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ARTICLE

Nowhere-zero 3-flow and $\mathbb{Z}_3$-connectedness in graphs with four edge-disjoint spanning trees

Miaomiao Han | Hong-Jian Lai | Jiaao Li

Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA

Correspondence
Jiaao Li, Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA.
Email: joli@mix.wvu.edu

Abstract

Given a zero-sum function $\beta : V(G) \to \mathbb{Z}_3$ with $\sum_{v\in V(G)} \beta(v) = 0$, an orientation D of G with $d^+_D(v) - d^-_D(v) = \beta(v)$ in $\mathbb{Z}_3$ for every vertex $v \in V(G)$ is called a β-orientation. A graph G is $\mathbb{Z}_3$-connected if G admits a β-orientation for every zero-sum function β. Jaeger et al. conjectured that every 5-edge-connected graph is $\mathbb{Z}_3$-connected. A graph is $\langle\mathbb{Z}_3\rangle$-extendable at vertex v if any preorientation at v can be extended to a β-orientation of G for any zero-sum function β. We observe that if every 5-edge-connected essentially 6-edge-connected graph is $\langle\mathbb{Z}_3\rangle$-extendable at any degree five vertex, then the above-mentioned conjecture by Jaeger et al. holds as well. Furthermore, applying the partial flow extension method of Thomassen and of Lovász et al., we prove that every graph with at least four edge-disjoint spanning trees is $\mathbb{Z}_3$-connected. Consequently, every 5-edge-connected essentially 23-edge-connected graph is $\langle\mathbb{Z}_3\rangle$-extendable at any degree five vertex.

KEYWORDS

3-flow conjecture, edge-disjoint spanning trees, group connectivity

1 | INTRODUCTION

We consider finite graphs without loops, but with possible multiple edges, and follow [2] for undefined terms and notation. As in [2], $\kappa'(G)$ denotes the edge-connectivity of a graph G; and $d^+_D(v)$, $d^-_D(v)$ denote the out-degree and the in-degree of a vertex in a digraph D, respectively. For an $m \in \mathbb{Z}$, let $\mathbb{Z}_m$ be the set of integers modulo m, as well as the (additive) cyclic group on m elements. For vertex subsets $U, W \subseteq V(G)$, let $[U, W]_G = \{uw \in E(G) | u \in U, w \in W\}$; and for each $v \in V(G)$, define $E_G(v) = [\{v\}, V(G) - \{v\}]_G$. The subscript G may be omitted if G is understood from the context. An

edge cut $X = [S, V(G) - S]$ in a connected graph G is **essential** if at least two components of $G - X$ are nontrivial. A graph is **essentially k-edge-connected** if it does not have an essential edge cut with fewer than k edges.

For an integer $m > 1$, a graph G admits a **mod m-orientation** if G has an orientation D such that at every vertex $v \in V(G)$, $d_D^+(v) - d_D^-(v) \equiv 0 \pmod{m}$. Let $\mathcal{M}_m$ be the family of all graphs admitting a mod m-orientation. Let $k \geq 2$ be an integer and G be a graph with an orientation $D = D(G)$. For any vertex $v \in V(G)$, let $E_D^+(v)$ denote the set of all edges directed away from v, and let $E_D^-(v)$ denote the set of all edges directed into v. A function $f : E(G) \to \{\pm 1, \pm 2, \ldots, \pm(k-1)\}$ is called a **nowhere-zero k-flow** if

$$\sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e) = 0, \text{ for any vertex } v \in V(G).$$

The well-known 3-Flow Conjecture of Tutte is stated below.

Conjecture 1.1. *(Tutte [25]) Every 4-edge-connected graph admits a nowhere-zero 3-flow.*

Tutte [26] (see also Brylawski [3], Arrowsmith and Jaeger [1]) indicated that a graph G has a nowhere-zero k-flow if and only if G has a nowhere-zero $\mathbb{Z}_k$-flow. Moreover, a graph has a nowhere-zero 3-flow if and only if G has a mod 3-orientation (i.e. $G \in \mathcal{M}_3$).

Jaeger et al. [11] introduced the notion of $\mathbb{Z}_k$-connectedness as a generalization of nowhere-zero flows. In this article, we mainly focus on $\mathbb{Z}_3$-connectedness of graphs. A function $\beta : V(G) \to \mathbb{Z}_3$ is a zero-sum function of G if $\sum_{v \in V(G)} \beta(v) = 0$ in $\mathbb{Z}_3$. Let $Z(G, \mathbb{Z}_3)$ be the set of all zero-sum functions of G. An orientation D of G with $d_D^+(v) - d_D^-(v) = \beta(v)$ in $\mathbb{Z}_3$ for every vertex $v \in V(G)$ is called a **β-orientation**. Note that a mod 3-orientation of G is a β-orientation with $\beta(v) = 0$ for every vertex $v \in V(G)$. A graph G is **$\mathbb{Z}_3$-connected** if, for every $\beta \in Z(G, \mathbb{Z}_3)$, there is an orientation D such that $d_D^+(v) - d_D^-(v) \equiv \beta(v) \pmod{3}$ for every vertex $v \in V(G)$. The collection of all $\mathbb{Z}_3$-connected graphs is denoted by $\langle \mathbb{Z}_3 \rangle$. Jaeger et al. [11] proposed the following Conjecture.

Conjecture 1.2. *(Jaeger et al. [11]) Every 5-edge-connected graph is $\mathbb{Z}_3$-connected.*

A graph G with $z_0 \in V(G)$ is **$\mathcal{M}_3$-extendable at** z_0 if, for any pre-orientation D_{z_0} of $E_G(z_0)$ with $d_{D_{z_0}}^+(z_0) \equiv d_{D_{z_0}}^-(z_0) \pmod{3}$, D_{z_0} can be extended to a mod 3-orientation D of G. Kochol [12] showed that Conjecture 1.2 implies Conjecture 1.1.

Theorem 1.3. *(Kochol [12]) The following are equivalent.*

- **(i)** *Every 4-edge-connected graph has a nowhere-zero 3-flow.*
- **(ii)** *Every 5-edge-connected graph has a nowhere-zero 3-flow.*
- **(iii)** *Every 5-edge-connected essentially 6-edge-connected graph is $\mathcal{M}_3$-extendable at every degree 5 vertex.*
- **(iv)** *Every 4-edge-connected graph with each vertex of degree 4 or 5 is $\mathcal{M}_3$-extendable at every vertex.*

A graph is called **$\langle \mathbb{Z}_3 \rangle$-extendable at** z_0, if, for any $\beta \in Z(G, \mathbb{Z}_3)$ and any preorientation D_{z_0} of $E_G(z_0)$ with $d_{D_{z_0}}^+(z_0) - d_{D_{z_0}}^-(z_0) \equiv \beta(z_0) \pmod{3}$, D_{z_0} can be extended to a β-orientation D of G. In the next section, we shall prove the following proposition on extendability at vertex z_0.

Proposition 1.4. *Let G be a graph and $z_0 \in V(G)$.*

- **(i)** *G is $\langle \mathbb{Z}_3 \rangle$-extendable at z_0 if and only if $G - z_0$ is $\mathbb{Z}_3$-connected.*
- **(ii)** *If G is $\langle \mathbb{Z}_3 \rangle$-extendable at z_0, then G is $\mathbb{Z}_3$-connected.*

Thomassen [23] and Lovász et al. [19] utilized partial flow extensions to obtain breakthroughs in $\mathbb{Z}_3$-connectedness and modulo orientation problems. Lovász et al. [19,27] proved that every 6-edge-connected graph is $\mathbb{Z}_3$-connected. In fact, they have proved a stronger result.

