \in Y,\ |[X, y]_G| \leq 3. \tag{6}$$

Claim C below follows from a similar argument justifying Claim A in the proof of Theorem 1.6 with $p = 2$.

Claim C. $|Y| \geq 8$.

Claim C can be further extended to the following.

Claim D. $|Y| > 48$.

If $|Y| \leq 48$, then $|X| \geq 32$ as $|V(G)| \geq 80$. By Claim C, there exists a subset $Y_1 \subset Y$ with $|Y_1| = 8$. Let $X_1 = \{x \in X : [x, Y_1]_G = \emptyset\}$. By (6) and as $|Y_1| = 8$, we have $|X_1| \geq |X| - 3 \times 8 \geq 8$. Hence, $[X_1, Y_1]_{G^c} \cong K_{|X_1|,8} \in \mathcal{M}_5^o$ by Lemma 2.4. Let $X_2 = \{x \in X : |[x, Y_1]_{G^c}| \geq 4\}$. By definition, $X_1 \subset X_2$. If $|X_2| \leq |X| - 5$, then $|[X, Y_1]_{G^c}| \leq |X| \cdot |Y_1| - 5 \times 5$, so $|[X, Y_1]_G| \geq 25 > 3 \times 8$, contrary to (6). Therefore, $|X_2| \geq |X| - 4$. Moreover, $[X_2, Y_1]_{G^c} \in \mathcal{M}_5^o$ by Lemma 2.2(ii)–(iii). It follows that $X_2 \cup Y_1 \in \mathcal{X}$ and $|X_2 \cup Y_1| \geq |X| - 4 + |Y_1| > |X|$, contrary to the maximality of $|X|$. This proves Claim D.

Define $\mathcal{Y} = \{Y' \subset Y : G[Y'] \in \mathcal{M}_5^o \text{ or } G^c[Y'] \in \mathcal{M}_5^o\}$. Choose $Y' \in \mathcal{Y}$ with $|Y'|$ maximized. By Claim D, $|Y| \geq 48$ and so we have $\max\{|E(G[Y])|, |E(G^c[Y])|\} \geq$

$\frac{1}{4}|Y|(|Y|-1) \geq 12(|Y|-1)$. By Lemma 3.1, $|Y'| > 1$. By Lemma 2.2(iv), both $|X| \geq 9$ and $|Y'| \geq 9$.

Case 1. $G[Y'] \in \mathcal{M}_5^o$. Then $|[X, Y']_G| \leq 3$ by Lemma 2.2 (ii)–(iii). Therefore, $[X, Y']_{G^c}$ contains a spanning subgraph isomorphic to $K^{-3}_{|X|,|Y'|}$. By Lemma 3.5, we have $[X, Y']_{G^c} \in \mathcal{M}_5^o$. Thus $X \cup Y' \in \mathcal{X}$ and $|X \cup Y'| > X$, contrary to the choice of X.

Case 2. $G^c[Y'] \in \mathcal{M}_5^o$. Let $X' = \{x \in X : |[x, Y']_{G^c}| \geq 4\}$. By Lemma 2.2(ii)–(iii), $[X', Y']_{G^c} \in \mathcal{M}_5^o$. We are to show that $|X'| \geq |X| - 4$. By (6), we have $|[X, Y']_G| \leq 3|Y'|$. It follows that

$$|X| \cdot |Y'| - 3|Y'| \leq |X| \cdot |Y'| - |[X, Y']_G| = |[X, Y']_{G^c}| \leq |X'| \cdot |Y'| + 3(|X| - |X'|).$$

Thus, $|X| - |X'| \leq 3 + \frac{9}{|Y'|-3}$. Since $|Y'| \geq 9$, $|X| - |X'| \leq 4$. Now, we have $[X', Y']_{G^c} \in \mathcal{M}_5^o$ and $|X' \cup Y'| \geq |X| - 4 + 9 > |X|$, contrary to the maximality of $|X|$. This completes the proof of Theorem 1.7. □

4. Modulo orientation on bipartite graph. In this section, we are to justify Lemmas 2.3 and 2.4, which are needed in the proofs of Theorems 1.6 and 1.7. Throughout this section, we always denote $G = K_{m,n}$ and use (X, Y) to denote the vertex bipartition of $K_{m,n}$, where $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$. For any $S \subseteq V(G)$, let $x_S = |S \cap X|$ and $y_S = |S \cap Y|$.

