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ANTI-RAMSEY NUMBERS OF PATHS AND CYCLES IN HYPERGRAPHS*

RAN GU†, JIAAO LI‡, AND YONGTANG SHI§

Abstract. The anti-Ramsey problem was introduced by Erdős, Simonovits, and Sós in 1970s. The anti-Ramsey number of a hypergraph $\mathcal{H}$, $ar(n, s, \mathcal{H})$, is the smallest integer c such that in any coloring of the edges of the s-uniform complete hypergraph on n vertices with exactly c colors, there is a copy of $\mathcal{H}$ whose edges have distinct colors. In this paper, we determine the anti-Ramsey numbers of linear paths and loose paths in hypergraphs for sufficiently large n and give bounds for the anti-Ramsey numbers of Berge paths. Similar exact anti-Ramsey numbers are obtained for linear/loose cycles, and bounds are obtained for Berge cycles. Our main tools are the path extension technique and stability results on hypergraph Turán problems of paths and cycles.

Key words. anti-Ramsey, rainbow, hypergraphs, paths, cycles, Turán number

AMS subject classifications. 05D05, 05C35, 05C65

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1. Introduction.

The *anti-Ramsey number* of a graph G, denoted by $ar(n, G)$, is the minimum number of colors needed to color the edges of the complete graph K_n so that, in any coloring, there exists a copy of G whose edges have distinct colors. The *Turán number* of a graph G, denoted by $ex(n, G)$, is the maximum number of edges in a graph on n vertices that does not contain G as a subgraph. It is easy to observe that

$$2 + ex(n, \{H - e, e \in E(H)\}) \leq ar(n, H) \leq ex(n, H) + 1 \tag{1.1}$$

for any graph H.

In 1973, Erdős, Simonovits, and Sós [11] showed a remarkable result that $ar(n, K_p) = ex(n, K_{p-1}) + 2$ for sufficiently large n. Montellano-Ballesteros and Neumann-Lara [35] extended this result to all values of n and p with $n > p \geq 3$. In [11], it was shown that $ar(n, H) - ex(n, \{H - e, e \in E(H)\}) = o(n^2)$ when $n \to \infty$. Furthermore, Jiang [29] proved that if H is a graph such that each edge is incident to a vertex of degree

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†College of Science, Hohai University, Nanjing, Jiangsu Province 210098, China (rangu@hhu.edu.cn).
‡Corresponding author. School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, China (lijiaao@nankai.edu.cn).
§Center for Combinatorics and LPMC, Nankai University, Tianjin 300071, China (shi@nankai.edu.cn).

two, then $ar(n,H) - ex(n,\{H-e, e \in E(H)\}) = O(n)$. A history of results and open problems on this topic was given by Fujita, Magnant, and Ozeki [18].

A *hypergraph* $\mathcal{H}$ consists of a set $V(\mathcal{H})$ of vertices and a family $E(\mathcal{H})$ of nonempty subsets of $V(\mathcal{H})$ called edges of $\mathcal{H}$. If each edge of $\mathcal{H}$ has exactly s vertices, then $\mathcal{H}$ is *s-uniform* and $\mathcal{H}$ is called an *s-graph*. A *complete s-uniform hypergraph* is a hypergraph whose edge set consists of all s-subsets of the vertex set. In an edge-coloring of a (hyper)graph $\mathcal{H}$, a sub(hyper)graph $\mathcal{F} \subseteq \mathcal{H}$ is *rainbow* if all edges of $\mathcal{F}$ have distinct colors.

The anti-Ramsey number and Turán number are naturally extended from graphs to hypergraphs. The anti-Ramsey number of an s-uniform hypergraph $\mathcal{H}$, denoted by $ar(n,s,\mathcal{H})$, is the minimum number of colors needed to color the edges of a complete s-uniform hypergraph on n vertices so that there exists a rainbow $\mathcal{H}$ in any coloring. The Turán number of $\mathcal{H}$, denoted by $ex(n,s,\mathcal{H})$, is the maximum number of edges in an s-uniform hypergraph on n vertices that contains no $\mathcal{H}$. Özkahya and Young [39] investigated the anti-Ramsey number of matchings in hypergraphs, where a *matching* is a set of edges in a (hyper)graph in which no two edges have a common vertex. A *k-matching*, denoted by M_k, is a matching with k edges. Özkahya and Young [39] gave the lower and upper bounds for $ar(n,s,M_k)$ in terms of $ex(n,s,M_{k-1})$. They proved that

$$ex(n,s,M_{k-1}) + 2 \le ar(n,s,M_k) \le ex(n,s,M_{k-1}) + k,$$

where the lower bound holds for every n and the upper bound holds for $n \ge sk + (s-1)(k-1)$. For $s = 2$, Schiermeyer [40] proved that $ar(n,2,M_k) = ex(n,2,M_{k-1}) + 2$ for $k \ge 2$ and $n \ge 3k+3$, and this condition was further released to all $n \ge 2k+1$ by Chen, Li, and Tu [5] and by Fujita et al. [17], independently.

In fact, for k-matchings, the Turán number $ex(n,s,M_k)$ is still not known for $k \ge 3$ and $s \ge 3$. Let $[n]$ denote the set $\{1,2,\dots,n\}$ and $\binom{[n]}{k}$ denote the set consisting of all the k-sets of $[n]$. Erdős put forward a conjecture in 1965 that $ex(n,s,M_k) = \max\{|A_s|, |B_s(n)|\}$, where $A_s = \binom{[sk-1]}{s}$ and $B_s(n) = \{F \in \binom{[n]}{s} | F \cap [k-1] \ne \emptyset\}$. This conjecture is true for $s = 2$, which was shown by Erdős and Gallai [10]. In [9], Erdős proved that there exists a constant $n_0(s,k)$ such that for $n > n_0(s,k)$, the conjecture holds. Then Bollobás, Daykin, and Erdős [1] improved the bound for $n_0(k,s)$ such that $n_0(k,s) \le 2s^3(k-1)$. It was improved to $n_0(k,s) \le 3s^2(k-1)$ by Huang, Loh, and Sudakov [28] later.

For the anti-Ramsey number of k-matching, Özkahya and Young [39] conjectured that when $k \ge 3$, $ar(n,s,M_k) = ex(n,s,M_{k-1}) + 2$ if $n > sk$ and

$$ar(n,s,M_k) = \begin{cases} ex(n,s,M_{k-1}) + 2 & \text{if } k \le c_s, \\ ex(n,s,M_{k-1}) + s + 1 & \text{if } k \ge c_s, \end{cases}$$

if $n = sk$, where c_s is a constant dependent on s. They proved that the conjecture is true when $k = 2,3$ for sufficiently large n. Later, Frankl and Kupavskii [16] proved that $ar(n,s,M_k) = ex(n,s,M_{k-1}) + 2$ for $n \ge sk + (s-1)(k-1)$ and $k \ge 3$. For more results on matchings, we refer the reader to [13, 15].

For paths, Simonovits and Sós [41] proved that $ar(n, P_{2t+3+\epsilon}) = tn - \binom{t-1}{2} + 1 + \epsilon$ for large n, where $\epsilon = 0,1$ and P_k is a path on k vertices. Comparing with the Turán number of paths

$$ex(n, P_k) \le (k-2)n/2 \tag{1.2}$$

given by Erdős and Gallai [10], it follows that $ar(n, P_k) = ex(n, P_{k-1}) + O(1)$ when k is odd, and $ar(n, P_k) = ex(n, P_{k-2}) + O(1)$ when k is even. For a cycle C_k of order k, Erdős, Simonovits, and Sós [11] conjectured that $ar(n, C_k) = n(\frac{k-2}{2} + \frac{1}{k-1}) + O(1)$. This conjecture was confirmed by Montellano-Ballesteros and Neumann-Lara [36], and they gave the exact value of $ar(n, C_k)$ for all $n \geq k \geq 3$. It would be interesting to investigate the relation between the anti-Ramsey number and the Turán number for paths and cycles in hypergraphs. The Turán numbers of paths and cycles are extensively studied; see [12, 19, 20, 33] or section 2 below for details. Motivated by this, we will study the anti-Ramsey numbers of paths and cycles and compare it with the Turán numbers of paths and cycles in hypergraphs.

There are several possible ways to define paths and cycles in hypergraphs as generalization of paths and cycles in graphs from different aspects.

Definition 1.1. *Let $\mathcal{H}$ be an s-uniform hypergraph.*

(i) *A* Berge path *of length k in $\mathcal{H}$ is a family of k distinct edges $e_1, \dots, e_k$ and $k+1$ distinct vertices $v_1, \dots, v_{k+1}$ such that for each $1 \leq i \leq k$, e_i contains v_i and v_{i+1}. Let $\mathcal{B}_k$ denote the family of Berge paths of length k. A* Berge cycle *of length k in $\mathcal{H}$ is a cyclic list of k distinct edges $e_1, \dots, e_k$ and k distinct vertices $v_1, \dots, v_k$ such that e_i contains v_i and v_{i+1} for each $1 \leq i \leq k$, where $v_{k+1} = v_1$. Denote the family of all Berge cycles of length k by $\mathcal{BC}_k$.*

(ii) *A* loose path *of length k in $\mathcal{H}$ is a collection of distinct edges $\{e_1, e_2, \dots, e_k\}$ such that consecutive edges intersect in at least one element and nonconsecutive edges are disjoint. Denote the family of loose paths of length k by $\mathcal{P}_k$. A* loose cycle *is defined similarly in a cyclic order, and denote the family of all loose cycles of length k by $\mathcal{C}_k$.*

(iii) *A* linear path *of length k in $\mathcal{H}$ is a collection of distinct edges $\{e_1, e_2, \dots, e_k\}$ such that consecutive edges intersect in exactly one element and nonconsecutive edges are disjoint. Let $\mathbb{P}_k$ denote the linear path of length k. A* linear cycle *is defined similarly in a cyclic order, and let $\mathbb{C}_k$ denote the collection of linear path of length k.*

We first give the exact anti-Ramsey numbers of short paths $\mathbb{P}_i$, $\mathcal{B}_i$, $\mathcal{P}_i$ for $i = 2, 3$.

Theorem 1.1. (i) *For $s \geq 3$ and $n \geq 3s-4$, $ar(n, s, \mathbb{P}_2) = 2$.*
(ii) *For $s \geq 4$ and sufficiently large n, $ar(n, s, \mathbb{P}_3) = \binom{n-2}{s-2} + 2$.*
(iii) *For $n \geq 3s-4$, $ar(n, s, \mathcal{B}_2) = ar(n, s, \mathcal{P}_2) = 2$.*
(iv) *For $n \geq 4s-3$, $ar(n, s, \mathcal{B}_3) = ar(n, s, \mathcal{P}_3) = 3$.*

For linear paths and loose paths, we obtain the exact anti-Ramsey numbers for sufficiently large n.

Theorem 1.2. *For any integer k, if $k = 2t \geq 4$ and $s \geq 3$, then for sufficiently large n,*

$$ar(n, s, \mathbb{P}_k) = \binom{n}{s} - \binom{n-t+1}{s} + 2;$$

if $k = 2t+1 > 5$ and $s \geq 4$, then for sufficiently large n,

$$ar(n, s, \mathbb{P}_k) = \binom{n}{s} - \binom{n-t+1}{s} + \binom{n-t-1}{s-2} + 2.$$

Theorem 1.3. *For any integer k, if $k = 2t \geq 4$ and $s \geq 3$, then for sufficiently large n,*

$$ar(n, s, \mathcal{P}_k) = \binom{n}{s} - \binom{n-t+1}{s} + 2;$$

if $k = 2t + 1 \geq 5$ *and* $s \geq 3$*, then for sufficiently large* n*,*

$$ar(n, s, \mathcal{P}_k) = \binom{n}{s} - \binom{n-t+1}{s} + 3.$$

We remark that, due to some technique obstruction, our proof of Theorem 1.2 does not work directly for the case $k = 5$ or the case k is odd and $s = 3$. However, those special cases are handled in Theorem 1.3 for loose path with a refined analysis.

The methods developed in proving Theorems 1.2 and 1.3, with additional effort and some new ideas, allow us to determine the anti-Ramsey numbers of linear cycles and loose cycles as well if k and s are not too small. We obtain the following exact results for linear cycles and loose cycles.

THEOREM 1.4. *For any integer* k*, if* $k = 2t \geq 8$ *and* $s \geq 4$*, then for sufficiently large* n*,*

$$ar(n, s, \mathbb{C}_k) = ar(n, s, \mathbb{P}_k) = \binom{n}{s} - \binom{n-t+1}{s} + 2;$$

if $k = 2t + 1 \geq 11$ *and* $s \geq k + 3$*, then for sufficiently large* n*,*

$$ar(n, s, \mathbb{C}_k) = ar(n, s, \mathbb{P}_k) = \binom{n}{s} - \binom{n-t+1}{s} + \binom{n-t-1}{s-2} + 2.$$

THEOREM 1.5. *For any integer* k*, if* $k = 2t \geq 8$ *and* $s \geq 4$*, then for sufficiently large* n*,*

$$ar(n, s, \mathcal{C}_k) = ar(n, s, \mathcal{P}_k) = \binom{n}{s} - \binom{n-t+1}{s} + 2;$$

if $k = 2t + 1 \geq 11$ *and* $s \geq k + 3$*, then for sufficiently large* n*,*

$$ar(n, s, \mathcal{C}_k) = ar(n, s, \mathcal{P}_k) = \binom{n}{s} - \binom{n-t+1}{s} + 3.$$

For a Berge path $\mathcal{B}_k$, Györi, Katona, and Lemons in [25] proved that $ex(n, s, \mathcal{B}_k) \leq \frac{n}{k}\binom{k}{s}$ when $k > s + 1 > 3$ and that $ex(n, s, \mathcal{B}_k) \leq \frac{n(k-1)}{s+1}$ when $2 < k \leq s$, which are sharp for infinitely many n. Then Davoodi et al. [8] proved that $ex(n, s, \mathcal{B}_{s+1}) \leq n$. We apply those results to obtain the bounds for the anti-Ramsey number $ar(n, s, \mathcal{B}_k)$ as follows.