Theorem 1.5. *(Lovász et al. [19] and Wu [27]) Every 6-edge-connected graph is $\langle \mathbb{Z}_3 \rangle$-extendable at any vertex of degree at most 7.*

Analogous to Theorem 1.3(iii) of Kochol, it is natural to suggest the following strengthening of Conjecture 1.2, which eliminates nontrivial 5-edge-cut, and whose truth would imply Conjecture 1.2, as to be shown in Section 3 of this article.

Conjecture 1.6. *Every 5-edge-connected essentially 6-edge-connected graph is $\langle \mathbb{Z}_3 \rangle$-extendable at any vertex of degree 5.*

The main results of this article are the following.

Theorem 1.7. *Every graph with 4 edge-disjoint spanning trees is $\mathbb{Z}_3$-connected.*

Thomassen [23] resolved the weak 3-flow conjecture by showing high edge-connectivity (8-edge-connected) guarantees the existence of nowhere-zero 3-flows. Analogously, a natural question is to ask whether a higher essentially edge-connectivity ensures the existence of nowhere-zero 3-flows. It is straightforward to check that the graph $K_{3,t}^+$ ($t \geq 4$) admits no mod 3-orientation, where $K_{3,t}^+$ denotes the graph obtained from complete bipartite graph $K_{3,t}$ by adding a new edge joining two vertices of degree t. This indicates a 3-edge-connected graph with arbitrary high essentially edge-connectivity may not admit a nowhere-zero 3-flow. The next theorem partially answers the question about existence and shows that 5-edge-connectivity with certain high essentially edge-connectivity 23 is sufficient for admitting a nowhere-zero 3-flow. This also approaches Theorem 1.3(iii) of Kochol, and provides some supporting evidence to Conjecture 1.6.

Theorem 1.8. *Each of the following holds.*

(a) *Every 5-edge-connected essentially 23-edge-connected graph is $\mathcal{M}_3$-extendable at any degree five vertex.*

(b) *Every 5-edge-connected essentially 23-edge-connected graph is $\langle \mathbb{Z}_3 \rangle$-extendable at any degree five vertex.*

Theorems 1.7 and 1.8 are immediate corollaries of a technical theorem, stated below as Theorem 1.9, which would be proved via utilizing a method of Thomassen [23] and Lovász et al. in [19].

Following Catlin [4], let $F(G, k)$ denote the minimum number of additional edges that must be added to G to result in a supergraph G' of G that has k edge-disjoint spanning trees. In particular, G has k edge-disjoint spanning trees if and only if $F(G, k) = 0$. It is known ([16,28]) that if G is $\mathbb{Z}_3$-connected, then it contains two edge-disjoint spanning trees (i.e. $F(G, 2) = 0$). A cut-edge is called a **bridge**. The following provides a sufficient condition for graphs to be $\mathbb{Z}_3$-connected through the number of edge-disjoint spanning trees.

Theorem 1.9. *Let G be a graph.*

(i) *Suppose that $F(G, 4) \leq 3$. Then G is $\mathbb{Z}_3$-connected, unless G contains a bridge. (Thus, G is $\mathbb{Z}_3$-connected if and only if $\kappa'(G) \geq 2$.)*

(ii) *Suppose that $F(G, 4) = 0$. Then for any vertex $v \in V(G)$ with $d_G(v) \leq 7$, if $\kappa'(G - v) \geq 2$, then G is $\langle \mathbb{Z}_3 \rangle$-extendable at v.*

Prerequisites will be presented in the next section. In Section 3, we will study the relationship among Conjectures 1.1, 1.2, and 1.6. Theorems 1.9, 1.7, and 1.8 will be proved in a subsequent section.

2 | PREREQUISITES

In this section, we will justify Proposition 1.4 and present other preliminaries. For a graph G and a vertex $z \in V(G)$, define $N_G(z) = \{v \in V(G) : zv \in E(G)\}$. For notation convenience, the algebraic manipulations in the proof of Proposition 1.4 will be over $\mathbb{Z}_3$.

Proof of Proposition 1.4. As Part (ii) is straightforward, we only prove Part (i). Suppose that a graph G is $\langle \mathbb{Z}_3 \rangle$-extendable at vertex z_0. Let D_{z_0} be a fixed preorientation of $E_G(z_0)$. We also use D_{z_0} to denote the digraph induced by the oriented edges of D_{z_0}. Define

$$b(v) = d^+_{D_{z_0}}(v) - d^-_{D_{z_0}}(v) \text{ for each } v \in N_G(z_0) \cup \{z_0\}. \tag{1}$$

Then $b(z_0) + \sum_{v \in N_G(z_0)} b(v) = 0$.

We are to prove $G - z_0$ is $\mathbb{Z}_3$-connected. For any $\beta \in \mathbb{Z}(G - z_0, \mathbb{Z}_3)$, define

$$\beta'(v) = \begin{cases} \beta(v) + b(v), & \text{if } v \in N_G(z_0); \\ b(z_0), & \text{if } v = z_0; \\ \beta(v), & \text{otherwise.} \end{cases}$$

Then $\sum_{v \in V(G)} \beta'(v) = \sum_{v \in V(G - z_0)} \beta(v) + (b(z_0) + \sum_{v \in N_G(z_0)} b(v)) = 0$, and so $\beta' \in Z(G, \mathbb{Z}_3)$. Since G is $\langle \mathbb{Z}_3 \rangle$-extendable at z_0, there exists an orientation D' of G such that $d^+_{D'}(v) - d^-_{D'}(v) = \beta'(v)$ for any vertex $v \in V(G)$ and D' agrees with D_{z_0} on $E_G(z_0)$. Let D be the restriction of D' on $G - z_0$. By the definition of β', we have $d^+_D(v) - d^-_D(v) = \beta(v)$ for any vertex $v \in V(G - z_0)$, and so $G - z_0$ is $\mathbb{Z}_3$-connected.

Conversely, assume that $G - z_0$ is $\mathbb{Z}_3$-connected. Let $\beta' \in Z(G, \mathbb{Z}_3)$, and D_{z_0} be a preorientation of $E_G(z_0)$ with $d^+_{D_{z_0}}(z_0) - d^-_{D_{z_0}}(z_0) = \beta'(z_0)$. Define $b(v)$ as in (1), and

$$\beta(v) = \begin{cases} \beta'(v) - b(v), & \text{if } v \in N_G(z_0); \\ \beta'(v), & \text{otherwise.} \end{cases}$$

As $\sum_{v \in V(G - z_0)} \beta(v) = \sum_{v \in V(G)} \beta'(v) = 0$, we have $\beta \in Z(G - z_0, \mathbb{Z}_3)$. Since $G - z_0 \in \langle \mathbb{Z}_3 \rangle$, there exists an orientation D' of $G - z_0$ satisfying $d^+_{D'}(v) - d^-_{D'}(v) = \beta'(v)$ for any vertex $v \in V(G - z_0)$. Combine D' and D_{z_0} to obtain an orientation D of G. Then for any vertex $v \in V(G)$, depending on $v = z_0$ or not, we always have $d^+_D(v) - d^-_D(v) = \beta'(v)$, and so G is $\langle \mathbb{Z}_3 \rangle$-extendable at z_0. This completes the proof of Proposition 1.4. ■

Let G be a graph and $\beta \in Z(G, \mathbb{Z}_3)$. Define an integer valued mapping $\tau : 2^{V(G)} \mapsto \{0, \pm 1, \pm 2, \pm 3\}$ as follows: for each vertex $x \in V(G)$,

$$\tau(x) \equiv \begin{cases} \beta(x) & \pmod 3; \\ d(x) & \pmod 2. \end{cases}$$

For a vertex set $A \subset V(G)$, denote $\beta(A) \equiv \sum_{v\in A}\beta(v) \pmod 3$, $d(A) = |[A, V(G) - A]|$ and define $\tau(A)$ to be

$$\tau(A) \equiv \begin{cases} \beta(A) & (\text{mod } 3); \\ d(A) & (\text{mod } 2). \end{cases}$$

Theorem 2.1. *(Lovász, Thomassen, Wu, and Zhang, Theorem 3.1 of [19]) Let G be a graph, $\beta \in Z(G, \mathbb{Z}_3)$ and $z_0 \in V(G)$. If D_{z_0} is a preorientation of $E_G(z_0)$, and if*

(i) $|V(G)| \geq 3$,

(ii) $d(z_0) \leq 4 + |\tau(z_0)|$ *and* $d^+(z_0) - d^-(z_0) \equiv \beta(z_0) \pmod 3$, *and*

(iii) $d(A) \geq 4 + |\tau(A)|$ *for each nonempty* $A \subseteq V(G) - \{z_0\}$ *with* $|V(G) - A| \geq 2$,

then D_{z_0} can be extended to a β-orientation of the entire graph G.