NOTATION 4.1. *For a mapping $b \in Z(G, \mathbb{Z}_{2p+1})$, let $L(b)$ be defined as in Definition 2.7. For each $\ell \in L(b)$ and $k \in \mathbb{Z}$, define $N_k(\ell) = \{v \in V : \ell(v) = k\}$, $N_+(\ell) = \{v \in V : \ell(v) > 0\}$, $N_-(\ell) = \{v \in V : \ell(v) < 0\}$, $M_1(\ell) = \max\{\ell(v) : v \in V(G)\}$, and $M_2(\ell) = \min\{\ell(v) : v \in V(G)\}$. Throughout the rest of this paper, when ℓ is understood from the context, we often use N_k, N_+, N_-, M_1, M_2 instead. The norm of ℓ is defined to be*

$$\|\ell\| = \max\left\{\sum_{v \in X \cap N_+} \ell(v), \sum_{v \in Y \cap N_+} \ell(v)\right\}.$$

By Definition 2.7(ii), for any $b \in Z(G, \mathbb{Z}_{2p+1})$, if $\ell \in L(b)$, then

$$\sum_{M_2 \leq t \leq M_1} t|N_t(\ell)| = \sum_{-4p-2 \leq t \leq 4p+2} t|N_t(\ell)| = \sum_{v \in V(G)} \ell(v) = 0. \tag{7}$$

DEFINITION 4.2. *Let $b \in Z(G, \mathbb{Z}_{2p+1})$ and $\ell_1 \in L(b)$ be given. Assume that there are vertices $u_1, \ldots, u_t, v_1, \ldots, v_t$ in G such that $|\ell_1(u_i) - \ell_1(v_i)| = 4p+2$ for $1 \leq i \leq t$. We define $\ell_2 = \ell_1(u_1, \ldots, u_t; v_1, \ldots, v_t)$, a switch of ℓ_1, as follows:*

$$\ell_2(x) = \begin{cases} \ell_1(v_i) & \text{if } x = u_i, \text{ where } 1 \leq i \leq t; \\ \ell_1(u_i) & \text{if } x = v_i, \text{ where } 1 \leq i \leq t; \\ \ell_1(x) & \text{otherwise.} \end{cases}$$

It is routine to verify that $\ell_2 \in L(b)$. The following observation follows from Definition 4.2.

OBSERVATION 4.3. Let G be a graph, k be an integer with $0 \leq k \leq 4p-2$, $b \in Z(G, \mathbb{Z}_{2p+1})$, and $\ell_1 \in L(b)$.

(i) If $\{u_1, \ldots, u_t\} \subseteq N_k(\ell_1)$ and $\{v_1, \ldots, v_t\} \subseteq N_{k-4p-2}(\ell_1)$, then $\ell_2 = l_1(u_1, \ldots, u_t; v_1, \ldots, v_t)$ is a switch of ℓ_1.

(ii) If $G = K_{n,n}$ is a complete bipartite graph with bipartition (X, Y) and if $|N_k(\ell_1)| \leq |N_{k-4p-2}(\ell_1)| + 1$, then ℓ_1 has a switch $\ell_2 \in L(b)$ satisfying either $N_k(l_2) \subseteq X$ or $N_k(l_2) \subseteq Y$.

(iii) If ℓ_2 is a switch of ℓ_1, then $M_1(\ell_1) = M_1(\ell_2)$ and $M_2(\ell_1) = M_2(\ell_2)$.