THEOREM 1.6. *If* $k > 2s + 1$*, then for sufficiently large* n*,*

$$\frac{2n}{k}\binom{\lfloor k/2 \rfloor}{s} \leq ar(n, s, \mathcal{B}_k) \leq \frac{n}{k-1}\binom{k-1}{s} + 1.$$

If $s + 2 \leq k \leq 2s + 1$*, then for sufficiently large* n*,*

$$\frac{n}{s+1}\left\lfloor \frac{k-2}{2} \right\rfloor \leq ar(n, s, \mathcal{B}_k) \leq \frac{n}{k-1}\binom{k-1}{s} + 1.$$

If $k \leq s + 1$*, then for sufficiently large* n*,*

$$\frac{n}{s+1}\left\lfloor \frac{k-2}{2} \right\rfloor \leq ar(n, s, \mathcal{B}_k) \leq \frac{(k-2)n}{s+1} + 1.$$

Theorem 1.6 indicates that the anti-Ramsey number $ar(n, s, \mathcal{B}_k)$ varies for different s and k. This may suggest that determining the exact value of $ar(n, s, \mathcal{B}_k)$ would be very difficult. Note that the Turán number of Berge path is still not clear at this moment.

However, it seems that the anti-Ramsey numbers of Berge cycles have different behavior with Berge paths, and we obtain the following bounds, similar to the Özkahya and Young result [39] on matchings.

PROPOSITION 1.7. *For any fixed integers $s \geq 4$, $k \geq 3$,*

$$ex(n, s, \mathcal{B}_{k-1}) + 2 \leq ar(n, s, \mathcal{BC}_k) \leq ex(n, s, \mathcal{B}_{k-1}) + k.$$

The next section will be focused on introducing results on Turán numbers of paths and cycles in hypergraphs, which are needed tools to derive our main results. A useful lemma obtained from the stability results on Turán problems will be given in the next section as well, which will be frequently used to find certain desired paths in later proofs. The proof of the main results will be presented in later sections.

2. Preliminaries. Note that the s-uniform Berge path $\mathcal{B}_2$ and loose path $\mathcal{P}_2$ are the same definitions, and the determination of $ex(n, s, \mathcal{P}_2)$ is trivial. In [20], Füredi, Jiang, and Seiver determined $ex(n, s, \mathcal{P}_k)$ for $s \geq 3$.

THEOREM 2.1 (see [20]). *Let s, t be positive integers with $s \geq 3$. For sufficiently large n, we have*

$$ex\left(n, s, \mathcal{P}_{2t+1}\right) = \binom{n}{s} - \binom{n-t}{s}$$

and

$$ex\left(n, s, \mathcal{P}_{2t+2}\right) = \binom{n}{s} - \binom{n-t}{s} + 1.$$

For $\mathcal{P}_{2t+1}$, the unique extremal family consists of all the s-subsets of $[n]$ which meet some fixed set S of size t. For $\mathcal{P}_{2t+2}$, the unique extremal family consists of all the above edges plus one additional s-set disjoint from S.

The determination of $ex(n, s, \mathbb{P}_k)$ is nontrivial even for $k = 2$. Frankl [12] gave the value of $ex(n, s, \mathbb{P}_2)$ for $s \geq 4$ and sufficiently large n. Then Keevash, Mubayi, and Wilson [32] determined the value of $ex(n, 4, \mathbb{P}_2)$ for all n. Note that when $s = 3$, $ex(n, 3, \mathbb{P}_2) \leq n$, which can be achieved when n is divisible by 4 by taking $n/4$ vertex disjoint copies of $K_4^{(3)}$ (i.e., the complete 3-graph on 4 vertices). For $k \geq 3$, Füredi, Jiang, and Seiver [20] provided the exact Turán number of $\mathbb{P}_k$ for sufficiently large n, where $s \geq 4$, $k \geq 3$. Kostochka, Mubayi, and Verstraëte [33] considered $ex(n, s, \mathbb{P}_k)$ for $s \geq 3$, $k \geq 4$ and sufficiently large n. Later, Jackowska, Polcyn, and Ruciński [31] determined $ex(n, 3, \mathbb{P}_3)$ for all n. We summarize those results (only for the sufficiently large n) as follows.

THEOREM 2.2 (see [12, 20, 31, 33, 32]). *For sufficiently large n, we have*

1. $ex\left(n, s, \mathbb{P}_2\right) = \binom{n-2}{s-2}$ *for $s \geq 4$, and* $ex\left(n, 3, \mathbb{P}_2\right) \leq n$*;*
2. $ex\left(n, s, \mathbb{P}_{2t+1}\right) = \binom{n}{s} - \binom{n-t}{s}$ *for $s \geq 3$ and $t \geq 1$;*
3. $ex\left(n, s, \mathbb{P}_{2t+2}\right) = \binom{n}{s} - \binom{n-t}{s} + \binom{n-t-2}{s-2}$ *for $s \geq 3$ and $t \geq 1$.*

The unique extremal family for $\mathbb{P}_2$ consists of all the s-subsets of $[n]$ containing some two fixed vertices for $s \geq 4$. For $\mathbb{P}_{2t+1}$, the unique extremal family consists of all the s-subsets of $[n]$ which meet some fixed set S of size t. For $\mathbb{P}_{2t+2}$, the unique extremal family consists of all the above edges plus all the s-sets in $[n] \backslash S$ containing some two fixed vertices not in S.

For linear cycles, Frankl and Füredi [14] showed that the unique extremal n-vertex s-graph ($s \geq 3$) containing no $\mathbb{C}_3$ consists of all edges containing some fixed vertex x for large enough n. For $s = 3$, Csákány and Kahn [7] obtained the same result for all $n \geq 6$. Füredi and Jiang [19] and Kostochka, Mubayi, and Verstraëte [33] determined the Turán number of $\mathbb{C}_k$ for all $k \geq 3$, $s \geq 3$ and sufficiently large n as follows.

THEOREM 2.3 (see [19, 33]). *Let s, t be positive integers with $s \geq 3$. For sufficiently large n, we have*

$$ex\,(n, s, \mathbb{C}_{2t+1}) = \binom{n}{s} - \binom{n-t}{s},$$

and for $(s, t) \neq (3, 1)$, we have

$$ex\,(n, s, \mathbb{C}_{2t+2}) = \binom{n}{s} - \binom{n-t}{s} + \binom{n-t-2}{s-2}.$$

For $\mathbb{C}_{2t+1}$, the only extremal family consists of all the s-sets in $[n]$ that meet some fixed t-set L. For $\mathbb{C}_{2t+2}$, the only extremal family consists of all the s-sets in $[n]$ that intersect some fixed t-set L plus all the s-sets in $[n] \setminus L$ that contain some two fixed elements.

For the exceptional case of $\mathbb{C}_4$ in 3-uniform hypergraphs, Kostochka, Mubayi, and Verstraëte [33] showed that

$$ex\,(n, 3, \mathbb{C}_4) = \binom{n}{3} - \binom{n-1}{3} + \max\left\{n-3, 4\left\lfloor\frac{n-1}{4}\right\rfloor\right\},$$

and they also characterized the extremal graphs.

The Turán number of a loose cycle was initially studied by Chvátal [6]. Then Mubayi and Verstraëte [38] proved that $ex\,(n, s, \mathcal{C}_3) = \binom{n}{s} - \binom{n-1}{s-1}$ for all $s \geq 3$ and $n \geq 3s/2$. Füredi and Jiang [19] determined $ex\,(n, s, \mathcal{C}_k)$ for $k \geq 3$, $s \geq 4$ and sufficiently large n. This confirms (in a stronger form) a conjecture proposed by Mubayi and Verstraëte [37] for $k \geq 3$, $s \geq 4$. Kostochka, Mubayi, and Verstraëte [33] extended the results above and determined $ex\,(n, s, \mathcal{C}_k)$ for all $s \geq 3$ and large n. We summarize their results as follows.

THEOREM 2.4 (see [19, 33]). *Let $t \geq 2$, $s \geq 3$ be fixed integers. For sufficiently large n, we have*

$$ex\,(n, s, \mathcal{C}_{2t+1}) = \binom{n}{s} - \binom{n-t}{s},$$

$$ex\,(n, s, \mathcal{C}_{2t+2}) = \binom{n}{s} - \binom{n-t}{s} + 1,$$

and

$$ex\,(n, s, \mathcal{C}_4) = \binom{n}{s} - \binom{n-1}{s} + \left\lfloor\frac{n-1}{s}\right\rfloor.$$

For $\mathcal{C}_{2t+1}$ ($t \geq 2$), the only extremal family consists of all the s-sets in $[n]$ that meet some fixed t-set L. For $\mathcal{C}_{2t+2}$ ($t \geq 2$), the only extremal family consists of all the s-sets in $[n]$ that intersect some fixed t-set L plus one additional s-set outside L. For $\mathcal{C}_4$, the only extremal family consists of all the s-sets in $[n]$ that intersect some fixed t-set L plus $\lfloor\frac{n-1}{s}\rfloor$ disjoint edges outside L.

For the Turán number of a Berge cycle, Győri and Lemons proved that $ex(n,3,\mathcal{BC}_{2k+1}) \le O(k^4)\cdot n^{1+1/k}$ in [27] and $ex(n,3,\mathcal{BC}_{2k}) \le O(k^2)\cdot ex(n,C_{2k})$ in [26]. Füredi and Özkahya [21] obtained better constant factors (depending on k). Further improvements were obtained for even k by Gerbner, Methuku, and Vizer [24] and also by Gerbner, Methuku, and Palmer [23] and for odd k by Gerbner [22]. For $s \ge 4$, Győri and Lemons [26] showed that $ex(n,s,\mathcal{BC}_{2k+1}) \le O_s(k^{s-2}) \cdot ex(n,3,\mathcal{BC}_{2k+1})$ and $ex(n,s,\mathcal{BC}_{2k}) \le O_s(k^{s-1}) \cdot ex(n,C_{2k})$, i.e., $ex(n,s,\mathcal{BC}_k) = O(n^{1+1/\lfloor k/2 \rfloor})$. The constant factors were improved by Jiang and Ma [30] and for even k by Gerbner, Methuku, and Vizer [24].

There are many other results on the Turán numbers of paths and cycles in graphs [2, 3, 34, 42] or hypergraphs [4, 21, 25]. The reader is referred to these references for details.

The following stability result on linear paths and linear cycles will be needed in our proofs. Let $\partial\mathcal{H}$ denote the $(s-1)$-graph consisting of sets contained in some edge of $\mathcal{H}$.

THEOREM 2.5 (see [33]). *For fixed $s \ge 3$ and $k \ge 4$, let $\ell = \left\lfloor \frac{k-1}{2} \right\rfloor$ and $\mathcal{H}$ be an n-vertex s-graph with $|\mathcal{H}| \sim \ell\binom{n}{s-1}$ containing no $\mathbb{P}_k$ or containing no $\mathbb{C}_k$. Then there exists $G^* \subset \partial\mathcal{H}$ with $|G^*| \sim \binom{n}{s-1}$ and a set L of ℓ vertices of $\mathcal{H}$ such that $L \cap V(G^*) = \emptyset$ and $e \cup \{v\} \in \mathcal{H}$ for any $(s-1)$-edge $e \in G^*$ and any $v \in L$. In particular, $|\mathcal{H} - L| = o(n^{s-1})$.*

Notice that the stability result above considers the case $k \ge 4$. For $k = 3$, $s \ge 4$, Füredi, Jiang, and Seiver [20] provided another version of stability result on linear paths, and as the authors in [31] pointed out, the similar stability result holds for $k = 3$ and $s = 3$ as well. We rewrite their results for $k = 3$ with the similar notations in Theorem 2.5 (in a slightly weaker form). Note that when $k = 3$, $\ell = \left\lfloor \frac{k-1}{2} \right\rfloor = 1$.

THEOREM 2.6. *For fixed $s \ge 3$, let $\mathcal{H}$ be an n-vertex s-graph with $|\mathcal{H}| \sim \binom{n}{s-1}$ containing no $\mathbb{P}_3$. Then there exists $G^* \subset \partial\mathcal{H}$ with $|G^*| > \frac{1}{2}\binom{n}{s-1}$ and a vertex v of $\mathcal{H}$ such that $v \notin V(G^*)$ and $e \cup \{v\} \in \mathcal{H}$ for any $(s-1)$-edge $e \in G^*$. In particular, $|\mathcal{H} - v| = o(n^{s-1})$.*

Considering the structure of the $(s-1)$-graph G^*, we present the following lemma, which is frequently used in our proofs.

LEMMA 2.7. *For fixed $s \ge 3$ and $k \ge 3$, let $t = \left\lfloor \frac{k-1}{2} \right\rfloor$ and $\mathcal{H}$ be an n-vertex s-graph with $|\mathcal{H}| \sim t\binom{n}{s-1}$, which contains no $\mathbb{P}_k$ or contains no $\mathbb{C}_k$ when $k \ge 4$. Let $G^* \subset \partial\mathcal{H}$ be the $(s-1)$-graph as defined in Theorem* 2.5 *or* 2.6 *above. Let W be a set of d vertices in $\mathcal{H}$, where d is a fixed constant. Then for sufficiently large n, there are* $\max\{t-1,1\}$ *pairs of $(s-1)$-edges in G^*, say $\{a_i, b_i\}$, $i = 1, \dots, t-1$, such that each of the following holds:*

(i) *For every i, a_i and b_i have exactly one common vertex (i.e., $|a_i \cap b_i| = 1$);*
(ii) *for any $i \ne j$, $a_i \cup b_i$ and $a_j \cup b_j$ are vertex disjoint; and, moreover,*
(iii) *all these $(s-1)$-edges are disjoint from W.*

Proof. The number of $(s-1)$-edges incident with some vertices in W is at most $|W| \cdot \binom{n-1}{s-2}$, so in G^* the number of $(s-1)$-edges disjoint from W is at least

$$|G^*| - d\binom{n-1}{s-2} > \begin{cases} \binom{n}{s-3} & \text{for } s \ge 4, \\ n & \text{for } s = 3. \end{cases}$$

By Theorem 2.2 and (1.2), we get that $|G^*| - d\binom{n-1}{s-2} > ex(n, s-1, \mathbb{P}_2)$ for sufficiently large n. So we can find a pair $\{a_1, b_1\}$ of $(s-1)$-edges with exactly one common vertex. Since

$$|G^*| - d\binom{n-1}{s-2} - (t-1)(2s-3)\binom{n-1}{s-2} > ex(n, s-1, \mathbb{P}_2),$$

we can repeat the argument above to find $\{a_2, b_2\}, \ldots, \{a_{t-1}, b_{t-1}\}$ satisfying the properties described in Lemma 2.7. ∎

Given a path P, if a vertex v belongs to more than one edge in P, we call v a *cross vertex* of P or say v is a $cross(P)$ vertex. If v belongs to exactly one edge in P, we call v a *free vertex* of P or say v is a $free(P)$ vertex.