The following is an application of Theorem 2.1.

Lemma 2.2. *Let G be a 6-edge-connected graph. Each of the following holds.*

(i) *If $v \in V(G)$ with $d(v) \leq 7$, then $G - v \in \langle \mathbb{Z}_3 \rangle$.*

(ii) *If $E_1 \subset E(G)$ with $|E_1| \leq 3$, then $G - E_1 \in \langle \mathbb{Z}_3 \rangle$.*

Proof. (i) we may assume that $d_G(v) = 7$ to prove the lemma. Otherwise, pick an edge $e \in E_G(v)$ and add an edge parallel to e, which results in still a 6-edge-connected graph. Take an arbitrary $\beta' \in Z(G - v, \mathbb{Z}_3)$. We shall show that $G - v$ has a β'-orientation. Define $\beta(v) = 3$. We shall apply Theorem 2.1 by viewing v as z_0 in Theorem 2.1. Since $d(v) = 7$, we have $|\tau(v)| = 3$, and thus we can orient the edges $E_G(v)$ with an orientation D_v so that $d^+_{D_v}(v) = 5$ and $d^-_{D_v}(v) = 2$. Define $b(x) = d^+_{D_v}(x) - d^-_{D_v}(x)$ for each $x \in N_G(v)$ and set

$$\beta(x) = \begin{cases} \beta'(x) + b(x), & \text{if } x \in N_G(v); \\ \beta(v), & \text{if } x = v; \\ \beta'(x), & \text{otherwise.} \end{cases} \tag{2}$$

Then $\beta \in Z(G, \mathbb{Z}_3)$. As $\kappa'(G) \geq 6$, conditions (i)–(iii) of Theorem 2.1 are satisfied, and so by Theorem 2.1, G has a β-orientation D. Let D' be the restriction of D on $G - v$. By (2), D' is a β'-orientation of $G - v$. This proves (i).

(ii) Since $\mathbb{Z}_3$-connectedness is preserved under adding edges, we may assume that $|E_1| = 3$. In the graph G, subdivide each edge in E_1 with an internal vertex, say z_1, z_2, z_3. Identify z_1, z_2, z_3 to form a new vertex z_0 in the resulted graph G'. By the construction of G', we have $\kappa'(G') \geq 6$. By Lemma 2.2 (i), $G - E_1 = G' - z_0 \in \langle \mathbb{Z}_3 \rangle$. ∎

For an edge set $X \subseteq E(G)$, the **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X, and then deleting the resulting loops. If H is a subgraph of G, then we use G/H for $G/E(H)$. For a vertex set $W \subset V(G)$ such that $G[W]$ is connected, we also use G/W for $G/G[W]$.

Lemma 2.3. *(Proposition 2.1 of [13]) Let G be a graph. Each of the following holds.*

(i) *If $G \in \langle \mathbb{Z}_3 \rangle$ and $e \in E(G)$, then $G/e \in \langle \mathbb{Z}_3 \rangle$.*

(ii) *If $H \subseteq G$ and if $H, G/H \in \langle \mathbb{Z}_3 \rangle$, then $G \in \langle \mathbb{Z}_3 \rangle$.*

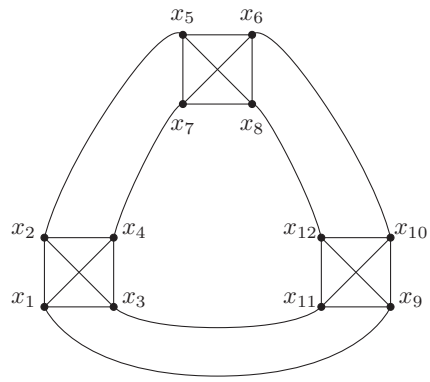

FIGURE 1 The graph J : a 4-edge-connected $\langle \mathbb{Z}_3 \rangle$-reduced graph

3 | RELATIONSHIP AMONG THE CONJECTURES

A graph is called $\langle \mathbb{Z}_3 \rangle$**-reduced** if it does not have any nontrivial $\mathbb{Z}_3$-connected subgraphs. By definition, K_1 is $\langle \mathbb{Z}_3 \rangle$-reduced. The potential minimal counterexamples of Conjectures 1.1 and 1.2 must be $\langle \mathbb{Z}_3 \rangle$-reduced graphs. As an example, it is routine to verify that the 4-edge-connected non-$\mathbb{Z}_3$-connected graph J constructed by Jaeger et al. [11] (see Figure 1) is indeed a $\langle \mathbb{Z}_3 \rangle$-reduced graph. Applying Theorem 2.1, we obtain the following.

Lemma 3.1. *Every $\langle \mathbb{Z}_3 \rangle$-reduced graph has minimal degree at most 5.*

Proof. Suppose, to the contrary, that there is a $\langle \mathbb{Z}_3 \rangle$-reduced graph G with $\delta(G) \geq 6$. As a cycle of length 2 is $\mathbb{Z}_3$-connected, G has no parallel edges and $|V(G)| \geq 4$. If $\kappa'(G) \geq 6$, then G is $\mathbb{Z}_3$-connected by Theorem 1.5, contradicting that G is a $\langle \mathbb{Z}_3 \rangle$-reduced graph. For a vertex subset $W \subset V(G)$, let $W^c = V(G) - W$. Among all those edge-cuts $[W, W^c]$ of size at most 5 in G, choose the one with $|W|$ minimized. Let v_c denote the vertex onto which W^c is contracted in G/W^c. Obtain a graph G' from G/W^c by adding $6 - d_{G/W^c}(v_c)$ edges between W and v_c. Then $\kappa'(G') \geq 6$ by the choice of W. By Lemma 2.2 (i), $G[W] = G' - v_c$ is $\mathbb{Z}_3$-connected, a contradiction. ■

Very recently, Lemma 3.1 has already an application in [17] to verify Tutte's 3-flow conjecture for graphs with independent number at most four. We believe that the following strengthening of Lemma 3.1 holds as well, whose truth implies Conjecture 1.2, as will be shown below in Proposition 3.3.

Conjecture 3.2. *Every $\langle \mathbb{Z}_3 \rangle$-reduced graph has minimal degree at most 4.*

The following proposition reveals some relationship among the conjectures.