4.1. Proof of Lemma 2.4. Let $G = K_{8,8}$. By contradiction and by Lemma 2.8, assume that there exists a $b \in Z(G, \mathbb{Z}_5)$ such that for any $\ell \in L(b)$, G has no orientation realizing ℓ. By replacing b with $-b$ if necessary, we may assume there is an $\ell \in L(b)$ satisfying $|M_1(\ell)| \geq |M_2(\ell)|$. Hence we may choose an $\ell_0 \in L(b)$ such that $\|\ell_0\| = \max\{\|\ell\| : \ell \in L(b) \text{ and } |M_1(\ell)| \geq |M_2(\ell)|\}$. By Lemma 2.6, there exists an $S \subset V$ such that

$$\left|\sum_{v \in S} \ell_0(v)\right| > |\partial_G(S)|. \tag{8}$$

We shall show that there is an orientation D realizing ℓ_0 to obtain a contradiction. Throughout the proof, we may choose different S satisfying (8) with additional properties for some specific purposes in different steps, and let $\overline{S} = V(G) - S$. In the following, we use M_1, M_2, N_k, N_+, N_- to denote $M_1(\ell_0), M_2(\ell_0), N_k(\ell_0), N_+(\ell_0), N_-(\ell_0)$, respectively. Since $K_{8,8}$ is Eulerian, by Definition 2.7(ii), any $\ell \in L(b)$ is even integer valued, and so M_1 and M_2 are even integers. By definition, $M_1 - 10 \leq M_2 \leq M_1 \leq 8$.

Claim 1. $M_1 \leq 6$.

If $M_1 = 8$, by Definition 2.7(ii), $M_2 \geq 8 - 10 = -2$. Then for any $v \in V(G) - N_8$, we have $-2 \leq \ell(v) \leq 6$. Thus by (7),

$$8|N_8| + 6(16 - |N_8| - |N_{-2}|) + (-2)|N_{-2}| \geq \sum_{v \in V(G)} \ell_0(v) \geq 8|N_8| + (-2)|N_{-2}|.$$

By $\sum_{v \in V(G)} \ell_0(v) = 0$ and algebraic manipulation, we have both $|N_8| + 48 \geq 4|N_{-2}|$ and $|N_{-2}| \geq 4|N_8|$, implying that $|N_8| \leq 3$ and $|N_{-2}| \leq 12$. Thus

$$\|\ell_0\| \leq \sum_{v \in N_+} \ell_0(v) = -\sum_{v \in N_-} \ell_0(v) = 2|N_{-2}| \leq 24. \tag{9}$$

We will show that

$$N_8 \cap X = \emptyset \quad \text{or} \quad N_8 \cap Y = \emptyset. \tag{10}$$

If not, we may assume $\{x_1, y_1\} \subseteq N_8$. Since $\sum_{v \in N_+} \ell_0(v) \leq 24$, we have $\|\ell_0\| \leq \sum_{v \in N_+} \ell_0(v) - 8 \leq 16$ in this case. Moreover, since $|N_{-2}| \geq 4|N_8| \geq 8$, it follows that $X \cap N_{-2} \neq \emptyset$ and $Y \cap N_{-2} \neq \emptyset$. We may assume that $x_2, y_2 \in N_{-2}$. Let $\ell_1 = \ell_0(x_2; y_1)$ and $\ell_2 = \ell_0(x_1; y_2)$. By Definition 4.2,

$$\max\{\|\ell_1\|, \|\ell_2\|\} \geq \max\left\{\sum_{v \in X \cap N_+} \ell_0(v) + 8, \sum_{v \in Y \cap N_+} \ell_0(v) + 8\right\} = \|\ell_0\| + 8,$$

contrary to the maximality of $\|\ell_0\|$. Therefore, (10) follows.

Choose $S \subset V$ satisfying (8) with $|S|$ minimized. By (8) and as $|S|$ is minimized, we have

$$24 \geq \left|\sum_{v \in S} \ell_0(v)\right| > |\partial_G(S)| = x_S(8 - y_S) + y_S(8 - x_S). \tag{11}$$

Hence $|S| \leq 3$. Clearly, $|S| = 1$ is impossible since $|\partial_G(S)| = 8 = M_1$ in this case. If $|S| = 2$, then $|S \cap X| = |S \cap Y| = 1$ and $\ell_0(v) = 8$ for any $v \in S$, contrary to (10).