3. Short path—Proof of Theorem 1.1. (i) Let $\mathcal{H}$ be a complete s-uniform hypergraph on n vertices. It is clear that $ar(n, s, \mathbb{P}_2) \geq 2$. Suppose that there exists a 2-coloring of $\mathcal{H}$ without a rainbow $\mathbb{P}_2$. Then there must be two edges e_1 and e_2 satisfying that the colors of e_1 and e_2 are different and $|e_1 \cap e_2| > 1$. Let $u \in e_1 \backslash e_2$ and $v \in e_2 \backslash e_1$. Consider the edge e_3 consisting of u, v and $s-2$ vertices in $V(\mathcal{H}) \backslash V(e_1 \cup e_2)$. Since there is no rainbow $\mathbb{P}_2$, e_3 cannot be colored with either of the two colors, a contradiction. So any 2-coloring of $\mathcal{H}$ admits a rainbow $\mathbb{P}_2$.

(ii) Let $\mathcal{H}$ be a complete s-uniform hypergraph on n vertices. Consider the following coloring of $\mathcal{H}$ with $\binom{n-2}{s-2}+1$ colors. Take two vertices u and v in $\mathcal{H}$; then the number of edges containing both u and v is $\binom{n-2}{s-2}$. Coloring each of these edges with different colors and the remaining edges of $\mathcal{H}$ with an additional color, we can see that this coloring of $\mathcal{H}$ yields no rainbow $\mathbb{P}_3$. Thus, $ar(n, s, \mathbb{P}_3) \geq \binom{n-2}{s-2}+2$.

To prove that $ar(n, s, \mathbb{P}_3) \leq \binom{n-2}{s-2}+2$, we argue by contradiction. Suppose that there exists a coloring of $\mathcal{H}$ without a rainbow $\mathbb{P}_3$, which uses $\binom{n-2}{s-2}+2$ colors. Let $\mathcal{G}$ be a spanning subgraph of $\mathcal{H}$ with $\binom{n-2}{s-2}+2$ edges such that each color appears on exactly one edge of $\mathcal{G}$. Since $|\mathcal{G}| = \binom{n-2}{s-2}+2 > ex(n, s, \mathbb{P}_2)$ for sufficiently large n, there is a linear path P of length two with edges e_1 colored by α_1 and e_2 colored by α_2 in $\mathcal{G}$. Let v be the common vertex of e_1 and e_2. Since $\mathcal{H}$ contains no rainbow $\mathbb{P}_3$, any edge which contains only one vertex from $(e_1 \cup e_2) \setminus \{v\}$ must be colored with α_1 or α_2 in $\mathcal{H}$.

Denote by $\mathcal{F}$ the subgraph obtained by deleting e_1 and e_2 from $\mathcal{G}$. If there is a linear path P' of length two with edges f_1 and f_2 in $\mathcal{F}$, let us say the colors of f_1 and f_2 are β_1 and β_2, respectively. If f_1 or f_2 contains a $free(P)$ vertex w of $e_1 \cup e_2$ and w is not a $cross(P')$ vertex in $f_1 \cup f_2$, then the edge consisting of w and some $s-1$ vertices in $V(\mathcal{H}) \backslash V(e_1 \cup e_2 \cup f_1 \cup f_2)$ along with f_1 and f_2 form a rainbow $\mathbb{P}_3$. Suppose $f_1 \cup f_2$ contains exactly one $free(P)$ vertex w of $e_1 \cup e_2$ and w is the $cross(P')$ vertex. Take an edge e_3 consisting of a $free(P)$ vertex $x \neq w$ of e_1, a $free(P')$ vertex y of f_1, and $s-2$ vertices of $V(\mathcal{H}) \setminus V(e_1 \cup e_2 \cup f_1 \cup f_2)$; then the color of e_3 is either α_1 or α_2. Hence, the path with edges e_3, f_1, and f_2 is a rainbow $\mathbb{P}_3$. If $f_1 \cup f_2$ contains no $free(P)$ vertex of $e_1 \cup e_2$, then the edge e_4 formed by a $free(P)$ vertex x of e_1, a $free(P')$ vertex y of f_1, and $s-2$ vertices of $V(\mathcal{H}) \setminus V(e_1 \cup e_2 \cup f_1 \cup f_2)$ must be colored with α_1 or α_2. So the path with edges e_4, f_1, and f_2 is a rainbow $\mathbb{P}_3$, a contradiction. Therefore, we can assume that there is no $\mathbb{P}_2$ in $\mathcal{F}$. By Theorem 2.2, $\mathcal{F}$ consists of all the $\binom{n-2}{s-2}$ edges containing two fixed vertices x and y. Note that $\{x, y\} \nsubseteq e_1$ and $\{x, y\} \nsubseteq e_2$ since $e_1, e_2 \notin \mathcal{F}$.

We divide our discussion into the following cases depending on the relationship between vertices x, y and edges e_1, e_2.

Case 1. x, $y \notin e_1 \cup e_2$. Consider the edge e consisting of x, y, a $free(P)$ vertex of $e_1 \cup e_2$, and $s-3$ vertices in $V(\mathcal{H}) \setminus V(e_1 \cup e_2 \cup \{x, y\})$. Then, by the structure of $\mathcal{F}$, we have $e \in \mathcal{F}$, and thus e has a different color with α_1 and α_2. Therefore, we find a rainbow $\mathbb{P}_3$ with edges e, e_1, and e_2 in $\mathcal{H}$.

Case 2. $x \in e_1 \cup e_2$, $y \notin e_1 \cup e_2$, and x is not the $cross(P)$ vertex in $e_1 \cup e_2$. The edge e, which consists of x, y, and $s-2$ vertices in $V(\mathcal{H}) \setminus V(e_1 \cup e_2 \cup \{x, y\})$, has a different color with α_1 and α_2. Hence, e, e_1, e_2 form a rainbow $\mathbb{P}_3$ in $\mathcal{H}$. Note that if $y \in e_1 \cup e_2$, $x \notin e_1 \cup e_2$, and y is not the $cross(P)$ vertex in $e_1 \cup e_2$, we can also find a rainbow $\mathbb{P}_3$ in $\mathcal{H}$ similarly.

Case 3. x is a $cross(P)$ vertex in $e_1 \cup e_2$, $y \notin e_1 \cup e_2$. Let e be an edge with a $free(P)$ vertex in e_1 and $s-1$ vertices in $V(\mathcal{H}) \setminus V(e_1 \cup e_2 \cup \{x, y\})$. If e has color α_2, then we have a rainbow $\mathbb{P}_2$ with edges e and e_1. Similar to Case 2, we can find a rainbow $\mathbb{P}_3$ in $\mathcal{H}$. So the color of e is α_1. Pick an edge e' consisting of x, y, a vertex z in $e \setminus e_1$, and $s-3$ vertices in $V(\mathcal{H}) \setminus V(e_1 \cup e_2 \cup \{x, y, z\})$; then e, e', e_2 form a rainbow $\mathbb{P}_3$ in $\mathcal{H}$. And by symmetry, if y is a $cross(P)$ vertex in $e_1 \cup e_2$ and $x \notin e_1 \cup e_2$, we can find a rainbow $\mathbb{P}_3$ in $\mathcal{H}$ as well.

Case 4. $x \in e_1 \setminus e_2$, $y \in e_2 \setminus e_1$. Take an edge e consisting of a $free(P)$ vertex $w \neq x$ in e_1 and $s-1$ vertices in $V(\mathcal{H}) \setminus V(e_1 \cup e_2)$ and the color of e is α_1 or α_2. If the color of e is α_1, then the edge e' consisting of w, x, y, and $s-3$ vertices of $V(\mathcal{H}) \setminus V(e_1 \cup e_2 \cup e \cup \{x, y\})$ along with e and e_2 form a rainbow $\mathbb{P}_3$. If the color of e is α_2, then the edge e'' consisting of x, y, and $s-2$ vertices of $V(\mathcal{H}) \setminus V(e_1 \cup e_2 \cup e \cup \{x, y\})$, along with e_1 and e form a rainbow $\mathbb{P}_3$.

We have examined all the cases in the above discussion. In conclusion, any coloring of $\mathcal{H}$ with $\binom{n-2}{s-2} + 2$ colors admits a rainbow $\mathbb{P}_3$. Hence, we have that $ar(n, s, \mathbb{P}_3) = \binom{n-2}{s-2} + 2$.

(iii) Since $ar(n, s, \mathcal{B}_2) \leq ar(n, s, \mathcal{P}_2) \leq ar(n, s, \mathbb{P}_2)$, we can obtain that $ar(n, s, \mathcal{B}_2) = ar(n, s, \mathcal{P}_2) = 2$ for $n \geq 3s - 4$.

(iv) Let $\mathcal{H}$ be a complete s-uniform hypergraph on n vertices. Consider a 3-coloring of $\mathcal{H}$ such that there is no rainbow $\mathcal{P}_3$ in $\mathcal{H}$. Since $ar(n, s, \mathcal{P}_2) = 2 < 3$ by (iii), there is a rainbow loose path P of length 2 with edges e_1 and e_2, colored by, say, α_1 and α_2. Suppose that the number of $free(P)$ vertices in e_1 is a, so the number of $free(P)$ vertices in e_2 is equal to a. Let

$$p = \begin{cases} \lfloor s/2 \rfloor & \text{if } s - a > \lfloor s/2 \rfloor, \\ s - a & \text{if } s - a \leq \lfloor s/2 \rfloor. \end{cases}$$

Assume that there is an edge f with color $\alpha_3 \notin \{\alpha_1, \alpha_2\}$ such that $f \cap (e_1 \cup e_2) = \emptyset$. Consider an edge e consisting of all the $free(P)$ vertices in e_1, p vertices in f, and $s-a-p$ vertices in $V(\mathcal{H}) \setminus V(e_1 \cup e_2 \cup f)$. Note that the color of e is either α_1 or α_2. If e is colored with α_2, then e_1, e, f is a rainbow $\mathcal{P}_3$. So e can only be colored with α_1. Similarly, let

$$q = \begin{cases} \lceil s/2 \rceil & \text{if } s - a > \lceil s/2 \rceil, \\ s - a & \text{if } s - a \leq \lceil s/2 \rceil. \end{cases}$$

Consider the edge e' consisting of all the $free(P)$ vertices in e_2, q vertices in $V(f) \setminus V(e)$, and $s-a-q$ vertices in $V(\mathcal{H}) \setminus V(e_1 \cup e_2 \cup f \cup e)$; then e' is colored with α_2. Thus, e, f, e' is a rainbow $\mathcal{P}_3$.

So each of the edges colored with α_3 contains vertices in $e_1 \cup e_2$. Take an edge h with color α_3. Note that $h \cap e_1 \neq \emptyset$ and $h \cap e_2 \neq \emptyset$.

Case 1. Either e_1 or e_2 contains a $free(P)$ vertex not belonging to h. Without loss of generality, suppose that there are b $free(P)$ vertices in $e_1 \setminus h$, where $b \geq 1$. Take an edge e with b $free(P)$ vertices in $e_1 \setminus h$ and $s-b$ vertices in $V(\mathcal{H}) \setminus V(e_1 \cup e_2 \cup h)$. Then e is colored with α_1 or α_2. If e is colored with α_2, then e, e_1, h form a rainbow $\mathcal{P}_3$. So e is colored with α_1. Then we have a rainbow $\mathcal{P}_2$ with edges h and e_2, which are colored with α_3 and α_2, respectively. And we have an edge e colored with α_1 and $e \cap (h \cup e_2) = \emptyset$. It is the same situation as we analyzed before; we can also find a rainbow $\mathcal{P}_3$ in $\mathcal{H}$.

Case 2. All the $free(P)$ vertices in $e_1 \cup e_2$ belong to h. Recall that there are a $free(P)$ vertices in e_1. Take an edge e consisting of a $free(P)$ vertices in e_1 and $s-a$ vertices in $V(\mathcal{H}) \setminus V(e_1 \cup e_2 \cup h)$. If e is colored with α_2, then we have a loose path P' of length two with edges e and e_1, which are colored with α_2 and α_1, respectively, and e contains at least one $free(P')$ vertex not belonging to h. It is just similar to Case 1, in which we can find a rainbow $\mathcal{P}_3$ in $\mathcal{H}$. Thus, e is colored with α_1. Similarly, take an edge e' consisting of a $free(P)$ vertices in e_2 and $s-a$ vertices $V(\mathcal{H}) \setminus V(e_1 \cup e_2 \cup h \cup e)$; we can obtain that e' is colored with α_2. Now there is a rainbow $\mathcal{P}_3$ consisting of edges e, h, and e'.

Therefore, $ar(n,s,\mathcal{P}_3) \leq 3$ for $n \geq 4s-3$. Since $ar(n,s,\mathcal{P}_3) \geq 3$ trivially holds, we have that $ar(n,s,\mathcal{P}_3) = 3$ for $n \geq 4s-3$.

Since $ar(n,s,\mathcal{B}_3) \leq ar(n,s,\mathcal{P}_3)$, we obtain that $ar(n,s,\mathcal{B}_3) = ar(n,s,\mathcal{P}_3) = 3$ for $n \geq 4s-3$. ∎

4. Linear path—Proof of Theorem 1.2. Let $\mathcal{H}$ be a complete s-uniform hypergraph on n vertices. For the lower bounds, we construct a coloring of $\mathcal{H}$ by using the extreme s-graphs in Theorem 2.2.