Proposition 3.3. *Each of the following holds.*

(i) *Conjecture 1.6 implies validity of Conjecture 3.2.*

(ii) *Conjecture 3.2 implies validity of Conjecture 1.2.*

Proof. We shall prove (ii) first. Assume that Conjecture 3.2 holds. Then by the validity of Conjecture 3.2, every graph with minimum degree at least five is not $\langle \mathbb{Z}_3 \rangle$-reduced. Let G be a counterexample to Conjecture 1.2 with $|V(G)|$ minimized. Since $\delta(G) \geq \kappa'(G) \geq 5$, G is not $\langle \mathbb{Z}_3 \rangle$-reduced, and so G contains a nontrivial $\mathbb{Z}_3$-connected subgraph H. Since $\kappa'(G/H) \geq \kappa'(G) \geq 5$, and since $|V(G)| > |V(G/H)|$, the minimality of G implies that G/H is $\mathbb{Z}_3$-connected. By Lemma 2.3 (ii), G must be

$\mathbb{Z}_3$-connected as well, contrary to the assumption that G is a counterexample of Conjecture 1.2. This proves (ii).

To prove (i), we use arguments similar to those in the proof of Lemma 3.1. By contradiction, we assume that Conjecture 1.6 holds but there is a counterexample G to Conjecture 3.2 with $|V(G)|$ minimized and with $\delta(G) \geq 5$. By the validity of Conjecture 1.6, G must have an essential edge-cut of size at most 5. Among all those essential edge-cuts $[W, W^c]$ of size at most 5, choose the one with $|W|$ minimized. Let v_c denote the vertex onto which W^c is contracted in G/W^c. Adding some edges between W and v_c such that v_c has degree 5 in the new graph, and we still denote it G/W^c. Then we have $|W| \geq 2$, and the minimality of $|W|$ forces that G/W^c is an essentially 6-edge-connected graph. By the assumption that Conjecture 1.6 holds, G/W^c is $\langle \mathbb{Z}_3 \rangle$-extendable at v_c. By Proposition 1.4, $G[W] = G/W^c - v_c \in \langle \mathbb{Z}_3 \rangle$, contradicting that G is $\langle \mathbb{Z}_3 \rangle$-reduced. ■

In the rest of this section, we study the relationship between $\langle \mathbb{Z}_3 \rangle$-extendability and edge deletions. Theorem 3.4 below indicates that deleting one or two adjacent edges does not make Conjecture 1.2 stronger. Theorem 3.5 and Proposition 3.6 below also describe the strength of Conjecture 3.2 and Conjecture 1.6 via edge deletions.

Theorem 3.4. *The following statements are equivalent.*

(i) *Every 5-edge-connected graph is $\mathbb{Z}_3$-connected.*

(ii) *Every 5-edge-connected graph with two adjacent edges deleted is $\mathbb{Z}_3$-connected.*

Theorem 3.5. *The following statements are equivalent.*

(i) *Every $\langle \mathbb{Z}_3 \rangle$-reduced graph has minimal degree at most 4.*

(ii) *Every 5-edge-connected graph with any two edges deleted is $\mathbb{Z}_3$-connected.*

Proposition 3.6. *The following statements are equivalent.*

(i) *Every 5-edge-connected essentially 6-edge-connected graph is $\langle \mathbb{Z}_3 \rangle$-extendable at any vertex of degree 5.*

(ii) *Every 5-edge-connected graph is $\langle \mathbb{Z}_3 \rangle$-extendable at any vertex of degree 5.*

(iii) *Every 5-edge-connected graph with three incident edges of a degree 5 vertex deleted is $\mathbb{Z}_3$-connected.*

We shall justify Theorem 3.4 and Theorem 3.5 by utilizing Kochol's method in [12]. In [12], Kochol applies $\mathcal{M}_3$-extension on a degree 5 vertex and converts it into degree 3 vertices, which helps him establish Theorem 1.3. Unlike mod 3-orientations, direct application of the method above does not seem to help on $\langle \mathbb{Z}_3 \rangle$-extension for certain β-orientation. We observe that some edge deletions behave similarly as extension, as showed in Proposition 1.4 and the theorems above. This is part of the reason why we would like to prove Theorem 1.9 in the form of edge deletions.

A lemma is needed to prove Theorems 3.4 and 3.5.

Definition 3.7. Let G_1 be a graph with $e = u_1v_1 \in E(G_1)$, and $G_2(u_2, v_2)$ be a graph with distinguished (and distinct) vertices of u_2, v_2. Let $G_1 \oplus_e G_2$ be a graph obtained from the disjoint union of $G_1 - e$ and G_2 by identifying u_1 and u_2 to form a vertex u, and by identifying v_1 and v_2 to form a vertex v. Thus for $i \in \{1, 2\}$, we can view $u = u_i$ and $v = v_i$ in G_i. Note that even if e and u_2, v_2 are given, $G_1 \oplus_e G_2$ may not be unique. Thus we use $G_1 \oplus_e G_2$ to denote any one of the resulting graphs.

Lemma 3.8. *Let G_1 and G_2 be nontrivial graphs with $e \in E(G_1)$.*

(i) *If G_1 and G_2 are not $\mathbb{Z}_3$-connected graphs, then $G_1 \oplus_e G_2$ is not $\mathbb{Z}_3$-connected.*

(ii) *If G_1 and G_2 are $\langle \mathbb{Z}_3 \rangle$-reduced graphs, then $G_1 \oplus_e G_2$ is a $\langle \mathbb{Z}_3 \rangle$-reduced graph.*

Proof. (i) The proof is similar to those of Lemma 1 in [12] and of Lemma 2.5 in [6]. Let $G = G_1 \oplus_e G_2$. We shall adopt the notation in Definition 3.7. Fix $i \in \{1, 2\}$. Since G_i is not $\mathbb{Z}_3$-connected, there exists a $\beta_i \in Z(G_i, \mathbb{Z}_3)$ such that G_i does not have a β_i-orientation. Define $\beta : V(G) \mapsto \mathbb{Z}_3$ as follows:

$$\beta(x) = \begin{cases} \beta_1(x), & \text{if } x \in V(G_1) - \{u_1, v_1\}; \\ \beta_2(x), & \text{if } x \in V(G_2) - \{u_2, v_2\}; \\ \beta_1(x) + \beta_2(x), & \text{if } x \in \{u, v\}. \end{cases}$$

As $\sum_{z \in V(G)} \beta(z) = \sum_{i=1}^{2} \sum_{z \in V(G_i)} \beta_i(z)$, we have $\beta \in Z(G, \mathbb{Z}_3)$. It remains to show that G does not have a β-orientation. By contradiction, assume that G has a β-orientation D. Let D_2 be the restriction of D on $E(G_2)$. Then $d^+_{D_2}(x) - d^-_{D_2}(x) = \beta_2(x)$ in $\mathbb{Z}_3$ for any $x \in V(G_2) - \{u_2, v_2\}$. Since G_2 does not have a β_2-orientation, we must have $d^+_{D_2}(u) - d^-_{D_2}(u) \neq \beta_2(u)$ in $\mathbb{Z}_3$. Thus, we have either

$$d^+_{D_2}(u) - d^-_{D_2}(u) = \beta_2(u) + 1 \text{ and } d^+_{D_2}(v) - d^-_{D_2}(v) = \beta_2(v) - 1, \tag{3}$$

or

$$d^+_{D_2}(u) - d^-_{D_2}(u) = \beta_2(u) - 1 \text{ and } d^+_{D_2}(v) - d^-_{D_2}(v) = \beta_2(v) + 1. \tag{4}$$

Let D'_1 be the restriction of D on $E(G_1) - e$. If (3) holds, then both $d^+_{D'_1}(u) - d^-_{D'_1}(u) = \beta_1(u) - 1$ and $d^+_{D'_1}(v) - d^-_{D'_1}(v) = \beta_1(v) + 1$. Obtain an orientation D_1 of G_1 from D'_1 by orienting $e = u_1v_1$ from u_1 to v_1. If (4) holds, then both $d^+_{D'_1}(u) - d^-_{D'_1}(u) = \beta_1(u) + 1$ and $d^+_{D'_1}(v) - d^-_{D'_1}(v) = \beta_1(v) - 1$. Obtain an orientation D_1 of G_1 from D'_1 by orienting $e = u_1v_1$ from v_1 to u_1. In either case, D_1 is a β_1-orientation of G_1, contrary to the choice of β_1. (ii) follows from (i) by the definition of $\langle \mathbb{Z}_3 \rangle$-reduced graph. This proves the lemma. ∎

Similar operations as Definition 3.7 are developed in [8] to construct infinite families of $4p$-edge-connected graphs without mod $(2p + 1)$-orientation for every $p \geq 3$, which disproves Jaeger's circular flow conjecture. Now we are ready to prove Theorem 3.4 below.