Therefore, we must have $|S| = 3$ and $S \cap X \neq \emptyset, S \cap Y \neq \emptyset$. Hence, by (10),

$$8+8+6 \geq \left|\sum_{v\in S} \ell_0(v)\right| > |\partial_G(S)| = 20.$$

As ℓ_0 is even valued, $|\sum_{v\in S} \ell_0(v)| = 22$. By (10), we may assume that $S = \{x_1, y_1, y_2\}$ with $\ell_0(y_1) = \ell_0(y_2) = 8$ and $\ell_0(x_1) = 6$. Hence by (9), $\|\ell_0\| \leq \sum_{v\in N_+} \ell_0(v) - \min\{\ell_0(x_1), \ell_0(y_1)\} \leq 24 - 6 = 18$. Since $|N_{-2}| = \frac{1}{2}\sum_{v\in N_+} \ell_0(v) \geq \frac{1}{2}|\sum_{v\in S} \ell_0(v)| \geq \frac{22}{2} = 11$, it follows that $|N_{-2} \cap X| \geq 4$. Without loss of generality, we assume that $x_3, x_4 \in N_{-2} \cap X$ and let $\ell_2 = \ell_0(x_3, x_4; y_1, y_2)$. Then we have $\ell_2 \in L(b)$ with $\|\ell_2\| \geq 8+8+6 > 18$, contrary to the maximality of $\|\ell_0\|$. Therefore, Claim 1 holds.

To continue presenting our arguments, we note that by definition, for any $T \subset V$, we have

$$\begin{aligned}(12)\qquad |\partial_G(T)| - \left|\sum_{v\in T} \ell_0(v)\right| &\geq x_T(8-y_T) + y_T(8-x_T) - M_1(x_T+y_T)\\ &= (8-M_1)(x_T+y_T) - 2x_Ty_T.\end{aligned}$$

Claim 2. $M_1 = 6$.

If not, then $M_1 \leq 4$. Pick any $T \subset V$. By the symmetry between T and $V(G)-T$, we may assume that $|T| \leq 8$. By (12), we have

$$\begin{aligned}|\partial_G(T)| - \left|\sum_{v\in T} \ell_0(v)\right| &\geq x_T(8-y_T) + y_T(8-x_T) - M_1(x_T+y_T)\\ &\geq x_T(8-y_T) + y_T(8-x_T) - 4(x_T+y_T)\\ &= 8 - 2(2-x_T)(2-y_T)\\ &\geq 8 - \frac{1}{2}(x_T+y_T-4)^2 \geq 0.\end{aligned}$$

By Lemma 2.6, there exists an orientation D realizing ℓ_0, leading to a contradiction. Therefore, Claim 2 holds.

For any S satisfying (8), by (12) with $T = S$ and $M_1 = 6$, we have $0 > x_S + y_S - x_Sy_S$, and so $x_S \geq 2$, $y_S \geq 2$, $x_{\overline{S}} \leq 6$, and $y_{\overline{S}} \leq 6$. By swapping S and $\overline{S}$ in the arguments above, we also have $x_{\overline{S}} \geq 2$, $y_{\overline{S}} \geq 2$, $x_S \leq 6$, and $y_S \leq 6$. Hence we have

$$(13)\qquad 6 \geq x_S \geq 2 \quad\text{and}\quad 6 \geq y_S \geq 2.$$

To estimate $|N_6|$ and $|N_{-4}|$, it follows from (7) that

$$6|N_6|+4(16-|N_6|-|N_{-4}|)-4|N_{-4}| \geq 0 = \sum_{v\in V(G)} l_0(v) \geq 6|N_6|-2(16-|N_6|-|N_{-4}|)-4|N_{-4}|,$$

which implies

$$(14)\qquad 4|N_6| - 16 \leq |N_{-4}| \leq 8 + |N_6|/4 \quad\text{and}\quad |N_6| \leq 6.$$

Case 1. $|N_6| = 6$. Then by (14), $8 \leq |N_{-4}| \leq 9$, and so both $|N_+| = |N_6| = 6$ and $\sum_{v\in N_+} \ell_0(v) = -\sum_{v\in N_-} \ell_0(v) = 36$. Since $|N_6 \cup N_{-4}| \geq 2|N_6|$, by Observation 4.3(ii), there exists an ℓ_1, a switch of ℓ_0, satisfying $N_6(\ell_1) \subset X$ or $N_6(\ell_1) \subset Y$. Hence $\|\ell_0\| \geq \|\ell_1\| = 36$, forcing $\|\ell_0\| = 36$. Thus either $N_6 \subset X$ or $N_6 \subset Y$, and so we may assume $\ell_0(x_i) = 6$ for $1 \le i \le 6$. Utilizing the symmetry between S and $\bar{S} = V - S$, we may choose $S \subset V$ satisfying (8) with $\sum_{v\in S} \ell_0(v) > 0$. Let $L' = \{\ell_0(x_7), \ell_0(x_8), \ell_0(y_1), \ldots, \ell_0(y_8)\}$ be a multiset. As $\ell_0(x_i) = 6$ for $1 \le i \le 6$, it follows by Definition 2.7 that either nine members in L' are "-4" and one of them is "0" or eight members in L' are "-4" and two of them are "-2". In either case, we have