PROPOSITION 4.1. (a) *For $k = 2t$, we have*

$$\min\{ar(n,s,\mathbb{P}_{2t}), ar(n,s,\mathbb{C}_{2t})\} \geq \binom{n}{s} - \binom{n-t+1}{s} + 2.$$

(b) *For $k = 2t+1$, we have*

$$\min\{ar(n,s,\mathbb{P}_{2t+1}), ar(n,s,\mathbb{C}_{2t+1})\} \geq \binom{n}{s} - \binom{n-t+1}{s} + \binom{n-t-1}{s-2} + 2.$$

Proof. (a) If $k = 2t$, we pick a vertex set S with $t-1$ vertices. Take all the edges that meet S and color each of these edges with different colors. Then color the remaining edges of $\mathcal{H}$ with one additional color. This gives a coloring of $\mathcal{H}$ with $\binom{n}{s} - \binom{n-t+1}{s} + 1$ colors. Since each vertex is contained in at most two edges of a rainbow linear path and a rainbow linear cycle, it is easy to see that any rainbow linear path or rainbow linear cycle in $\mathcal{H}$ has length at most $2(t-1)+1 < 2t$. So we have $ar(n,s,\mathbb{P}_{2t}) \geq \binom{n}{s} - \binom{n-t+1}{s} + 2$ and $ar(n,s,\mathbb{C}_{2t}) \geq \binom{n}{s} - \binom{n-t+1}{s} + 2$.

(b) If $k = 2t+1$, we pick a copy of the extreme $\mathbb{P}_{2t}$-free graph obtained in Theorem 2.2. Then color each edge of this extreme $\mathbb{P}_{2t}$-free graph with a distinct color, and color the remaining edges of $\mathcal{H}$ with one additional color to obtain a coloring of $\mathcal{H}$ with $\binom{n}{s} - \binom{n-t+1}{s} + \binom{n-t-1}{s-2} + 1$ colors. It is routine to check that there is no rainbow $\mathbb{P}_{2t+1}$ and no rainbow $\mathbb{C}_{2t+1}$ in the above coloring, and thus $ar(n,s,\mathbb{P}_{2t+1}) \geq \binom{n}{s} - \binom{n-t+1}{s} + \binom{n-t-1}{s-2} + 2$ and $ar(n,s,\mathbb{C}_{2t+1}) \geq \binom{n}{s} - \binom{n-t+1}{s} + \binom{n-t-1}{s-2} + 2$. ∎

For the upper bounds, let

$$D = \begin{cases} \binom{n}{s} - \binom{n-t+1}{s} + 2 & \text{if } k = 2t, \\ \binom{n}{s} - \binom{n-t+1}{s} + \binom{n-t-1}{s-2} + 2 & \text{if } k = 2t+1. \end{cases}$$

We argue by contradiction and suppose that there is a coloring of $\mathcal{H}$ using D colors yielding no rainbow $\mathbb{P}_k$. Let $\mathcal{G}$ be a spanning subgraph of $\mathcal{H}$ with D edges such that each color appears on exactly one edge of $\mathcal{G}$. By Theorem 2.2, we obtain that there is a linear path P of length $k-1$ in $\mathcal{G}$. Denote by $e_1, e_2, \ldots, e_{k-1}$ the edges of P and by $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$ the colors of $e_1, e_2, \ldots, e_{k-1}$, respectively.

Since $\mathcal{H}$ contains no rainbow $\mathbb{P}_k$, we obtain the following fact.

Observation 4.1. Let v be a $free(P)$ vertex in $e_1 \cup e_{k-1}$. Then for any edge g satisfying $g \cap P = \{v\}$, the edge g must be colored with a color of $\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}\}$.

Denote by $\mathcal{F}$ the subgraph obtained by deleting $e_1, e_2, \ldots, e_{k-1}$ from $\mathcal{G}$. We divide the remaining proof into two cases according to the parity of k.

4.1. Completing the proof when $k = 2t$ is even. In this subsection, we assume that $k = 2t \geq 4$ is even.

CLAIM 4.1. *When $k = 2t \geq 4$, there is no $\mathbb{P}_{k-1}$ in $\mathcal{F}$.*

Proof. By contradiction, suppose there is a linear path P' of length $k-1$ in $\mathcal{F}$. Denote the edges of P' by $f_1, f_2, \ldots, f_{k-1}$ with colors $\beta_1, \beta_2, \ldots, \beta_{k-1}$, respectively. Since there is no rainbow $\mathbb{P}_k$ in $\mathcal{H}$, every edge g with $g \cap V(P') = \{u\}$, where u is a $free(P')$ vertex in $f_1 \cup f_{k-1}$, must be colored with a color of $\{\beta_1, \beta_2, \ldots, \beta_{k-1}\}$. We obtain an s-graph $\mathcal{F}_e$ by deleting $f_1, f_2, \ldots, f_{k-1}$ and all the edges containing at least two vertices of $P \cup P'$ from $\mathcal{F}$. Let c denote the number of vertices of $P \cup P'$. Then $c \leq 2[(k-1)s - (k-2)]$, and so we have

$$|\mathcal{F}_e| \geq |\mathcal{F}| - (k-1) - \sum_{i=2}^{s} \binom{c}{i}\binom{n-c}{s-i} > ex(n, s, \mathbb{P}_{k-2})$$

for sufficiently large n. Thus, we have a linear path P'' of length $k-2$ in $\mathcal{F}_e$. Denote by $h_1, h_2, \ldots, h_{k-2}$ the edges of P''. Moreover, every edge in P'' contains at most one vertex from $P \cup P'$. So it follows from Observation 4.1 that P'' contains no $free(P)$ vertex of e_1, e_{k-1} and no $free(P')$ vertex of f_1, f_{k-1}. Take an edge e consisting of a $free(P)$ vertex x of e_1, a $free(P'')$ vertex of $h_1 \setminus V(P \cup P')$ (since $s \geq 3$, such vertex does exist), and $s-2$ vertices in $V(\mathcal{H}) \setminus V(P \cup P' \cup P'')$; then e is colored with one color in $\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}\}$ by Observation 4.1. Take another edge e' consisting of a $free(P')$ vertex of $f_1 \setminus \{x\}$, a $free(P'')$ vertex of $h_{k-2} \setminus V(P \cup P')$, and $s-2$ vertices in $V(\mathcal{H}) \setminus (P \cup P' \cup P'' \cup e)$; then Observation 4.1 indicates e' is colored with one color in $\{\beta_1, \beta_2, \ldots, \beta_{k-1}\}$. Hence, the path with edges $e, h_1, h_2, \ldots, h_{k-2}, e'$ is a rainbow $\mathbb{P}_k$, a contradiction. This proves the claim. ∎

Note that $|\mathcal{F}| \sim (t-1)\binom{n}{s-1}$. By Claim 4.1, Theorems 2.5 and 2.6 are applied to $\mathcal{F}$. So we can find an $(s-1)$-graph $G^* \subset \partial\mathcal{F}$ with $|G^*| \sim \binom{n}{s-1}$ for $k \geq 6$ and with $|G^*| \geq \frac{1}{2}\binom{n}{s-1}$ for $k = 4$, and there is a vertex set $L = \{v_1, v_2, \ldots, v_{t-1}\}$ such that $L \cap V(G^*) = \emptyset$ and $e \cup \{v\} \in \mathcal{F}$ for any $(s-1)$-edge $e \in G^*$ and any $v \in L$. Moreover, $|\mathcal{F} - L| = o(n^{s-1})$. We point out that all the vertices of L are not $free(P)$ vertices in $e_1 \cup e_{k-1}$. Otherwise, let W be the vertex set of P. By Lemma 2.7, we can find an $(s-1)$-edge disjoint with W in G^*, and it gives an s-edge in $\mathcal{F}$ containing only a $free(P)$ vertex of P. This edge together with P form a rainbow $\mathbb{P}_k$, a contradiction.

CLAIM 4.2. *When $k = 2t \geq 4$, there is no edge in $\mathcal{F} - L$.*

Proof. Suppose to the contrary there exists an edge $h \in \mathcal{F} - L$ with, say, color λ. By Lemma 2.7, we can find two $(s-1)$-edges a_0, b_0 in G^* such that a_0 and b_0 have exactly one common vertex u and are disjoint from P and h. Let W be the vertex set of $P \cup h \cup a_0 \cup b_0$. By Lemma 2.7, we can find $(s-1)$-edges $\{a_i, b_i\}$ disjoint from W for $i = 1, \ldots, t-1$ such that for every i, a_i and b_i have exactly one common vertex, and for any $j \neq i$, $\{a_i, b_i\}$ and $\{a_j, b_j\}$ are vertex disjoint. Then

$$f_1 = a_0 \cup \{v_1\}, f_2 = \{v_1\} \cup a_1, f_3 = b_1 \cup \{v_2\}, f_4 = \{v_2\} \cup a_2, f_5 = b_2 \cup \{v_3\},$$
$$\ldots, f_{k-4} = \{v_{t-2}\} \cup a_{t-2}, f_{k-3} = b_{t-2} \cup \{v_{t-1}\}, f_{k-2} = \{v_{t-1}\} \cup a_{t-1}$$

form a $\mathbb{P}_{k-2}$ in $\mathcal{F}$, denoted by P'. In the rest of the paper, this kind of path is abbreviated as

$$P' = a_0 \bigoplus_{i=1}^{t-2} (\{v_i\} \oplus a_i \oplus b_i) \oplus \{v_{t-1}\} \oplus a_{t-1}.$$

Let $\beta_1, \beta_2, \ldots, \beta_{k-2}$ be the colors of $f_1, f_2, \ldots, f_{k-2}$, respectively. Note that $b_0 \cup \{v_1\}$ and $b_{t-1} \cup \{v_{t-1}\}$ are edges of $\mathcal{F}$, so both of them have colors distinct from any other edges in $\mathcal{F}$. The edges, which consist of one $free(P')$ vertex in f_1, one $free(P)$ vertex in e_1, and $s-2$ vertices disjoint with P and P', must be colored with colors from $\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}\}$ by Observation 4.1. Let f be an edge consisting of the $free(P')$ vertex u in f_1, a vertex in h, and $s-2$ vertices disjoint with P, P', b_0, and h. Then the color of f is in $\{\lambda, \beta_1, \beta_2, \ldots, \beta_{k-2}\}$ because otherwise $h \cup f \cup P'$ is a rainbow $\mathbb{P}_k$. If the color of f is λ, then we can extend $f \cup P'$ to a rainbow $\mathbb{P}_k$ with an additional edge containing one $free(P')$ vertex in f_{k-2} and a $free(P)$ vertex in e_{k-1}.

Assume the color of f is β_j for some j. Let W be the vertex set of $f \cup P' \cup h \cup b_0$. By Lemma 2.7, we can find $(s-1)$-edges $\{a'_i, b'_i\}$ in G^*, which are disjoint from W for $i = 1, \ldots, t-1$. Furthermore,

$$h \oplus f \oplus b_0 \bigoplus_{i=1}^{t-2} (\{v_i\} \oplus a'_i \oplus b'_i) \oplus \{v_{t-1}\} \oplus a'_{t-1}$$

is a rainbow $\mathbb{P}_k$, a contradiction. This shows that $\mathcal{F} - L$ contains no edge; i.e., all the edges in $\mathcal{F}$ contain vertices in L. ∎

Notice that

$$|\mathcal{F}| = D - (k-1) = \binom{n}{s} - \binom{n-t+1}{s} - k + 3$$

and that there are $\binom{n}{s} - \binom{n-t+1}{s}$ edges in $\mathcal{H}$ which intersect L. Therefore, Claim 4.1 implies that $\mathcal{F}$ contains no isolated vertices, and

(4.1) there are only $k-3$ edges containing vertices in L which are not belonging to $\mathcal{F}$.

We will derive the final contradiction from the following claim.

CLAIM 4.3. *When $k = 2t \geq 4$, there exist at most one edge in P which is disjoint with L.*

Proof. Suppose that there are two edges e_i and e_j $(j > i)$ in P, which are disjoint with L. If $j > i+1$, we find an edge f in $\mathcal{F}$ containing a vertex in e_i and disjoint with e_j and an edge g in $\mathcal{F}$ containing a vertex in e_j and disjoint with e_i and f. Let $f \cap L = v_p$, $g \cap L = v_q$. Without loss of generality, we suppose that $v_p = v_1$, $v_q = v_{t-1}$. Let W consist of the vertices in $e_i \cup e_j \cup f \cup g$. By Lemma 2.7, we can find $(s-1)$-edges $\{a'_i, b'_i\}$ disjoint from W for $i = 1, \dots, t-1$. Then

$$e_i \oplus f \oplus a'_1 \oplus b'_1 \bigoplus_{i=2}^{t-2} (\{v_i\} \oplus a'_i \oplus b'_i) \oplus g \oplus e_j$$

is a rainbow $\mathbb{P}_k$ in $\mathcal{H}$, a contradiction.

If $j = i+1$, we find an edge h in $\mathcal{F}$ such that h contains exactly one vertex in $e_j \setminus e_i$ and is disjoint with e_i. Without loss of generality, we suppose that $h \cap L = v_1$. Let W consist of the vertices in $e_i \cup e_j \cup h$. By Lemma 2.7, we can find $(s-1)$-edges $\{a''_i, b''_i\}$ disjoint from W for $i = 1, \dots, t-1$. Then

$$e_i \oplus e_j \oplus h \oplus a''_1 \oplus b''_1 \bigoplus_{i=2}^{t-2} (\{v_i\} \oplus a''_i \oplus b''_i) \oplus \{v_{t-1}\} \oplus a''_{t-1}$$

is a rainbow $\mathbb{P}_k$ in $\mathcal{H}$, a contradiction. $\square$

Since P has $k-1$ edges, Claim 4.3 shows there are at least $k-2$ edges containing vertices of L in P. As $\mathcal{F} = \mathcal{G} - E(P)$, we conclude that there are at least $k-2$ edges containing vertices in L which are not belonging to $\mathcal{F}$, contradicting (4.1). This completes the proof for even k.