Proof of Theorem 3.4. It suffices to prove that (i) implies (ii). By contradiction, assume that (i) holds and that there exists a graph Γ with $\kappa'(\Gamma) \geq 5$ and with two distinct adjacent edges $vv_1, vv_2 \in E(\Gamma)$, where v_1 and v_2 may or may not be distinct, such that $\Gamma - \{vv_1, vv_2\} \notin \langle \mathbb{Z}_3 \rangle$. As $\kappa'(\Gamma) \geq 5$, $|E_\Gamma(v)| \geq 5$. Let $K \cong K_4$ with $V(K) = \{w_1, w_2, w_3, w_4\}$.

We assume first that $v_1 \neq v_2$ in Γ, and use $L(v_1, v_2)$ to denote $\Gamma - \{vv_1, vv_2\}$ with v_1 and v_2 being two distinguished vertices. For $1 \leq j \leq 2$, let $\phi_j : L_j(v^j_1, v^j_2) \mapsto L(v_1, v_2)$ be a graph isomorphism with $\phi_j(v^j) = v$, $\phi_j(v^j_1) = v_1$ and $\phi_j(v^j_2) = v_2$. Define $J(v^1, v^2) = K \oplus_{w_1w_2} L_1(v^1_1, v^1_2) \oplus_{w_3w_4} L_2(v^2_1, v^2_2)$. Let $J^k(v^1, v^2)$, $(1 \leq k \leq 3)$, be three isomorphic copies of $J(v^1, v^2)$, and define $G(\Gamma) = K \oplus_{w_1w_2} J^1(v^1, v^2) \oplus_{w_2w_3} J^2(v^1, v^2) \oplus_{w_3w_4} J^3(v^1, v^2)$, as depicted in Figure 2. By the definition of $G(\Gamma)$, $G(\Gamma)$ contains six subgraphs H_i, $(1 \leq i \leq 6)$, each of which is isomorphic to $\Gamma - \{vv_1, vv_2\}$.

It is known that $K \notin \langle \mathbb{Z}_3 \rangle$. As $\Gamma - \{vv_1, vv_2\} \notin \langle \mathbb{Z}_3 \rangle$, it follows from Lemma 3.8 that $J(v^1, v^2) \notin \langle \mathbb{Z}_3 \rangle$, and so by repeated applications of Lemma 3.8, $G(\Gamma) \notin \langle \mathbb{Z}_3 \rangle$.

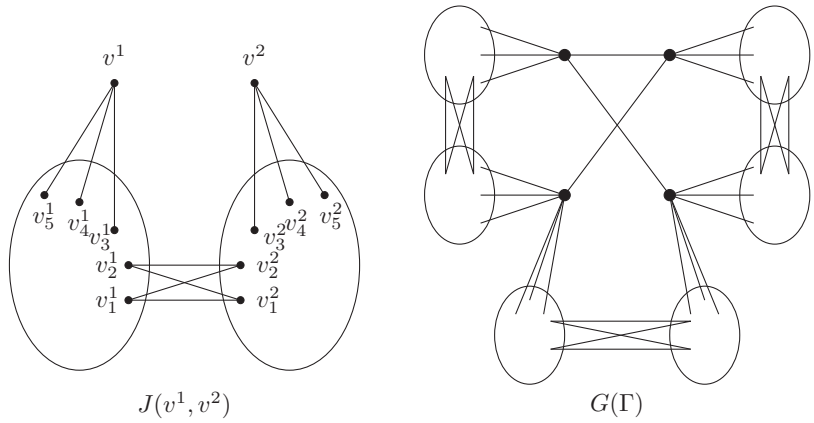

FIGURE 2 The construction in Theorem 3.4

Let $W \subseteq E(\Gamma)$ be a minimum edge cut of $G(\Gamma)$. If for any i, $|W \cap E(H_i)| = 0$, then W is an edge cut of the graph $G(\Gamma)/(\cup_{i=1}^{6} H_i)$, and so it is straightforward to check that $|W| \geq 5$. Hence we assume that for some i, $W \cap E(H_i) \neq \emptyset$. Then $\Gamma - \{vv_1, vv_2\}$ contains an edge subset W_i' corresponding to $W \cap E(H_i)$ under the isomorphism between $\Gamma - \{vv_1, vv_2\}$ and H_i. If W_i' does not separate the neighbors of v and $\{v_1, v_2\}$ in Γ, then W_i' is an edge cut of Γ, and so $|W| \geq |W_i'| \geq \kappa'(\Gamma) \geq 5$. Hence by symmetry, we assume that v and v_1 are in different components of $\Gamma - W_i'$. Since $\kappa'(\Gamma) \geq 5$, we have $|W_i'| \geq \kappa'(\Gamma - \{vv_1, vv_2\}) = 5 - 2 = 3$. By the definition of $G(\Gamma)$, $G(\Gamma) - E(H_i)$ contains 2 edge-disjoint (v, v_1)-paths, which implies that $|W - E(H_i)| \geq 2$, and so $|W| = |W \cap E(H_i)| + |W - E(H_i)| \geq 3 + 2 = 5$. We conclude that $\kappa'(G(\Gamma)) \geq 5$. By Theorem 3.4(i), we have $G(\Gamma) \in \langle \mathbb{Z}_3 \rangle$, which leads to a contradiction to the fact that $G(\Gamma) \notin \langle \mathbb{Z}_3 \rangle$.

Next we assume that $v_1 = v_2$. Then for $j = 1, 2$, $v_1^j = v_2^j$ in $L_j(v_1^j, v_2^j)$. In this case, we differently define $J(v^1, v^2)$ to be the graph obtained from the disjoint union of $L_1(v_1^1, v_2^1)$ and $L_2(v_1^2, v_2^2)$ by identifying v_1^1 with v_1^2. Since $L_1(v_1^1, v_2^1)$ is a block of $J(v^1, v^2)$, $J(v^1, v^2) \notin \langle \mathbb{Z}_3 \rangle$. We again define $G(\Gamma) = K \oplus_{w_1w_2} J^1(v^1, v^2) \oplus_{w_2w_3} J^2(v^1, v^2) \oplus_{w_3w_4} J^3(v^1, v^2)$. Then by Lemma 3.8, $G(\Gamma) \notin \langle \mathbb{Z}_3 \rangle$. By a similar argument as shown above, we again conclude that $\kappa'(G(\Gamma)) \geq 5$, and so by Theorem 3.4(i), $G(\Gamma) \in \langle \mathbb{Z}_3 \rangle$. This contradiction establishes the theorem. ■

We need the following splitting theorem of Mader [20] before proceeding the next proof. For two distinct vertices x, y, let $\lambda_G(x, y)$ be the maximum number of edge-disjoint paths connecting x and y in G. The following Mader's theorem asserts that local edge-connectivity is preserved under splitting.