$$0 < \sum_{v\in S} \ell_0(v) \le 6x_S + (-4)(y_S - 2) - 4. \tag{15}$$

Therefore, by (13) and (15), a contradiction is obtained:

$$\begin{aligned} 0 > |\partial_G(S)| - \left|\sum_{v\in S} \ell_0(v)\right| &\ge x_S(8 - y_S) + y_S(8 - x_S) - [6x_S + (-4)(y_S - 2) - 4] \\ &= 8 - 2(6 - x_S)(1 - y_S) \ge 8. \end{aligned}$$

Case 2. $|N_6| = 5$. Thus, $4 \le |N_{-4}| \le 9$ by (14). Hence

$$\sum_{v\in N_+} \ell_0(v) = -\sum_{v\in N_-} \ell_0(v) \le 36 \text{ and } |N_+| \le 7. \tag{16}$$

Since $|N_6 \cup N_{-4}| \ge 9$ and $|N_6| = 5$, by Observation 4.3(ii), there exists an ℓ_1, which is a switch of l_0, such that $N_6(\ell_1) \subset X$ or $N_6(\ell_1) \subset Y$. Thus $\|\ell_0\| \ge \|\ell_1\| \ge 6 \times 5 = 30$. We first show that

$$\text{either } N_6 \subset X \text{ or } N_6 \subset Y. \tag{17}$$

If both $N_6 \cap X \ne \emptyset$ and $N_6 \cap Y \ne \emptyset$, then by (16) and the maximality of $\|\ell_0\|$, we have $\|\ell_0\| = 30 = 6 + 6 + 6 + 6 + 4 + 2$, and so we may assume that $\ell_0(x_i) = 6$ for $1 \le i \le 4$, $\ell_0(x_5) = 4, \ell_0(x_6) = 2$ and $\ell_0(y_1) = 6$. It follows that $\sum_{v\in N_+} \ell_0(v) = -\sum_{v\in N_-} \ell_0(v) = 6\times 5 + 4 + 2 = 36$, and so $|N_+| = 7$ and $|N_-| = |N_{-4}| = 9$, implying that $x_7 \in N_{-4}$. Set $\ell_2 = \ell_0(x_7; y_1)$ to be a switch of ℓ_0. Then $\|\ell_2\| = 36 > \|\ell_0\|$, contrary to the maximality of $\|\ell_0\|$. This justifies (17).

By (17), we may assume that $N_6 \subset X$ and $\ell_0(x_i) = 6$ with $1 \le i \le 5$. By symmetry between S and $\overline{S}$, we may choose $S \subset V$ satisfying (8) with $|S| \le 8$. Thus, by (16), $6x_S + (4 + 2) \ge |\sum_{v\in S} l_0(v)| > |\partial_G(S)| = x_S(8 - y_S) + y_S(8 - x_S)$, and so $2x_S y_S - 2x_S - 8y_S + 6 > 0$, which amounts to $(x_S - 4)(y_S - 1) \ge 2$. As x_S and y_S are nonnegative integers with $x_S + y_S = |S| \le 8$, $(x_S, y_S) \in \{(5, 3), (6, 2)\}$. In either case, $|S| = 8$ and $6x_S + (4 + 2) = |\sum_{v\in S} \ell_0(v)|$. However, this implies $\sum_{v\in N_+} \ell_0(v) = -\sum_{v\in N_-} \ell_0(v) = 36$. Hence we must have $x_S = 5$ and $|N_-| \ge 9$, and so by $|S| = 8$, $\ell_0(S)$ contains five "6," one "4," and one "2," plus a negative value. It follows that $6x_S + (4 + 2) = |\sum_{v\in S} \ell_0(v)| \le 6x_S + (4 + 2) - 2$, a contradiction.