4.2. Completing the proof when $k = 2t+1$ is odd. In this subsection, we assume that $k = 2t+1$ is odd.

Recall that $\mathcal{F}$ denotes the subgraph obtained from $\mathcal{G}$ by deleting $e_1, e_2, \dots, e_{k-1}$. If there is no $\mathbb{P}_{k-1}$ in $\mathcal{F}$, then we can use Theorem 2.5 to characterize the structure of $\mathcal{F}$. However, this may not be the case when k is odd. Fortunately, we can prove that after deleting a few edges, the remaining subgraph of $\mathcal{F}$ contains no $\mathbb{P}_{k-1}$.

CLAIM 4.4. *If there is a linear path P_1 of length $k-1$ in $\mathcal{F}$, then $\mathcal{F} - E(P_1)$ contains no $\mathbb{P}_{k-1}$.*

Proof. Suppose to the contrary that there is a linear path P_2 of length $k-1$ in $\mathcal{F} - E(P_1)$. Notice that the colors used in P_1 and P_2 are pairwise distinct by the selection of $\mathcal{F}$. Let $f_1, f_2, \dots, f_{k-1}$ be the edges of P_1 with colors $\beta_1, \beta_2, \dots, \beta_{k-1}$, respectively. Denote by $g_1, g_2, \dots, g_{k-1}$ the edges of P_2 with colors $\gamma_1, \gamma_2, \dots, \gamma_{k-1}$, respectively.

Let c denote the number of vertices of $P \cup P_1 \cup P_2$. Then we have $c \le 3[(k-1)s - (k-2)]$. Note that the number of edges which contain at least two vertices in $P \cup P_1 \cup P_2$ is at most $\sum_{i=2}^{s} \binom{c}{i}\binom{n-c}{s-i}$. Since

$$\begin{aligned}
&|\mathcal{F}| - \sum_{i=2}^{s} \binom{c}{i}\binom{n-c}{s-i} \\
&= \binom{n}{s} - \binom{n-t+1}{s} + \binom{n-t-1}{s-2} + 2 - (k-1) - \sum_{i=2}^{s} \binom{c}{i}\binom{n-c}{s-i} \\
&> ex(n, s, \mathbb{P}_{k-3})
\end{aligned}$$

for sufficiently large n, there exists a linear path P_3 of length $k-3$ such that every edge in P_3 has at most one vertex of $P \cup P_1 \cup P_2$. Hence, all the $free(P)$ vertices in $e_1 \cup e_{k-1}$, $free(P_1)$ vertices in $f_1 \cup f_{k-1}$, and $free(P_2)$ vertices in $g_1 \cup g_{k-1}$ are not in P_3 by Observation 4.1. Denote by $h_1, h_2, \ldots, h_{k-3}$ the edges of P_3. Consider an edge e, which consists of a $free(P)$ vertex x in e_1, a $free(P_3)$ vertex in $h_1 \setminus (P_1 \cup P_2)$, and $s-2$ vertices disjoint with $P \cup P_1 \cup P_2 \cup P_3$; it follows from Observation 4.1 that the color of e is in $\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}\}$. And consider an edge e', which consists of a $free(P_1)$ vertex $y \neq x$ in $f_1 \cup f_{k-1}$ (we can find such a vertex y since $s > 3$), a $free(P_3)$ vertex in $h_{k-3} \setminus (P_1 \cup P_2)$, and $s-2$ vertices disjoint with $P_1 \cup P_2 \cup P_3 \cup P \cup e$; then the color of e' is from $\{\beta_1, \beta_2, \ldots, \beta_{k-1}\}$ by Observation 4.1. Moreover, $e \cup P_3 \cup e'$ is a rainbow $\mathbb{P}_{k-1}$. Now consider another edge e'', which consists of a $free(P_2)$ vertex $z \neq x, y$ in $g_1 \cup g_{k-1}$, a vertex in $e' \setminus (P_1 \cup P_3)$, and $s-2$ vertices disjoint with $P_1 \cup P_2 \cup P_3 \cup P \cup e \cup e'$; then e'' has a color appearing in $e \cup P_3 \cup e'$ by Observation 4.1. However, to prevent extending P_2 to a rainbow $\mathbb{P}_k$, the color of e'' should be one of $\{\gamma_1, \gamma_2, \ldots, \gamma_{k-3}\}$, a contradiction. ∎

So if $\mathcal{F}$ has a $\mathbb{P}_{k-1}$, denote by $\mathcal{F}_0$ the subgraph obtained by deleting all the $k-1$ edges of that $\mathbb{P}_{k-1}$ in $\mathcal{F}$. Then we have that $\mathcal{F}_0$ is $\mathbb{P}_{k-1}$-free by Claim 4.4 and

$$|\mathcal{F}_0| = \binom{n}{s} - \binom{n-t+1}{s} + \binom{n-t-1}{s-2} + 2 - 2(k-1).$$

Since $|\mathcal{F}_0| \sim (t-1)\binom{n}{s-1}$ and by Theorem 2.5, we can find an $(s-1)$-graph $G^* \subset \partial\mathcal{F}_0$ with $|G^*| \sim \binom{n}{s-1}$ and a set L of $t-1$ vertices of $\mathcal{F}_0$ such that $L \cap V(G^*) = \emptyset$ and $e \cup \{v\} \in \mathcal{F}_0$ for any $(s-1)$-edge $e \in G^*$ and any $v \in L$. Moreover, $|\mathcal{F}_0 - L| = o(n^{s-1})$.

If $\mathcal{F}$ does not contain a $\mathbb{P}_{k-1}$, then by Theorem 2.5 again, we can find an $(s-1)$-graph $G^* \subset \partial\mathcal{F}$ with $|G^*| \sim \binom{n}{s-1}$ and a set L of $t-1$ vertices of $\mathcal{F}$ such that $L \cap V(G^*) = \emptyset$ and $e \cup \{v\} \in \mathcal{F}$ for any $(s-1)$-edge $e \in G^*$ and any $v \in L$. Additionally, $|\mathcal{F} - L| = o(n^{s-1})$. Since the number of edges meeting L is at most $\binom{n}{s} - \binom{n-t+1}{s}$, we have $|\mathcal{F} - L| \geq |\mathcal{F}| - \left[\binom{n}{s} - \binom{n-t+1}{s}\right] = \binom{n-t-1}{s-2} + 2 - (k-1) > k-1$. Now we delete any $k-1$ edges of $\mathcal{F} - L$ from $\mathcal{F}$ and still denote the remaining subgraph $\mathcal{F}_0$.

Therefore, in either case, we can find an $(s-1)$-graph $G^* \subset \partial\mathcal{F}_0$ with $|G^*| \sim \binom{n}{s-1}$ and a set L of $t-1$ vertices of $\mathcal{F}_0$ such that $L \cap V(G^*) = \emptyset$ and $e \cup \{v\} \in \mathcal{F}_0$ for any $(s-1)$-edge $e \in G^*$ and any $v \in L$. Moreover, $|\mathcal{F}_0 - L| = o(n^{s-1})$. We select a G^* with the maximum number of $(s-1)$-edges. Let the vertices in L be $v_1, v_2, \ldots, v_{t-1}$. We point out that all the v_i for $i = 1, 2, \ldots, t-1$ are not $free(P)$ vertices in $e_1 \cup e_{k-1}$. Otherwise, let W be the vertex set of P. Then by Lemma 2.7, we can find an $(s-1)$-edge disjoint with W in G^*, and this together with v_i will form an s-edge which extends P to a rainbow $\mathbb{P}_k$.

Since the number of edges containing vertices of L is at most $\binom{n}{s} - \binom{n-t+1}{s}$ in $\mathcal{F}_0$, we have $|\mathcal{F}_0 - L| > \binom{n-t-1}{s-2} + 2 - 2(k-1)$. We further claim the following.

Claim 4.5. *$|\mathcal{F}_0 - L| \leq \binom{n-t-1}{s-2} + \binom{2s-2}{s} + \binom{2s-1}{s-1}n$.*

Proof. By contradiction, assume that $|\mathcal{F}_0 - L| > \binom{n-t-1}{s-2} + \binom{2s-2}{s} + \binom{2s-1}{s-1}n$. Note that the number of vertices of $\mathcal{F}_0 - L$ is $n-t+1$; by Theorem 2.2, we can find a $\mathbb{P}_2$ in $\mathcal{F}_0 - L$, denoted by P_1. Let h_1, h_2 be the edges of P_1 with colors γ_1, γ_2, respectively. The number of edges containing at least $s-1$ vertices in P_1 is less than $\binom{2s-1}{s} + \binom{2s-1}{s-1}n$. Since $|\mathcal{F}_0 - L| - \left[\binom{2s-2}{s} + \binom{2s-1}{s-1}n\right] > ex(n-t+1, s, \mathbb{P}_2)$ for sufficiently large n, there is another linear path P_2 of length two in $\mathcal{F}_0 - L$ such that each edge of which has at least two vertices not in P_1. Let h_3, h_4 be the edges of P_2

with colors γ_3, γ_4. So h_4 contains a $free(P_2)$ vertex $x \notin P_1$. Furthermore, one of h_1, h_2 contains a $free(P_1)$ vertex not belonging to P_2. Let us say h_2 has a $free(P_1)$ vertex $y \notin P_2$. Take two $(s-1)$-edges a_0, b_0 in G^* that are disjoint from P_1, P_2 and P such that $a_0 \cap b_0 = u$. Let W be the vertex set of $P \cup P_1 \cup P_2 \cup a_0 \cup b_0$. By Lemma 2.7, we can find $(s-1)$-edges $\{a_i, b_i\}$ disjoint from W for $i = 1, \ldots, t-1$, and so

$$P' = a_0 \bigoplus_{i=1}^{t-2} (\{v_i\} \oplus a_i \oplus b_i) \oplus \{v_{t-1}\} \oplus a_{t-1}$$

is a $\mathbb{P}_{2t-2} = \mathbb{P}_{k-3}$ in $\mathcal{F}_0$. Denote by $f_1, f_2, \ldots, f_{k-3}$ the edge of P' with colors $\beta_1, \beta_2, \ldots, \beta_{k-3}$ appearing in each of those edges in P', respectively. Note that $f_1 = a_0 \cup \{v_1\}$.

Consider the edge g consisting of x, y, u and $s-3$ vertices disjoint with P, P_1, P_2, and b_0. Then the color of g is in $\{\gamma_1, \ldots \gamma_4, \beta_1, \ldots, \beta_{k-3}\}$; otherwise, we can easily extend P' to a rainbow $\mathbb{P}_k$ by adding g, h_1 and h_2. If the color of g is in $\{\gamma_1, \gamma_2\}$, then $h_3 \cup h_4 \cup g \cup P'$ is a rainbow $\mathbb{P}_k$. If the color of g is in $\{\gamma_3, \gamma_4\}$, then $h_1 \cup h_2 \cup g \cup P'$ is a rainbow $\mathbb{P}_k$. So the color of g must be in $\{\beta_1, \ldots, \beta_{k-3}\}$. Let W be the vertex set of $P \cup P_1 \cup P_2 \cup P' \cup b_0$. By Lemma 2.7, we can find $(s-1)$-edges $\{a'_i, b'_i\}$ disjoint from W for $i = 1, \ldots, t-1$, and

$$h_1 \oplus h_2 \oplus g \oplus b_0 \bigoplus_{i=1}^{t-2} (\{v_i\} \oplus a'_i \oplus b'_i) \oplus \{v_{t-1}\} \oplus a'_{t-1}$$

is a rainbow $\mathbb{P}_k$, a contradiction. ∎

Now Claim 4.5 provides some further structural properties of $\mathcal{F}_0$.

CLAIM 4.6. (a) *There is no isolated vertex in* $\mathcal{F}_0$.

(b) *There are at most* $\binom{2s-2}{s} + \binom{2s-1}{s-1}n + 2(k-1) - 2$ *edges meeting* L *but not in* $\mathcal{F}_0$.

(c) *Every vertex in* $V(\mathcal{F}_0) \setminus L$ *belongs to* G^* *and is not an isolated vertex in* G^*.

Proof. (a) In fact, if $\mathcal{F}_0$ contains an isolated vertex, then the number of edges meeting L in $\mathcal{F}_0$ is at most $\binom{n-1}{s} - \binom{n-1-(t-1)}{s}$. Thus, we have

$$|\mathcal{F}_0 - L| \geq |\mathcal{F}_0| - \left[\binom{n-1}{s} - \binom{n-1-(t-1)}{s} \right] \geq O(n^{s-1}),$$

a contradiction to Claim 4.5. This indicates that $\mathcal{F}_0$ contains no isolated vertices.

(b) By Claim 4.5, there are

$$|\mathcal{F}_0| - |\mathcal{F}_0 - L| \geq \binom{n}{s} - \binom{n-t+1}{s} + 2 - 2(k-1) - \left[\binom{2s-2}{s} + \binom{2s-1}{s-1} n \right]$$

edges in $\mathcal{F}_0$ containing vertices in L. Since there are $\binom{n}{s} - \binom{n-t+1}{s}$ edges containing vertices of L in $\mathcal{H}$, we have that there are at most $\binom{2s-2}{s} + \binom{2s-1}{s-1}n + 2(k-1) - 2$ edges meeting L but not in $\mathcal{F}_0$.

(c) If there exists a vertex $v \in V(\mathcal{F}_0) \setminus L$ but $v \notin G^*$ or v is an isolated vertex in G^*, then we have at least $\binom{n-(t-1)-1}{s-2} > \binom{2s-2}{s} + \binom{2s-1}{s-1}n + 2(k-1) - 2$ edges meeting L but not belonging to $\mathcal{F}_0$, which is a contradiction to (b). Hence, every vertex in $V(\mathcal{F}_0) \setminus L$ is contained in some edges of G^*. ∎

Now we focus on the edges which are disjoint with L, namely, the edges in $(\mathcal{F} \cup P) - L = \mathcal{G} - L$ and, more generally, the edges in $\mathcal{H} - L$. Considering the relationship between edges in $\mathcal{H} - L$, we make the following claim.