Theorem 3.9. *(Mader [20]) Let G be a graph and let z be a nonseparating vertex of G with degree at least four and $|N_G(z)| \geq 2$. Then there exist two edges v_1z, v_2z in G such that, splitting v_1z, v_2z, the resulting graph $G' = G - v_1z - v_2z + v_1v_2$ satisfies $\lambda_{G'}(x, y) = \lambda_G(x, y)$ for any two vertices x, y different from z.*

Proof of Theorem 3.5. (i) ⇒ (ii). By contradiction, assume that (i) holds and that there exists a 5-edge-connected graph Γ with $|V(\Gamma)|$ minimized and with two distinct edges $u_1u_2, v_1v_2 \in E(\Gamma)$, where u_1 and v_1 may or may not be distinct, such that $G = \Gamma - \{u_1u_2, v_1v_2\} \notin \langle \mathbb{Z}_3 \rangle$. By the minimality of $|V(\Gamma)|$, G must be a $\langle \mathbb{Z}_3 \rangle$-reduced graph. For $i = 1, 2$, let $K^i \cong K_3$ with $V(K^i) = \{w_1^i, w_2^i, w_3^i\}$. Define $K(v_1, v_2) = K^1 \oplus_{w_1^1w_2^1} G(u_1, u_2)$ and $H(w_3^1, w_3^2) = K^2 \oplus_{w_1^2w_2^2} K(v_1, v_2)$. As K_3 and G are $\langle \mathbb{Z}_3 \rangle$-reduced graphs, by Lemma 3.8(ii), $H(w_3^1, w_3^2)$ is a $\langle \mathbb{Z}_3 \rangle$-reduced graph. Moreover, $H(w_3^1, w_3^2)$

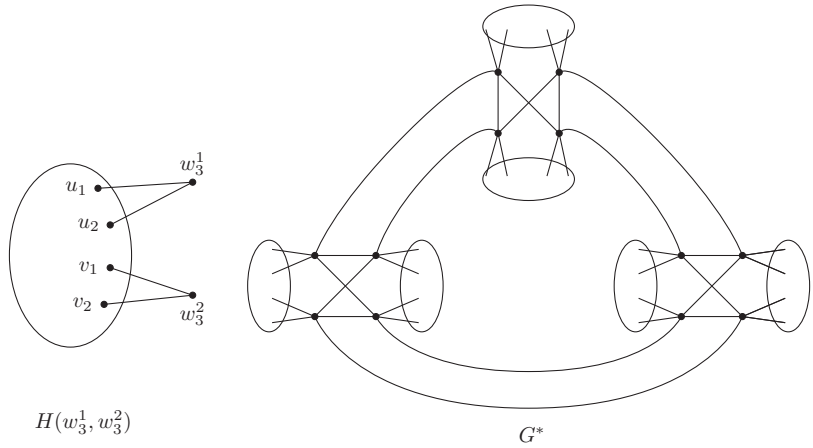

FIGURE 3 The construction in Theorem 3.5

has exactly two vertices of degree 2, namely w_3^1, w_3^2, and the other vertices of $H(w_3^1, w_3^2)$ have degree at least five.

Let J be the graph as depicted in Figure 1 with $V(J) = \{x_1, \ldots, x_{12}\}$. Obtain a graph G^* by attaching copies of $H(w_3^1, w_3^2)$ and applying $\oplus_e$ operation for each $e = x_{2i-1}x_{2i}$, $1 \le i \le 6$, as depicted in Figure 3. Then we have $\delta(G^*) \ge 5$. By the validity of (i), G^* is not $\langle \mathbb{Z}_3 \rangle$-reduced. On the other hand, as K_3 and G are $\langle \mathbb{Z}_3 \rangle$-reduced, it follows by Lemma 3.8(ii) that $H(w_3^1, w_3^2)$ is also $\langle \mathbb{Z}_3 \rangle$-reduced. As J and $H(w_3^1, w_3^2)$ are $\langle \mathbb{Z}_3 \rangle$-reduced, we conclude by Lemma 3.8(ii) that G^* is also $\langle \mathbb{Z}_3 \rangle$-reduced, contrary to the fact that G^* is not $\langle \mathbb{Z}_3 \rangle$-reduced, as implied by (i). This shows that (i) implies (ii).

(ii) $\Rightarrow$ (i). Assume that (ii) holds. Then (ii) implies that every 5-edge-connected graph is $\mathbb{Z}_3$-connected. Let G be a counterexample to (i). Then G is a $\langle \mathbb{Z}_3 \rangle$-reduced graph with $\delta(G) \ge 5$. If $\kappa'(G) \ge 5$, then by (ii), G itself is $\mathbb{Z}_3$-connected, contrary to the assumption that G is $\langle \mathbb{Z}_3 \rangle$-reduced. Hence $\kappa'(G) \le 4$. Since $\delta(G) \ge 5$, G must have an essential edge-cut of size at most 4. Among all essential edge-cuts $[W, W^c]$ of size at most 4, choose one with $|W|$ minimized. Since G is a $\langle \mathbb{Z}_3 \rangle$-reduced graph, $G[W]$ is also a $\langle \mathbb{Z}_3 \rangle$-reduced graph. Moreover, we claim that it is possible to add two new edges to $G[W]$ to result in a 5-edge-connected graph. If $|[W, W^c]| \le 3$, we obtain a graph $G[W]^+$ from $G[W]$ by appropriately adding two new edges (possibly parallel) joining vertices in W so that $\delta(G[W]^+) \ge 5$, and so by the minimality of $|W|$, we have $\kappa'(G[W]^+) \ge 5$. By the validity of (ii), we conclude that $G[W]$ is $\mathbb{Z}_3$-connected. Since $\delta(G) \ge 5$, $G[W]$ is a nontrivial subgraph of G. This contradicts the assumption that G is a $\langle \mathbb{Z}_3 \rangle$-reduced graph.

Hence we assume that $|[W, W^c]| = 4$. Let $H = G/W^c$ and z be the vertex onto which $G[W^c]$ is contracted, and denote $E_H(z) = \{e_1, e_2, e_3, e_4\}$ with $e_i = zv_i$, $1 \le i \le 4$. Since $E_H(z)$ may contain parallel edges, the v_i's do not have to be distinct. By the minimality of W and Menger's theorem, we have $\lambda_H(x, y) \ge 5$ for any two vertices $x, y \in V(H) - \{z\}$.

Suppose first that $H[E_H(z)]$ contains parallel edges. Assume that z and v_1 are joined by at least two edges. Define $H'' = H/H[\{z, v_1\}]$. By the minimality of W, we have $\kappa'(H'') \ge 5$. As $|E_H(z) - E(H[\{z, v_1\}])| \le 2$, it follows by (ii) that $G[W] = H'' - (E_H(z) - E(H[\{z, v_1\}]))$ is $\mathbb{Z}_3$-connected, contrary to the assumption that G is $\langle \mathbb{Z}_3 \rangle$-reduced.

Hence we assume that $H[E_H(z)]$ contains no parallel edges, and so the v_i's are four distinct vertices. By Theorem 3.9, we may assume that the graph $H' = H - v_1z - v_2z + v_1v_2$ satisfies $\lambda_{H'}(x, y) = \lambda_H(x, y) \ge 5$ for any two vertices $x, y \in V(H') - \{z\}$. This implies that the graph $H'' = H'/\{zv_3\}$

is 5-edge-connected. By (ii), $G[W] \cong H'' - \{v_1v_2, e_4\} \in \langle \mathbb{Z}_3 \rangle$, contrary to the assumption that G is $\langle \mathbb{Z}_3 \rangle$-reduced. ■

Proposition 3.6 indicates certain implications of Conjecture 1.6. The proof of Proposition 3.6 is similar to that of Proposition 3.3 and is omitted.

4 | PROOFS OF THEOREMS 1.7, 1.8, AND 1.9

Theorems 1.7, 1.8, and 1.9 will be proved in this section. We start with two lemmas.