Case 3. $|N_6| \le 4$. By Claim 2, $|N_6| \ge 1$, and so $N_- = N_{-2} \cup N_{-4}$. We have

$$\sum_{v\in N_+} \ell_0(v) \le 6|N_6| + 4(16 - |N_6| - |N_-|) \text{ and } -\sum_{v\in N_-} \ell_0(v) \le 4|N_-|. \tag{18}$$

It follows from (18) that $\sum_{v\in N_+} \ell_0(v) \le 2|N_6| + 64 - (-\sum_{v\in N_-} \ell_0(v))$, and so

$$\sum_{v\in N_+} \ell_0(v) = -\sum_{v\in N_-} \ell_0(v) \le |N_6| + 32 \le 36. \tag{19}$$

By the symmetry between S and $\overline{S}$, we may choose $S \subset V$ satisfying (8) with $|S| \le 8$. By (13), we have $6 \times 4 + 4(x_S + y_S - 4) \ge |\sum_{v\in S} \ell_0(v)| > |\partial_G(S)| = x_S(8-y_S) + y_S(8-x_S)$, and so $(x_S-2)(y_S-2) \ge 1$. This implies that $x_S \ge 3$ and $y_S \ge 3$, and so $\{x_S, y_S\} = \{3,3\}, \{3,4\}, \{3,5\},$ or $\{4,4\}$. By symmetry, we have the following four subcases.

Subcase 3.1. $x_S = 3$ and $y_S = 3$.

Then $32 = 6\times 4 + 4(x_S+y_S-4) \ge |\sum_{v\in S}\ell_0(v)| > x_S(8-y_S)+y_S(8-x_S) = 30$. Thus, we must have the multiset $\{\ell_0(v) : v \in S\} = \{6,6,6,6,4,4\}$. As $x_S = 3$ and $y_S = 3$, it follows from (19) that $\|\ell_0\| \le 36 - 6 - 4 - 4 = 22$, and $\sum_{v\in N_+}\ell_0(v) = -\sum_{v\in N_-}\ell_0(v) \ge |\sum_{v\in S}\ell_0(v)| = 32$. Hence $10 \ge |N_-| \ge \frac{32}{4} = 8$, and so $|N_{-4}| \ge 6$. Since $|N_6 \cup N_{-4}| \ge 10 > 2|N_6|$, by Observation 4.3(ii), there exists an ℓ_1, which is a switch of ℓ_0, such that $N_6(\ell_1) \subseteq X$ or $N_6(\ell_1) \subseteq Y$. Hence, $\|\ell_1\| \ge 6\times 4 > \|\ell_0\|$, contrary to the maximality of $\|\ell_0\|$.

Subcase 3.2. $x_S = 3$ and $y_S = 4$.

Then $36 = 6\times 4 + 4(x_S+y_S-4) \ge |\sum_{v\in S}\ell_0(v)| > x_S(8-y_S)+y_S(8-x_S) = 32$. Thus,

$$\sum_{v\in N_+}\ell_0(v) = -\sum_{v\in N_-}\ell_0(v) \ge \left|\sum_{v\in S}\ell_0(v)\right| \ge 34. \tag{20}$$

As $36 \ge |\sum_{v\in S}\ell_0(v)| \ge 34$ and by (19), we must have $|N_-| = 9$, $|N_+| = 7$ and $|N_{-4}| \ge 8$, implying $S = N_+$. By (20), $6|N_6| + 4|N_4| + 2|N_2| \ge 34$, forcing $|N_2| \le 1$.

If $|N_2| = 0$, then $\|\ell_0\| \le 36 - 4 - 4 - 4 = 24$ by (19). Hence we must have $|N_6| = 4, |N_4| = 3$ or $|N_6| = 3, |N_4| = 4$ by (20), and so either $|N_4 \cap X| \ge 2$ or $|N_4 \cap Y| \ge 2$. Without loss of generality, we assume that $|N_4 \cap X| \ge 2$. Since $|N_{-4}| \ge 8$ and $S = N_+$, we have $|[N_{-4}\cup N_6]\cap X| \ge |N_6|$. By Observation 4.3(i), there exists an $\ell_1 \in L(b)$, a switch of ℓ_0 by swapping vertices in $N_{-4}\cup N_6$ so that $N_6(\ell_1) \subset X$. As a result, $\|\ell_0\| \ge \|\ell_1\| \ge 6\times 3 + 4 + 4 = 26 > 24$, contrary to the fact that $\|l_0\| \le 24$.