CLAIM 4.7. *Assume that there exist three edges f, g, h in $\mathcal{H}-L$ with distinct colors such that one of the following holds:*
(i) *f, g, h form a $\mathbb{P}_3$;*
(ii) *f, g form a $\mathbb{P}_2$, and h is disjoint with $f \cup g$;*
(iii) *f, g, h are disjoint with each.*
Then we can find a rainbow $\mathbb{P}_k$ in $\mathcal{H}$.

Proof. Notice that for each $e \in \{f, g, h\}$, there is a unique edge e' in $\mathcal{G}$ having the same color with e. So we denote f', g', h' to be the edges in $\mathcal{G}$ with the same color with f, g, h, respectively. If the edge e is in $\mathcal{G}$ for some $e \in \{f, g, h\}$, we have $e' = e$.

(i) Assume that there are three edges f, g, h in $\mathcal{H} - L$ with distinct colors such that f, g, h form a $\mathbb{P}_3$. Realize that in G^*, there exists an $(s-1)$-edge a_0 containing a vertex x in $h \setminus g$ and disjoint with $(f' \cup g' \cup h' \cup f \cup g \cup h) \setminus \{x\}$. Let W consist of the vertices in $f' \cup g' \cup h' \cup f \cup g \cup h \cup a_0$. By Lemma 2.7, we can find $(s-1)$-edges $\{a_i, b_i\}$ disjoint from W for $i = 1, \dots, t-1$. Then

$$f \oplus g \oplus h \oplus a_0 \bigoplus_{i=1}^{t-2} (\{v_i\} \oplus a_i \oplus b_i) \oplus \{v_{t-1}\} \oplus a_{t-1}$$

is a rainbow $\mathbb{P}_k$ in $\mathcal{H}$.

(ii) Assume that there are three edges f, g, h in $\mathcal{H}-L$ with distinct colors such that f, g form a $\mathbb{P}_2$, h is disjoint with $f \cup g$. In G^*, there exists an $(s-1)$-edge a_0 containing a vertex x in $g \setminus f$ and disjoint with $(f' \cup g' \cup h' \cup g \cup h \cup f) \setminus \{x\}$. And there exists an $(s-1)$-edge b_0 containing a vertex y in h and disjoint with $(f' \cup g' \cup h' \cup f \cup g \cup h \cup a_0) \setminus \{y\}$. Let W consist of the vertices in $f' \cup g' \cup h' \cup f \cup g \cup h \cup a_0 \cup b_0$. By Lemma 2.7, we can find $(s-1)$-edges $\{a_i, b_i\}$ disjoint from W for $i = 1, \dots, t-1$. Then

$$f \oplus g \oplus a_0 \bigoplus_{i=1}^{t-2} (\{v_i\} \oplus a_i \oplus b_i) \oplus \{v_{t-1}\} \oplus b_0 \oplus h$$

is a rainbow $\mathbb{P}_k$ in $\mathcal{H}$.

(iii) Assume that there are three edges f, g, h in $\mathcal{H} - L$ with distinct colors such that they are disjoint with each other. In G^*, there exists an $(s-1)$-edge a_0 containing a vertex x in f and disjoint with $(f' \cup g' \cup h' \cup f \cup g \cup h) \setminus \{x\}$. And we can find $(s-1)$-edges a_0', b_0' in G^* such that the $(s-1)$-edge a_0' contains a vertex y_1 in g and disjoint with $(f' \cup g' \cup h' \cup f \cup g \cup h \cup a_0) \setminus \{y_1\}$, and the $(s-1)$-edge b_0' contains a vertex $y_2 \neq y_1$ in g and disjoint with $(f' \cup g' \cup h' \cup f \cup g \cup h \cup a_0 \cup a_0') \setminus \{y_2\}$. Moreover, there exists an $(s-1)$-edge b_0 containing a vertex z in h and disjoint with $(f' \cup g' \cup h' \cup f \cup g \cup h \cup a_0 \cup a_0' \cup b_0') \setminus \{z\}$. Let W consist of the vertices in $f' \cup g' \cup h' \cup f \cup g \cup h \cup a_0 \cup b_0 \cup a_0' \cup b_0'$. By Lemma 2.7, we can find $(s-1)$-edges $\{a_i, b_i\}$ disjoint from W for $i = 1, \dots, t-1$. Then

$$f \oplus a_0 \oplus \{v_1\} \oplus a_0' \oplus g \oplus b_0' \bigoplus_{i=2}^{t-2} (\{v_i\} \oplus a_{i-1} \oplus b_{i-1}) \oplus \{v_{t-1}\} \oplus b_0 \oplus h$$

is a rainbow $\mathbb{P}_k$ in $\mathcal{H}$. Note that in this case, we require that $k = 2t + 1 \geq 7$. ∎

As $\mathcal{H}$ is a counterexample which contains no rainbow $\mathbb{P}_k$, we know that each of the conditions (i), (ii), (iii) in Claim 4.7 cannot exist. Hence there are at most two edges in $P - L$ by Claim 4.7. In fact, if there are more than two edges in $P - L$, then one of conditions (i), (ii), (iii) in Claim 4.7 must occur. On the other hand, since $|L| = t-1$, there are at most $2(t-1) = k-3$ edges in P containing vertices in L. Therefore, there are exactly two edges in $P - L$, denoted by e_i and e_j. Since the number of edges meeting L in $\mathcal{F}$ is at most $\binom{n}{s} - \binom{n-t+1}{s} - (k-3)$, we have

$$|\mathcal{F} - L| \geq |\mathcal{F}| - \left[\binom{n}{s} - \binom{n-t+1}{s} - (k-3)\right] = \binom{n-t-1}{s-2}. \tag{4.2}$$

We shall derive the final contradiction depending on whether $\mathcal{F} - L$ contains a $\mathbb{P}_2$.

Case A. $\mathcal{F} - L$ contains a $\mathbb{P}_2$. We take such a $\mathbb{P}_2$ in $\mathcal{F} - L$ and denote its edges by h_1 and h_2 with colors γ_1 and γ_2, respectively. Select an edge e in $\mathcal{H} - L$ such that e is disjoint with $\{e_i, e_j, h_1, h_2\}$. If the color of e is α_i or α_j, then h_1, h_2 form a $\mathbb{P}_2$ and e is disjoint with them, which satisfies condition (ii) of Claim 4.7, and so we can find a rainbow $\mathbb{P}_k$ in $\mathcal{H}$. Assume instead that the color of e is not in $\{\alpha_i, \alpha_j\}$; then e, e_i, e_j have distinct colors. Furthermore, either e, e_i, e_j are pairwise disjoint or e_i, e_j form a $\mathbb{P}_2$ and e is disjoint with them. This satisfies one of the conditions (ii) and (iii) of Claim 4.7, by which we can find a rainbow $\mathbb{P}_k$ in $\mathcal{H}$.

Case B. $\mathcal{F} - L$ does not contain a $\mathbb{P}_2$. By (4.2) and Theorem 2.2, $\mathcal{F} - L$ is the extreme $\mathbb{P}_2$-free hypergraph on $n-t+1$ vertices. Namely, $\mathcal{F} - L$ consists of all the $\binom{n-t-1}{s-2}$ edges containing two fixed vertices x and y. Note that $\{x, y\} \nsubseteq e_i$, and $\{x, y\} \nsubseteq e_j$ since $e_i, e_j \notin \mathcal{F}$. If e_i, e_j are not consecutive in P, then we select an edge h in $\mathcal{F} - L$ such that h intersects $e_i \cup e_j$ as small as possible. Since $\mathcal{F} - L$ consists of all the $\binom{n-t-1}{s-2}$ edges containing x and y, we have that h intersects $e_i \cup e_j$ in at most two vertices, namely, some of x and y. Then e_i, e_j, h must satisfy one of the conditions (i), (ii), (iii) in Claim 4.7, in which we can find a rainbow $\mathbb{P}_k$ in $\mathcal{H}$. Assume instead that e_i, e_j are consecutive in P in the following.

If either $x \in e_i \setminus e_j$, $y \in e_j \setminus e_i$ or $e_i \cap e_j \in \{x, y\}$, then we can select an edge h in $\mathcal{F} - L$ such that $h \cap (\{e_i, e_j\} \setminus \{x, y\}) = \emptyset$. Take an edge e in $\mathcal{H}$ such that e is disjoint with $\{e_i, e_j, h\}$ and L. If the color of e is α_i or α_j, then e_j, h form a rainbow $\mathbb{P}_2$ and e is disjoint with them, or e_i, h form a rainbow $\mathbb{P}_2$ and e is disjoint with them. Thus, e, e_j, h or e, e_i, h satisfy condition (ii) of Claim 4.7, in which we can obtain a rainbow $\mathbb{P}_k$ in $\mathcal{H}$. If the color of e is neither α_i nor α_j, then e, e_i, e_j have distinct colors. Moreover, e_i, e_j form a $\mathbb{P}_2$, and e is disjoint with them. This shows that e, e_i, e_j satisfy condition (ii) of Claim 4.7, in which we can find a rainbow $\mathbb{P}_k$ in $\mathcal{H}$.

Finally, assume instead that either $\{x, y\} \cap \{e_i, e_j\} = \emptyset$ or $x \in e_i \setminus e_j$, $y \notin e_j$. Then we select an edge h in $\mathcal{F} - L$ such that h intersects $e_i \cup e_j$ as small as possible. Thus, h intersects $e_i \cup e_j$ in at most one vertex, namely, x, which is a free vertex in $e_i \cup e_j$. This indicates that e_i, e_j, h satisfy one of the conditions (i) and (ii) of Claim 4.7, and hence we can find a rainbow $\mathbb{P}_k$ in $\mathcal{H}$.

Therefore, we have established the upper bound. ∎

5. Loose path—Proof of Theorem 1.3. Let $\mathcal{H}$ be a complete s-uniform hypergraph on n vertices. The lower bound in Theorem 1.3 follows from a similar construction in Theorem 1.2 by applying the extreme s-graphs obtained from Theorem 2.1.

For the upper bound, if $k = 2t$, since $ar(n, s, \mathcal{P}_k) \leq ar(n, s, \mathbb{P}_k) = \binom{n}{s} - \binom{n-t+1}{s} + 2$, we are done.

If $k = 2t+1$, we consider, by contradiction, a coloring of $\mathcal{H}$ using $\binom{n}{s} - \binom{n-t+1}{s} + 3$ colors yielding no rainbow $\mathcal{P}_k$. Let $\mathcal{G}$ be a spanning subgraph of $\mathcal{H}$ with $\binom{n}{s} - \binom{n-t+1}{s} + 3$ edges such that each color appears on exactly one edge of $\mathcal{G}$. By Theorem 2.2, we obtain that there is a loose path P of length $k-1$ in $\mathcal{G}$. Denote by $e_1, e_2, \ldots, e_{k-1}$ the edges of P and by $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$ the colors of $e_1, e_2, \ldots, e_{k-1}$, respectively.

Denote by $\mathcal{F}$ the subgraph obtained by deleting $e_1, e_2, \ldots, e_{k-1}$ from $\mathcal{G}$. Similar to the proof of Theorem 1.2, we show that after deleting a few edges from $\mathcal{F}$, the remaining subgraph contains no $\mathbb{P}_{k-1}$. Actually, we prove something stronger. Call a loose path P' *bad* if the number of $free(P')$ vertices in the two end edges of P' is at least three. Since $s \geq 3$, it is easy to get that a linear path is also a bad loose path.

Claim 5.1. *There are no edge-disjoint bad loose paths of length $k-1$ in $\mathcal{F}$.*

Proof. By contradiction, suppose that there are two edge-disjoint bad loose paths P_1 and P_2 of length $k-1$ in $\mathcal{F}$. Denote by $f_1, f_2, \ldots, f_{k-1}$ the edges of P_1 with colors $\beta_1, \beta_2, \ldots, \beta_{k-1}$, respectively, and denote by $g_1, g_2, \ldots, g_{k-1}$ the edges of P_2 with colors $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$, respectively. Let c denote the number of vertices of $P \cup P_1 \cup P_2$. Then $c \leq 3[(k-1)s - (k-2)]$. Note that the number of edges which contain at least two vertices in $P \cup P_1 \cup P_2$ is at most $\sum_{i=2}^{s} \binom{c}{i}\binom{n-c}{s-i}$. Since

$$|\mathcal{F}| - \sum_{i=2}^{s} \binom{c}{i}\binom{n-c}{s-i} > ex(n, s, \mathbb{P}_{k-3})$$

for sufficiently large n, there exists a linear path P_3 of length $k-3$ such that every edge in P_3 has at most one vertex of $P \cup P_1 \cup P_2$. Hence, for the same reason as in Observation 4.1, all the $free(P)$ vertices in $e_1 \cup e_{k-1}$, $free(P_1)$ vertices in $f_1 \cup f_{k-1}$, and $free(P_2)$ vertices in $g_1 \cup g_{k-1}$ are not in P_3. Denote by $h_1, h_2, \ldots, h_{k-3}$ the edges of P_3. Consider the edge e, which consists of a $free(P)$ vertex x in e_1, a $free(P_3)$ vertex in $h_1 \setminus (P_1 \cup P_2)$, and $s-2$ vertices disjoint with $P \cup P_1 \cup P_2 \cup P_3$. Then the color of e must be from $\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}\}$. And consider the edge e', which consists of a $free(P_1)$ vertex $y \neq x$ in $f_1 \cup f_{k-1}$ (we can find such a vertex y since P_1 is bad), a $free(P_3)$ vertex in $h_{k-3} \setminus (P_1 \cup P_2)$, and $s-2$ vertices disjoint with $P_1 \cup P_2 \cup P_3 \cup P \cup e$; then the color of e' is from $\{\beta_1, \beta_2, \ldots, \beta_{k-1}\}$. Moreover, $e \cup P_3 \cup e'$ is a rainbow $\mathcal{P}_{k-1}$. Now consider another edge e'', which consists of a $free(P_2)$ vertex $z \neq x, y$ in $g_1 \cup g_{k-1}$, a vertex in $e' \setminus (P_1 \cup P_3)$, and $s-2$ vertices disjoint with $P_1 \cup P_2 \cup P_3 \cup P \cup e \cup e'$; then e'' must have a color appearing in the rainbow loose path $e \cup P_3 \cup e'$. However, to avoid extending P_2 to a rainbow $\mathcal{P}_k$, the color of e'' should be one of $\{\gamma_1, \gamma_2, \ldots, \gamma_{k-3}\}$, a contradiction. Therefore, $\mathcal{F}$ contains no edge-disjoint bad loose paths of length $k-1$. ∎

So if $\mathcal{F}$ has a bad $\mathcal{P}_{k-1}$, denote by $\mathcal{F}_0$ the subgraph obtained by deleting all the $k-1$ edges of that $\mathcal{P}_{k-1}$ in $\mathcal{F}$; if $\mathcal{F}$ does not contain a bad $\mathcal{P}_{k-1}$, then we delete any $k-1$ edges of it and denote the subgraph remained by $\mathcal{F}_0$. Then, in either case,

$$\mathcal{F}_0 \text{ contains no bad } \mathcal{P}_{k-1}. \tag{5.1}$$

Therefore, $\mathcal{F}_0$ is $\mathbb{P}_{k-1}$-free and $|\mathcal{F}_0| = \binom{n}{s} - \binom{n-t+1}{s} + 3 - 2(k-1)$.