Lemma 4.1. *(Lemma 3.1(i) in [13]) Let G be a graph, v be a vertex of G with degree at least four and $vv_1, vv_2 \in E_G(v)$. If $G' = G - vv_1 - vv_2 + v_1v_2$ is $\mathbb{Z}_3$-connected, then G is $\mathbb{Z}_3$-connected.*

Lemma 4.2. *Let G be a graph, v be a vertex of G with degree at least four and $vv_1, vv_2 \in E_G(v)$. If $G_1 = G - v + v_1v_2$ is $\mathbb{Z}_3$-connected, then G is $\mathbb{Z}_3$-connected.*

Proof. Let $G_2 = G - vv_1 - vv_2 + v_1v_2$. As $|[\{v\}, V(G) - \{v\}]_{G_2}| = d_G(v) - 2 \geq 2$, we have $G_2/G_1 \in \langle \mathbb{Z}_3 \rangle$. Since $G_1 \in \langle \mathbb{Z}_3 \rangle$ and $G_2/G_1 \in \langle \mathbb{Z}_3 \rangle$, it follows by Lemma 2.3 that $G_2 \in \langle \mathbb{Z}_3 \rangle$. By Lemma 4.1, $G_2 \in \langle \mathbb{Z}_3 \rangle$ implies that $G \in \langle \mathbb{Z}_3 \rangle$. ■

For an integer $k > 0$, it is known (see [22], or more explicitly, Lemma 3.1 of [14] or Lemma 3.4 of [18]) that if $F(H, k) > 0$ for any nontrivial proper subgraph H of G, then

$$F(G, k) = k(|V(G)| - 1) - |E(G)|. \tag{5}$$

Proof of Theorem 1.9. Assume that Theorem 1.9 (i) holds and that G is a graph with $F(G, 4) = 0$. If $v \in V(G)$ with $d_G(v) \leq 7$ satisfies $\kappa'(G - v) \geq 2$, then $F(G - v, 4) \leq 3$ and so by Theorem 1.9 (i), $G - v$ is $\mathbb{Z}_3$-connected. It follows from Proposition 1.4 that G is $\langle \mathbb{Z}_3 \rangle$-extendable at vertex v. Thus if (i) holds, then (ii) would follow as well. Hence it suffices to show that

$$\text{if } F(G, 4) \leq 3 \text{ and } \kappa'(G) \geq 2, \text{ then } G \in \langle \mathbb{Z}_3 \rangle. \tag{6}$$

We argue by contradiction and assume that

$$G \text{ is a counterexample to (6) with } |V(G)| + |E(G)| \text{ minimized.} \tag{7}$$

As (i) holds if $|V(G)| \leq 2$, we assume that $|V(G)| \geq 3$. By assumption, there exists a set E_1 of edges not in G with $|E_1| = F(G, 4)$ such that $G^+ = G + E_1$ contains four edge-disjoint spanning trees, denoted T_1, T_2, T_3, T_4.

Claim 1. Each of the following holds.

(i) For any nontrivial proper subgraph H of G, $H \notin \langle \mathbb{Z}_3 \rangle$ and $F(H, 4) \geq 3$.

(ii) G is 4-edge-connected.

Proof of Claim 1.

(i) Let H be a nontrivial proper subgraph of G. As $F(G/H, 4) \leq 3$ (see, for example, Lemma 2.1 of [18]), if $H \in \langle \mathbb{Z}_3 \rangle$, then by (7) and $\kappa'(G/H) \geq \kappa'(G) \geq 2$, we have $G/H \in \langle \mathbb{Z}_3 \rangle$, and so by Lemma 2.3, $G \in \langle \mathbb{Z}_3 \rangle$, contrary to (7). Hence we must have $H \notin \langle \mathbb{Z}_3 \rangle$. If $F(H, 4) \leq 2$, then by $\kappa'(H) \geq 2$ and (7), we have $H \in \langle \mathbb{Z}_3 \rangle$, contrary to the fact that $H \notin \langle \mathbb{Z}_3 \rangle$. This proves Claim 1(i).

(ii) To prove Claim 1(ii), assume that G has a minimum edge-cut W with $|W| \leq 3$. Let H_1, H_2 be the two components of $G - W$. By (i) and by (5), we have

$$F(H_1, 4) + F(H_2, 4) = \sum_{i=1}^{2} [4(|V(H_i)| - 1) - E(H_i)|] = F(G, 4) - 4 + |W| \leq |W| - 1 \leq 2.$$

This, together with the fact that W is a minimum edge-cut, implies that $\kappa'(H_i) \geq 2$ for each $i \in \{1, 2\}$. Since $|V(G)| \geq 3$, at least one of H_1 and H_2 is nontrivial, contrary to Claim 1(i). Thus Claim 1(ii) must hold. ■

Claim 2. $E(G^+) = \cup_{i=1}^{4} E(T_i)$.

Proof of Claim 2. Suppose that there exists $e \in E(G^+) - \cup_{i=1}^{4} E(T_i)$. The minimality of E_1 indicates that $E_1 \subseteq \cup_{i=1}^{4} E(T_i)$, and thus $e \in E(G)$. Let $G' = G - e$. Then G' is a spanning subgraph of G with $F(G', 4) = F(G, 4) \leq 3$ and $\kappa'(G') \geq 3$ by Claim 1(ii). As $G' \in \langle \mathbb{Z}_3 \rangle$ implies $G \in \langle \mathbb{Z}_3 \rangle$, Claim 2 follows from (7). ■

Claim 3. Each of the following holds.

(i) G^+ has no subgraph H^+ with $1 < |V(H^+)| < |V(G^+)|$ such that $F(H^+, 4) = 0$.

(ii) $\kappa'(G^+) \geq 5$ and G^+ does not have an essentially 5-edge-cut.

(iii) G^+ has no vertex of degree 5.

Proof of Claim 3.

(i) Argue by contradiction to show Claim 3(i) and choose a subgraph H^+ of G^+ with $1 < |V(H^+)| < |V(G^+)|$ and $F(H^+, 4) = 0$ such that $|V(H^+)|$ minimized. By Claim 2, if $X = V(H^+)$, then $H^+ = G^+[X]$. If $|X| = 2$, then by Claim 1(i), Claim 2 and $F(H^+, 4) = 0$, we conclude that $E(G[X])$ consists of a cut edge of G, contrary to Claim 1(ii). Hence we assume that $|X| \geq 3$. Let $H = H^+ - E_1$. Then $H = G[X]$. Since $F(H^+, 4) = 0$ and by Claim 2, $F(H, 4) \leq |E_1| = F(G, 4) \leq 3$. If H has a cut edge e, then by (5) and as $|V(H)| \geq 3$, one component of $H - e$ must be nontrivial and has 4 edge-disjoint spanning trees, contrary to the minimality of $|V(H^+)|$. Hence $\kappa'(H) \geq 2$, and so by (7), $H \in \langle \mathbb{Z}_3 \rangle$, contrary to Claim 1(i). This proves Claim 3(i).

(ii) If W is a minimal 4-edge-cut or an essential 5-edge-cut of G^+ with G_1^+ and G_2^+ being the two components of $G^+ - W$, then by (5), there exists a nontrivial $H^+ \in \{G_1^+, G_2^+\}$ with $F(H^+, 4) = 0$, contrary to Claim 3(i). This proves Claim 3(ii).