Therefore, we have $|N_2| = 1$, and so $\|\ell_0\| \le 36 - 4 - 4 - 2 = 26$ by (19). Hence $|N_6| = 4$ and $|N_4| = 2$ by (20). Without loss of generality, assume that $|[N_4 \cup N_2]\cap X| \ge 2$. Since $|N_{-4}| \ge 8$ and $S = N_+$, we have $|[N_{-4}\cup N_6]\cap X| \ge |N_6|$. By Observation 4.3(i), there exists an $\ell_2 \in L(b)$, which is a switch of ℓ_0 by swapping vertices in $N_{-4}\cup N_6$ so that $N_6(l_2) \subset X$. It follows that $\|\ell_0\| \ge \|\ell_2\| \ge 6\times 4+4+2 = 30 > 26$, contrary to the fact that $\|\ell_0\| \le 26$. This completes the proof for subcase 3.2.

Subcase 3.3. $x_S = 3$ and $y_S = 5$.

Then $|\sum_{v\in S}\ell_0(v)| > x_S(8-y_S) + y_S(8-x_S) = 34$. By (19), we have $36 \ge \sum_{v\in N_+}\ell_0(v) = -\sum_{v\in N_-}\ell_0(v) \ge |\sum_{v\in S}\ell_0(v)| \ge 36$, and so we must have $|N_-| = |N_{-4}| = 9$, $|N_6| = 4$, and $|N_4| = 3$. These, together with $|S| = 8$, lead to a contradiction that $36 \le |\sum_{v\in S}\ell_0(v)| \le 6\times 4 + 4\times 3 - 4 = 32$.

Subcase 3.4. $x_S = 4$ and $y_S = 4$.

Then $|\sum_{v\in S}\ell_0(v)| > x_S(8-y_S) + y_S(8-x_S) = 32$. By (19), we have $36 \ge \sum_{v\in N_+}\ell_0(v) = -\sum_{v\in N_-}\ell_0(v) \ge |\sum_{v\in S}\ell_0(v)| \ge 34$. Hence we have $|N_-| = 9$,$|N_+| = 7$ and $|N_{-4}| \ge 8$. Moreover, $32 < |\sum_{v\in S}\ell_0(v)| \le \sum_{v\in N_+}\ell_0(v) - 2 \le 34$ by (19) and $|S| = 8$. Therefore, we have $\sum_{v\in N_+}\ell_0(v) = -\sum_{v\in N_-}\ell_0(v) = 36$. Hence $|N_-| = |N_{-4}| = 9$, leading to a contradiction that $32 < |\sum_{v\in S}\ell_0(v)| \le \sum_{v\in N_+}\ell_0(v) - 4 = 32$. This completes the proof of Subcase 3.4, as well as the theorem.

4.2. Proof of Lemma 2.3. By contradiction, we assume that there exists a mapping $b \in Z(G, \mathbb{Z}_{2p+1})$ without an orientation satisfying it. Then for any $\ell \in L(b)$, there exist no orientations realizing ℓ. For each such ℓ, by Lemma 2.6, there exists a subset $S \subset V$ such that $|\sum_{v\in S} \ell(v)| > |\partial_G(S)|$. Choose $\ell_0 \in L(b)$. We continue using M_1, M_2, N_k, N_+, N_- to denote $M_1(\ell_0), M_2(\ell_0), N_k(\ell_0), N_+(\ell_0), N_-(\ell_0)$, respectively, and let $\overline{S} = V(G) - S$. We will show that there is an orientation D realizing ℓ_0 to obtain a contradiction.

By symmetry between b and $-b$, we may assume $|M_1(\ell_0)| \geq |M_2(\ell_0)|$. Since $4p$ and $16p^2$ are even, by Definition 2.7, M_1 and M_2 are even and $M_1 - 4p - 2 \leq M_2 \leq M_1 \leq 4p$.