Note that $|\mathcal{F}_0| \sim (t-1)\binom{n}{s-1}$; by Theorem 2.5, we can find an $(s-1)$-graph $G^* \subset \partial\mathcal{F}_0$ with $|G^*| \sim \binom{n}{s-1}$ and a set L of $t-1$ vertices of $\mathcal{F}_0$ such that $L \cap V(G^*) = \emptyset$ and $e \cup \{v\} \in \mathcal{F}_0$ for any $(s-1)$-edge $e \in G^*$ and any $v \in L$. Moreover, $|\mathcal{F}_0 - L| = o(n^{s-1})$. Select a G^* with the maximum number of $(s-1)$-edges. Let the vertices in L be $v_1, v_2, \ldots, v_{t-1}$. Note that all the v_i for $i = 1, 2, \ldots, t-1$ are not $free(P)$ vertices in $e_1 \cup e_{k-1}$. Otherwise, let W be the vertex set of P; by Lemma 2.7, we can find an $(s-1)$-edge disjoint with W in G^*, and then this together with v_i will form an s-edge which extends P to a rainbow $\mathcal{P}_k$.

We divide the edges of $\mathcal{F}_0 - L$ into two types. Let Q denote the set of $free(P)$ vertices in $e_1 \cup e_{k-1}$. For an edge $e \in \mathcal{F}_0 - L$, we call it of *Type I* if $Q \subseteq e$ and of *Type II* otherwise. Now we estimate the number of edges of each type.

CLAIM 5.2. *There is no $\mathcal{P}_2$ in $\mathcal{F}_0 - L$ whose edges are all of Type II. Therefore, the number of edges of Type II is at most $\lfloor n/s \rfloor$.*

Proof. Suppose that there is a $\mathcal{P}_2$ with two edges of Type II in $\mathcal{F}_0 - L$, whose edges are denoted by h_1 and h_2 with colors γ_1 and γ_2, respectively. We can take $(s-1)$-edges a_0, b_0 in G^* such that a_0 and b_0 have exactly one common vertex u and are disjoint from P, h_1, and h_2. Let W be the vertex set of $P \cup h_1 \cup h_2 \cup a_0 \cup b_0$. By Lemma 2.7, we can find $(s-1)$-edges $\{a_i, b_i\}$ disjoint from W for $i = 1, \ldots, t-1$ such that for every i, a_i and b_i have exactly one common vertex, and for any $j \neq i$, $\{a_i, b_i\}$ and $\{a_j, b_j\}$ are vertex disjoint. Then

$$P' = a_0 \bigoplus_{i=1}^{t-2} (\{v_i\} \oplus a_i \oplus b_i) \oplus \{v_{t-1}\} \oplus a_{t-1}$$

is a $\mathcal{P}_{2t-2} = \mathcal{P}_{k-3}$. Let the edges of P' be $f_1, f_2, \ldots, f_{k-3}$ and the colors of edges be $\beta_1, \beta_2, \ldots, \beta_{k-3}$, respectively.

Consider an edge g, which consists of a vertex in $h_2 \setminus h_1$, the common vertex u of a_0 and b_0, and $s-2$ vertices disjoint from P, P', h_1, h_2, b_0, and b_{t-1}. So to prevent extending P' to a rainbow $\mathcal{P}_k$, the color of g is in $\{\gamma_1, \gamma_2, \beta_1, \beta_2, \ldots, \beta_{k-3}\}$. If the color of g is γ_1, consider the edge e, which consists of a vertex in $Q \setminus h_2$, a vertex in a_{t-1}, and $s-2$ vertices disjoint from P, P', h_1, h_2, g, b_0, and b_{t-1}; then the color of e is from $\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}\}$, and hence $h_2 \cup g \cup P' \cup e$ is a rainbow $\mathcal{P}_k$.

If the color of g is γ_2, we pick a vertex w in h_1 such that if $|Q| < s$, let $w \in h_1 \setminus Q$, and if $|Q| \geq s$, let w be an arbitrary vertex in h_1. Consider the edge e' consisting of w, the common vertex of a_{t-1} and b_{t-1}, and $s-2$ vertices disjoint from P, P', h_1, h_2, g, b_0, and b_{t-1}. Then the color of e' is from $\{\gamma_1, \gamma_2, \beta_1, \beta_2, \ldots, \beta_{k-3}\}$ because otherwise $g \cup P' \cup e' \cup h_1$ is a rainbow $\mathcal{P}_k$. If the color of e' is γ_2, then $h_1 \cup e' \cup P'$ is a rainbow $\mathcal{P}_{k-1}$. We obtain a rainbow $\mathcal{P}_k$ by adding an edge e'', which consists of a vertex in $Q \setminus h_1$, a $free(P')$ vertex in $f_1 \setminus \{u\}$, and $s-2$ vertices disjoint from P, P', h_1, h_2, g, e', b_0, and b_{t-1}. If the color of e' is γ_1, consider the edge e''' consisting of a vertex in $Q \setminus (g \cup e')$, a vertex in $g \setminus (h_2 \cup \{u\})$, and $s-2$ vertices disjoint from P, P', h_1, h_2, g, e', b_0, and b_{t-1}. Then the color of e''' is from $\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}\}$, and hence $e''' \cup g \cup P' \cup e'$ is a rainbow $\mathcal{P}_k$. If the color of e' is β_j for some j, let W be the vertex set of $h_1 \cup h_2 \cup e' \cup a_0 \cup b_0 \cup a_{t-1} \cup b_{t-1} \cup g$; by Lemma 2.7, we can find $(s-1)$-edges $\{a'_i, b'_i\}$ in G^*, which are disjoint from W for $i = 1, \ldots, t-1$, and

$$g \oplus b_0 \bigoplus_{i=1}^{t-2} (\{v_i\} \oplus a'_i \oplus b'_i) \oplus \{v_{t-1}\} \oplus b_{t-1} \oplus e' \oplus h_1$$

is a rainbow $\mathcal{P}_k$.

So assume instead the color of g is one of $\{\beta_1, \beta_2, \ldots, \beta_{k-3}\}$. Let W be the vertex set of $h_1 \cup h_2 \cup b_0 \cup b_{t-1} \cup g$. By Lemma 2.7, we can find $(s-1)$-edges $\{a''_i, b''_i\}$ disjoint from W for $i = 1, \ldots, t-1$, and

$$h_1 \oplus h_2 \oplus g \oplus b_0 \bigoplus_{i=1}^{t-2} (\{v_i\} \oplus a''_i \oplus b''_i) \oplus \{v_{t-1}\} \oplus b_{t-1}$$

is a rainbow $\mathcal{P}_k$. Hence, there is no $\mathcal{P}_2$ with two edges of Type II, and we have proved Claim 5.2. ∎

Now, we move to Type I edges. Recall that Q denotes the set of $free(P)$ vertices in $e_1 \cup e_{k-1}$. So if $s \leq |Q| \leq 2(s-1)$, the number of Type I edges is at most 1; if

$2 \le |Q| \le s-1$, then a rough counting shows that the number of Type I edges is at most $\binom{n-(t-1)-|Q|}{s-|Q|} \le \binom{n-t-1}{s-2}$. We further prove that there is no isolated vertex in $\mathcal{F}_0$. Indeed, if $\mathcal{F}_0$ has an isolated vertex, then, combining with Claim 5.2,

$$|\mathcal{F}_0| \le \binom{n-1}{s} - \binom{n-1-(t-1)}{s} + \binom{n-t-1}{s-2} + \left\lfloor \frac{n}{s} \right\rfloor,$$

which is less than $\binom{n}{s} - \binom{n-t+1}{s} + 3 - 2(k-1)$ for sufficiently large n, a contradiction. For the edges of Type I, we have the following claim.

Claim 5.3. *The number of edges of Type I is at most* 1.

Proof. If $s \le |Q| \le 2(s-1)$, then Claim 5.3 follows, and so we may assume $2 \le |Q| \le s-1$. Suppose to the contrary there are at least two edges of Type I. Then we can find a $\mathcal{P}_2$ with two edges of Type I, denoted by h_1 and h_2. Pick vertices $x \in h_2 \setminus h_1$ and $y \in h_1 \setminus h_2$. The number of edges containing exactly one of $\{x, y\}$ and one vertex in L and disjoint with $(h_1 \cup h_2) \setminus \{x, y\}$ is at least

$$2(t-1)\binom{n-(t-1)-2s}{s-2} > \binom{n-(t-1)-|Q|}{s-2} + \lfloor \frac{n}{s} \rfloor + 2(k-1) - 3,$$

which is at least $\binom{n-(t-1)-|Q|}{s-|Q|} + \lfloor \frac{n}{s} \rfloor + 2(k-1) - 3$, and so some of them must belong to $\mathcal{F}_0$. Suppose $e \in \mathcal{F}_0$ is such an edge and $v_j \in e \cap L$. Let W be the vertex set of $h_1 \cup h_2 \cup e$. By Lemma 2.7, we can find $(s-1)$-edges $\{a_i, b_i\}$ disjoint from W for $i = 1, \ldots, t-1$, and

$$h_1 \oplus h_2 \oplus e \oplus \{a_1, b_1\} \bigoplus_{i=1}^{j-1} (\{v_i\} \oplus a_{i+1} \oplus b_{i+1}) \bigoplus_{i=j+1}^{t-2} (\{v_i\} \oplus a_i \oplus b_i) \oplus \{v_{t-1}\} \oplus a_{t-1}$$

is a bad $\mathcal{P}_{k-1}$ in $\mathcal{F}_0$, a contradiction to (5.1). ∎

By Claims 5.2 and 5.3, we get that, in $\mathcal{F}_0$, there are at most $\lfloor n/s \rfloor + 1$ edges disjoint with L. So there are at least $|\mathcal{F}_0| - (\lfloor n/s \rfloor + 1) = \binom{n}{s} - \binom{n-t+1}{s} + 3 - 2(k-1) - \lfloor n/s \rfloor - 1 = \binom{n}{s} - \binom{n-t+1}{s} - 2(k-1) - \lfloor n/s \rfloor + 2$ edges in $\mathcal{F}_0$ containing vertices in L. Since there are $\binom{n}{s} - \binom{n-t+1}{s}$ edges containing vertices of L in $\mathcal{H}$, we have that there are at most $\lfloor n/s \rfloor + 2(k-1) - 2$ edges meeting L but not in $\mathcal{F}_0$. If there exists a vertex $v \in V(\mathcal{F}_0) \setminus L$ but v is an isolated vertex in G^*, then we have at least $\binom{n-(t-1)-1}{s-2} > \lfloor n/s \rfloor + 2(k-1) - 2$ edges meeting L but not belonging to $\mathcal{F}_0$, which is a contradiction. Hence, every vertex in $V(\mathcal{F}_0) \setminus L$ is contained in some edges of G^*.

In fact, to find a rainbow $\mathcal{P}_k$ in $\mathcal{H}$, we can make use of the suitable edges in $\mathcal{F}_0 - L$ and $P - L$ to extend a $\mathcal{P}_{k-3}$. We shall prove the following claim, which is analogous to Claim 4.7.

Claim 5.4. (i) *For $k \ge 7$, if there are three edges f, g, h in $\mathcal{H} - L$ with distinct colors such that f, g, h form a $\mathcal{P}_3$ or f, g form a $\mathcal{P}_2$ and h is disjoint with $f \cup g$ or f, g, h are disjoint with each other, then we can find a rainbow $\mathcal{P}_k$ in $\mathcal{H}$.*

(ii) *For $k = 5$, if there exists either a rainbow $\mathcal{P}_3$ or a $\mathcal{P}_2$ plus a disjoint edge with all three edges having distinct colors in $\mathcal{H} - L$, then we can find a rainbow $\mathcal{P}_k$ in $\mathcal{H}$.*

Proof. Similar to the proof of Claim 4.7, we denote f', g', h' to be the edges in $\mathcal{G}$ with the same color of f, g, h, respectively. If the edge e is in $\mathcal{G}$ for some $e \in \{f, g, h\}$, we have $e' = e$.

(i) Let $k \geq 7$. Suppose that there are three edges f, g, h in $\mathcal{H} - L$ with distinct colors such that f, g, h form a $\mathcal{P}_3$. In G^*, there exists an $(s-1)$-edge a_0 containing a vertex x in $h \setminus g$ and disjoint with $(f' \cup g' \cup h' \cup f \cup g \cup h) \setminus \{x\}$. Let W consist of the vertices in $f' \cup g' \cup h' \cup f \cup g \cup h \cup a_0$. By Lemma 2.7, we can find $(s-1)$-edges $\{a_i, b_i\}$ disjoint from W for $i = 1, \ldots, t-1$. Then

$$f \oplus g \oplus h \oplus a_0 \bigoplus_{i=1}^{t-2} (\{v_i\} \oplus a_i \oplus b_i) \oplus \{v_{t-1}\} \oplus a_{t-1}$$

is a rainbow $\mathcal{P}_k$ in $\mathcal{H}$.