(iii) We argue by contradiction to show Claim 3(iii). Let v_0 be a vertex with $d_{G^+}(v_0) = 5$, $E_{G^+}(v_0) = \{e_1, e_2, e_3, e_4, e_5\}$, and v_i, $1 \leq i \leq 5$, be vertices with $e_i = v_0 v_i$. As $E_{G^+}(v_0)$ may contain parallel edges, the v_i's are not necessarily distinct. Since $F(G^+, 4) = 0$, we may assume that for $1 \leq i \leq 4$, $e_i \in E(T_i)$, and $e_5 \in E(T_1)$. By Claim 1(ii), $|E_1 \cap E_{G^+}(v_0)| \leq 1$, and so we may assume that $e_1 \in E(G)$. By symmetry among e_2, e_3, e_4 and by Claim 1(i)(ii), e_1 has at most one parallel edge, and thus we may assume $e_2 \in E(G)$ and $v_2 \neq v_1$. Let e_5'' be an edge linking v_1 and v_5 but not in $E(G)$. Define $G'' = G - v_0 + v_1 v_5$ if $E_1 \cap E_{G^+}(v_0) = \emptyset$, and $G'' = G - v_0 + v_1 v_2$ otherwise. Let

$$E_1'' = \begin{cases} E_1 & \text{if } E_1 \cap E_{G^+}(v_0) = \emptyset; \\ E_1 - E_{G^+}(v_0) & \text{if } |E_1 \cap E_{G^+}(v_0)| = 1 \text{ and } e_5 \notin E_1; \\ (E_1 - E_{G^+}(v_0)) \cup \{e_5''\} & \text{if } E_1 \cap E_{G^+}(v_0) = \{e_5\}. \end{cases}$$

As for $i \in \{2,3,4\}$, $T_i - v_0$ is a spanning tree of $G'' + E_1''$, and $(T_1 - v_0) + e_5''$ is a spanning tree of $G'' + E_1''$. It follows by $|E_1''| \le |E_1| = 3$ that $F(G'', 4) \le 3$. Note that $|V(G'')| + |E(G'')| < |V(G)| + |E(G)|$. If G'' has a cut edge, then as $d_G(v_0) \le d_{G^+}(v_0) = 5$, G has an edge-cut W' with $|W'| \le 3$, contrary to Claim 1(ii). Thus $\kappa'(G'') \ge 2$. By (7), $G'' \in \langle \mathbb{Z}_3 \rangle$. Hence $G \in \langle \mathbb{Z}_3 \rangle$ by Lemma 4.2, contrary to (7). This proves Claim 3. ■

By Claim 3, $\kappa'(G^+) \ge 6$, and so by Lemma 2.2(ii) and $F(G,4) \le 3$, we have $G = G^+ - E_1 \in \langle \mathbb{Z}_3 \rangle$, contrary to (7). The proof is completed. ■

Theorem 1.7 is an immediate corollary of Theorem 1.9, and we will prove Theorem 1.8 by a simple discharging argument.

The next lemma follows from arguments of Nash-Williams in [22]. A detailed proof can be found in Theorem 2.4 of [30].

Lemma 4.3. *Let G be a nontrivial graph and let $k > 0$ be an integer. If $|E(G)| \ge k(|V(G)| - 1)$, then G has a nontrivial subgraph H with $F(H,k) = 0$.*

Proof of Theorem 1.8. It suffices to show (b). We shall show that every 5-edge-connected essentially 23-edge-connected graph contains 4 edge-disjoint spanning trees. Then Theorem 1.8(b) follows from Theorem 1.9(ii).

Let G be a counterexample with $|E(G)|$ minimized. Then $F(G,4) > 0$ and $|V(G)| \ge 4$. If $|E(G)| \ge 4(|V(G)| - 1)$, by Lemma 4.3, there exists a nontrivial subgraph H with $F(H,4) = 0$. By definition of contraction, G/H is 5-edge-connected and essentially 23-edge-connected. By the minimality of G, G/H has 4 edge-disjoint spanning trees. As H has 4 edge-disjoint spanning trees, it follows that (see Lemma 2.1 of [18]) $F(G,4) = 0$, contrary to the choice of G. Hence we have

$$|E(G)| < 4(|V(G)| - 1). \tag{8}$$

Since $|V(G)| \ge 4$ and G is essentially 23-edge-connected, for any edge $uv \in E(G)$, we have

$$d(u) + d(v) \ge 23 + 2. \tag{9}$$

For integers $i, k \ge 1$, define $D_i(G) = \{v \in V(G) : d_G(v) = i\}$, $D_{\le k}(G) = \cup_{i \le k} D_i(G)$, and $D_{\ge k}(G) = \cup_{i \ge k} D_i(G)$. It follows from (9) that $D_{\le 8}$ is an independent set.

Each vertex begins with charge equal to its degree. If $d(v) \ge 9$ and $vu \in E(G)$, then v gives charge $\frac{d(v)-8}{d(v)}$ to u. Note that G may contain parallel edges and the charge runs through each edge adjacent to v. Clearly, if $v \in D_{\ge 8}$, then v will be left with charge $d(v)(1 - \frac{d(v)-8}{d(v)}) = 8$.

For any vertex $x \in D_{\le 7}$, denote $d(x) = i \in \{5,6,7\}$. By (9), x will end with charge at least

$$i + \sum_{vx \in E(G)} \frac{d(v)-8}{d(v)} \ge i + \frac{25-i-8}{25-i} i = \frac{(42-2i)i}{25-i} \ge \min\left\{8, \frac{180}{19}, \frac{98}{9}\right\} = 8,$$

a contradiction to (8). ■

We remark that there exist 5-edge-connected and essentially 22-edge-connected graphs that do not contain 4 edge-disjoint spanning trees. Lowing the constant 23 may require new ideas and more elaborate work. As shown in Propositions 3.3 and 3.6, lowing into six would imply Conjectures 1.1 and 1.2.

5 | TWO APPLICATIONS

Recall that a $\langle \mathbb{Z}_3 \rangle$-reduced graph is a graph without nontrivial $\mathbb{Z}_3$-connected subgraphs. The number of edges in a $\langle \mathbb{Z}_3 \rangle$-reduced graph is often useful in reduction method and some inductive arguments. Theorem 1.9, together with Lemma 4.3, establishes an upper bound for the density of a $\langle \mathbb{Z}_3 \rangle$-reduced graph.

Corollary 5.1. *Every $\langle \mathbb{Z}_3 \rangle$-reduced graph on $n \geq 3$ vertices has at most $4n - 8$ edges.*

As defined in [15], a graph G is **strongly $\mathbb{Z}_{2s+1}$-connected** if, for every $b : V(G) \to \mathbb{Z}_{2s+1}$ with $\sum_{v \in V(G)} b(v) = 0$, there is an orientation D such that for every vertex $v \in V(G)$, $d_D^+(G) - d_D^+(G) \equiv b(v) \pmod{2s+1}$. Strongly $\mathbb{Z}_{2s+1}$-connected graphs are known as contractible configurations for mod $(2s+1)$-orientations. The following has recently been obtained.

Proposition 5.2. *([16]) Every strongly $\mathbb{Z}_{2s+1}$-connected graph contains $2s$ edge-disjoint spanning trees.*

By the monotonicity of circular flow (see, for example, [7] or [31]), it follows that every graph with a mod 5-orientation also has a mod 3-orientation. It is not known, in general, whether a strongly $\mathbb{Z}_{2k+3}$-connected graph is also strongly $\mathbb{Z}_{2k+1}$-connected. As an application of Proposition 5.2, if a graph G is strongly $\mathbb{Z}_5$-connected graph, then $F(G, 4) = 0$; it then follows from Theorem 1.7 that $G \in \langle \mathbb{Z}_3 \rangle$. Hence we obtain the following corollary.

Corollary 5.3. *Every strongly $\mathbb{Z}_5$-connected graph is $\mathbb{Z}_3$-connected.*

ORCID

Jiaao Li http://orcid.org/0000-0002-4771-2094

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