Claim 1. $M_1 \leq 4p - 2$.

If not, then $M_1 = 4p$, and so we have

$$\begin{aligned} &4p|N_{4p}| + (4p-2)(16p^2 + 4p - |N_{4p}| - |N_{-2}|) + (-2)|N_{-2}| \\ &\geq 0 = \sum_{v\in V(G)} \ell_0(v) \geq 4p|N_{4p}| + (-2)|N_{-2}|. \end{aligned}$$

Algebraic manipulations lead to $2p \leq 2p|N_{4p}| \leq |N_{-2}| \leq (4p-2)(4p+1) + \frac{|N_{4p}|}{2p}$ and $|N_{4p}| \leq 8p - 2 + \frac{2}{2p+1}$. Hence, $|N_{4p}| \leq 8p-2$, $|N_{-2}| \leq (4p-2)(4p+1)+3$, and

$$\sum_{v\in N_+} \ell_0(v) = -\sum_{v\in N_-} \ell_0(v) = 2|N_{-2}| \leq 32p^2 - 8p + 2.$$

By symmetry between S' and $\overline{S'}$, we may choose $S' \subset V$ satisfying $|\sum_{v\in S'} \ell(v)| > |\partial_G(S')|$ with $x_{S'} \leq 2p$. As $x_{S'} \leq 2p$, we have

$$32p^2 - 8p + 2 \geq \left|\sum_{v\in S'} \ell_0(v)\right| > |\partial_G(S')| = 16p^2 x_{S'} + y_{S'}(4p - 2x_{S'}). \tag{21}$$

Hence $x_{S'} \leq 1$. Clearly, $x_{S'} = 0$ is impossible since $|\partial_G(S')| = 4py_{S'} \geq |\sum_{v\in S'} l_0(v)|$, contrary to choice of S'. We must have $x_{S'} = 1$, and so $y_{S'} \leq 4p+1$ by (21). Since $4p(x_{S'} + y_{S'}) \geq |\sum_{v\in S'} \ell_0(v)|$, it follows that

$$\begin{aligned} |\partial_G(S')| - \left|\sum_{v\in S'} \ell_0(v)\right| &\geq x_{S'}(16p^2 - y_{S'}) + y_{S'}(4p - x_{S'}) - 4p(x_{S'} + y_{S'}) \\ &= 16p^2 - 4p - 2y_{S'} \geq 16p^2 - 12p - 2 \geq 0, \end{aligned}$$

contrary to the assumption that $|\sum_{v\in S'} \ell_0(v)| > |\partial_G(S')|$. This proves Claim 1.

By Claim 1, we have $M_1 \leq 4p - 2$. Choose $S \subset V$ such that $|\sum_{v\in S} \ell_0(v)| > |\partial_G(S)|$. We may further assume $y_S \leq 8p^2$ by symmetry between S and $\overline{S}$. Since $(4p-2)(x_S + y_S) \geq |\sum_{v\in S} \ell_0(v)|$, it follows that

$$\begin{aligned} |\partial_G(S)| - \left|\sum_{v\in S} \ell_0(v)\right| &\geq x_S(16p^2 - y_S) + y_S(4p - x_S) - (4p-2)(x_S + y_S) \\ &= (2 - 2x_S)y_S + (16p^2 - 4p + 2)x_S. \end{aligned} \tag{22}$$

Note that $|\partial_G(S)| - |\sum_{v\in S} \ell_0(v)| \geq 0$ if $x_S = 0$ or $x_S = 1$. Thus, we have $y_S \leq 8p^2$ and $2 \leq x_S \leq 4p$. It follows from (22) that

$$\begin{aligned} |\partial_G(S)| - \left|\sum_{v\in S} l_0(v)\right| &\geq (2 - 2x_S)\cdot 8p^2 + (16p^2 - 4p + 2)x_S \\ &= 16p^2 - (4p-2)x_S \geq 16p^2 - 4p(4p-2) > 0, \end{aligned}$$

a contradiction to $|\sum_{v\in S} \ell_0(v)| > |\partial_G(S)|$. The proof is completed. □

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