If f, g form a $\mathcal{P}_2$, h is disjoint with $f \cup g$. In G^*, there exists an $(s-1)$-edge a_0 containing a vertex x in $g \setminus f$ and disjoint with $(f' \cup g' \cup h' \cup g \cup h \cup f) \setminus \{x\}$. And there exists an $(s-1)$-edge b_0 containing a vertex y in h and disjoint with $(f' \cup g' \cup h' \cup f \cup g \cup h \cup a_0) \setminus \{y\}$. Let W consist of the vertices in $f' \cup g' \cup h' \cup f \cup g \cup h \cup a_0 \cup b_0$. By Lemma 2.7, we can find $(s-1)$-edges $\{a_i, b_i\}$ disjoint from W for $i = 1, \ldots, t-1$. Then

$$f \oplus g \oplus a_0 \bigoplus_{i=1}^{t-2} (\{v_i\} \oplus a_i \oplus b_i) \oplus \{v_{t-1}\} \oplus b_0 \oplus h$$

is a rainbow $\mathcal{P}_k$ in $\mathcal{H}$.

Assume that the three edges f, g, h are disjoint with each other. In G^*, there exists an $(s-1)$-edge a_0 containing a vertex x in f and disjoint with $(f' \cup g' \cup h' \cup f \cup g \cup h) \setminus \{x\}$. And we can find $(s-1)$-edges a_0', b_0' in G^* such that a_0' contains a vertex y_1 in g and disjoint with $(f' \cup g' \cup h' \cup f \cup g \cup h \cup a_0) \setminus \{y_1\}$ and b_0' contains a vertex $y_2 \neq y_1$ in g and disjoint with $(f' \cup g' \cup h' \cup f \cup g \cup h \cup a_0 \cup a_0') \setminus \{y_2\}$. Moreover, there exists an $(s-1)$-edge b_0 containing a vertex z in h and disjoint with $(f' \cup g' \cup h' \cup f \cup g \cup h \cup a_0 \cup a_0' \cup b_0') \setminus \{z\}$. Let W consist of the vertices in $f' \cup g' \cup h' \cup f \cup g \cup h \cup a_0 \cup b_0 \cup a_0' \cup b_0'$. By Lemma 2.7, we can find $(s-1)$-edges $\{a_i, b_i\}$ disjoint from W for $i = 1, \ldots, t-1$. Then

$$f \oplus a_0 \oplus \{v_1\} \oplus a_0' \oplus g \oplus b_0' \bigoplus_{i=2}^{t-2} (\{v_i\} \oplus a_{i-1} \oplus b_{i-1}) \oplus \{v_{t-1}\} \oplus b_0 \oplus h$$

is a rainbow $\mathcal{P}_k$ in $\mathcal{H}$.

(ii) For $k = 5$, the proof is identical to (i) and thus omitted. ◻

Actually, in P, there are at most $2(t-1)$ edges containing vertices of L, so we can find at least two edges in $P - L$. However, there are not three edges of P satisfying the condition described in Claim 5.4, and so we derive that there are exactly two edges in $P - L$.

Case A. $k \geq 7$. If $|\mathcal{F}_0 - L| > 0$, then the two edges in $P - L$ must be consecutive by Claim 5.4. Let e_i and e_{i+1} be such two edges. We take an edge $h \in \mathcal{F}_0 - L$, then select an edge $g \in \mathcal{H} - L$ such that g is disjoint with P and h. If the color of g is α_j for some j, then the color of g is either different with e_i or different with e_{i+1}. Suppose the colors of g and e_i are different; then we have three edges e_i, g, h satisfying the condition of Claim 5.4, and so we can find a rainbow $\mathcal{P}_k$ in $\mathcal{H}$. If the color of g is different with both e_i and e_{i+1}, then the three edges e_i, e_{i+1}, and g satisfy condition of Claim 5.4, in which we can find a rainbow $\mathcal{P}_k$ in $\mathcal{H}$ similarly.

Assume instead that $|\mathcal{F}_0 - L| = 0$. Then all the $\binom{n}{s} - \binom{n-t+1}{s} + 3 - 2(k-1)$ edges in $\mathcal{F}_0$ contain vertices in L, and P has $k-3$ edges containing vertices in L. Since the number of edges containing vertices of L is at most $\binom{n}{s} - \binom{n-t+1}{s}$, in $\mathcal{F} \setminus \mathcal{F}_0$ there are at most $k-2$ edges containing vertices in L. Since $|\mathcal{F}\setminus\mathcal{F}_0|=k-1$, there is an edge f in $\mathcal{F}\setminus\mathcal{F}_0$ such that $f\cap L=\emptyset$. Furthermore, the color of f is different with any other edges in $\mathcal{F}$. Applying the same proof to the case that $|\mathcal{F}_0-L|>0$, by replacing the edge $h\in\mathcal{F}_0-L$ with f, we can find a rainbow $\mathcal{P}_k$ in $\mathcal{H}$ as well.

Case B. $k=5$. As noticed above, there are exactly two edges, say, e_i and e_j, in $P-L$.

If $|\mathcal{F}_0-L|=0$, then all the $\binom{n}{s}-\binom{n-t+1}{s}+3-2(k-1)$ edges in $\mathcal{F}_0$ contain vertices in L. Also P has $k-3$ edges containing vertices in L. Since the number of edges containing vertices of L is at most $\binom{n}{s}-\binom{n-t+1}{s}$, in $\mathcal{F}\setminus\mathcal{F}_0$ there are at most $k-2$ edges containing vertices in L. Since $|\mathcal{F}\setminus\mathcal{F}_0|=k-1$, there is an edge h in $\mathcal{F}\setminus\mathcal{F}_0$ such that $h\cap L=\emptyset$. Then the color of h is different with any other edges in $\mathcal{F}$ and different with the colors appearing in P. If $|\mathcal{F}_0-L|>0$, then there exists an edge $f_1\in\mathcal{F}_0-L$. We set f to be an edge such that $f=h$ if $|\mathcal{F}_0-L|=0$ and $f=f_1\in\mathcal{F}_0-L$ if $|\mathcal{F}_0-L|>0$. Then by Claim 5.4, f satisfies that

$$\text{either } f\cap e_i=\emptyset,\ f\cap e_j=\emptyset \quad\text{or}\quad f\cap e_i\neq\emptyset,\ f\cap e_j\neq\emptyset. \tag{5.2}$$

For the former case of (5.2), pick an edge $g\in\mathcal{H}-L$ such that $g\cap e_i\neq\emptyset$, $g\cap f\neq\emptyset$, and g is disjoint with e_j. Consider the color of g. If the color of g is α_i, then the three edges e_j, g, f are applied for Claim 5.4; if the color of g is α_j, then the three edges e_i, g, f are applied for Claim 5.4; if the color of g is different with both α_i and α_j, then the three edges e_i, e_j, g are applied for Claim 5.4. Therefore, we can always find a rainbow $\mathcal{P}_k$ in this case.

For the latter case of (5.2) that $f\cap e_i\neq\emptyset$ and $f\cap e_j\neq\emptyset$, we must have e_i and e_j are consecutive in P by Claim 5.4. Let g be an edge in $\mathcal{H}-L$ such that g is disjoint with e_i, e_j, and f. If the color of g is α_i or α_j, then the three edges f, e_j, g or the three edges f, e_i, g are applied for Claim 5.4. If the color of g is neither α_i nor α_j, then the three edges e_i, e_j, g are applied for Claim 5.4, and so we can still find a rainbow $\mathcal{P}_k$ in $\mathcal{H}$. This completes the proof of Theorem 1.3. ∎

6. Linear cycle—Proof of Theorem 1.4. Let $\mathcal{H}$ be a complete s-uniform hypergraph on n vertices. Denote by V the vertex set of $\mathcal{H}$. Let

$$g(n,s,k)=\begin{cases}\binom{n}{s}-\binom{n-t+1}{s}+2 & \text{if } k=2t,\\ \binom{n}{s}-\binom{n-t+1}{s}+\binom{n-t-1}{s-2}+2 & \text{if } k=2t+1.\end{cases}$$

To prove Theorem 1.4, we show that $g(n,s,k)$ is both the lower and the upper bound for $ar(n,s,\mathbb{C}_k)$.

The lower bound follows from Proposition 4.1 by constructing a coloring of $\mathcal{H}$ using the extreme s-graphs without a $\mathbb{P}_{k-1}$ in Theorem 2.2.

For the upper bound, we argue by contradiction and suppose that there is a coloring of $\mathcal{H}$ using $g(n,s,k)$ colors yielding no rainbow $\mathbb{C}_k$. Since $g(n,s,k)=ar(n,s,\mathbb{P}_k)$ and by Theorem 1.2, there is a rainbow linear path P of length k in $\mathcal{H}$. Let $\mathcal{G}$ be a spanning subgraph of $\mathcal{H}$ with $P\subset\mathcal{G}$ such that $|\mathcal{G}|=g(n,s,k)$ and each color appears on exactly one edge of $\mathcal{G}$. Denote by $e_1,e_2,\ldots,e_k$ the edges of P, and let $\mathcal{F}=\mathcal{G}-\bigcup_{i=1}^{k-1}e_i$. Clearly, $\mathcal{F}$ is $\mathbb{C}_k$-free. The following claim tells us more information about $\mathcal{F}$ when $k=2t+1$.

Claim 6.1. *When $k=2t+1$, if there is a linear path P_1 of length $k-1$ in $\mathcal{F}$, then $\mathcal{F}-E(P_1)$ is $\mathbb{P}_{k-1}$-free.*

Proof. Assume that there is a linear path P_1 of length $k-1$ in $\mathcal{F}$. Suppose, by contradiction, that there is a linear path P_2 of length $k-1$ in $\mathcal{F} - E(P_1)$. Denote the edges of P_1 by $f_1, f_2, \ldots, f_{k-1}$ and the edges of P_2 by $g_1, g_2, \ldots, g_{k-1}$, respectively. We obtain an s-graph $\mathcal{F}'$ by deleting edge set $E(P_1) \cup E(P_2)$ and all the edges containing at least two vertices of $\bigcup_{i=i}^{k-1} e_i \cup E(P_1) \cup E(P_2)$ from $\mathcal{F}$. Let c denote the number of vertices of $V(P) \cup V(P_1) \cup V(P_2)$. Then $c \le ks - (k-1) + 2[(k-1)s - (k-2)]$, and so we have

$$|\mathcal{F}'| \ge |\mathcal{F}| - 2(k-1) - \sum_{i=2}^{s} \binom{c}{i}\binom{n-c}{s-i} > ex(n, s, \mathbb{P}_{k-3})$$

for sufficiently large n. Thus, we have a linear path P_3 of length $k-3$ in $\mathcal{F}'$. Denote by $h_1, h_2, \ldots, h_{k-3}$ the edges of P_3. Note that there are at most $k-3$ vertices in $V(P_3) \cap (V(P) \cup V(P_1) \cup V(P_2))$. Since $s-1 \ge k-3+5$ and every path has two disjoint end edges, we can always choose distinct vertices $v_1, v_2, \ldots, v_6$ such that the following holds: $v_1 \in e_1$ and $v_2 \in e_{k-1}$ are $free(P)$ vertices, $v_3 \in f_1$ and $v_4 \in f_{k-1}$ are $free(P_1)$ vertices, $v_5 \in g_1$ and $v_6 \in g_{k-1}$ are $free(P_2)$ vertices, and $v_i \notin P_3$ for each $1 \le i \le 6$.

Select $u_1 \in h_1 \setminus (V(P) \cup V(P_1) \cup V(P_2))$ and $u_2 \in h_{k-3} \setminus (V(P) \cup V(P_1) \cup V(P_2))$. Consider the edge e' consisting of v_1, v_2, u_1 and $s-3$ vertices disjoint with $P \cup P_1 \cup P_2 \cup P_3$. Then e' has a color appearing in $\bigcup_{i=1}^{k-1} e_i$; otherwise, $\bigcup_{i=1}^{k-1} e_i \cup e'$ is a rainbow $\mathbb{C}_k$, a contradiction. Similarly, the edge e'', which consists of v_3, v_4, a vertex x in $e' \setminus \{v_1, v_2, u_1\}$, and $s-3$ vertices disjoint with $P \cup P_1 \cup P_2 \cup P_3 \cup e'$, is colored with a color appearing on P_1. In addition, consider the edge e''', which consists of v_5, v_6, u_2, a vertex in $e'' \setminus \{v_3, v_4, x\}$, and $s-4$ vertices disjoint with $P \cup P_1 \cup P_2 \cup P_3 \cup e' \cup e''$. We get that e''' is colored with a color appearing on P_2. Now it follows that $P_3 \cup e' \cup e'' \cup e'''$ forms a rainbow $\mathbb{C}_k$, a contradiction. This proves the claim. ∎

So when $k = 2t+1$, if $\mathcal{F}$ has a $\mathbb{P}_{k-1}$, then we denote by $\mathcal{F}_0$ the subgraph obtained by deleting all the $k-1$ edges of that $\mathbb{P}_{k-1}$ from $\mathcal{F}$, and so $\mathcal{F}_0$ is $\mathbb{P}_{k-1}$-free by Claim 6.1. If there is no $\mathbb{P}_{k-1}$ in $\mathcal{F}$, we delete any $k-1$ edges of $\mathcal{F}$ and denote the subgraph remained by $\mathcal{F}_0$. When $k = 2t$, we obtain $\mathcal{F}_0$ by deleting any $k-1$ edges from $\mathcal{F}$. In any case, we obtain a subgraph $\mathcal{F}_0$ with $|\mathcal{F}_0| = |\mathcal{F}| - [(k-1)s - (k-2)] \sim (t-1)\binom{n}{s-1}$. Moreover, $\mathcal{F}_0$ is $\mathbb{C}_k$-free for $k = 2t$, and $\mathcal{F}_0$ is $\mathbb{P}_{k-1}$-free for $k = 2t+1$. Thus, we can apply Theorem 2.5 to $\mathcal{F}_0$ whenever k is even or odd. By Theorem 2.5, we can find an $(s-1)$-graph $G^* \subset \partial\mathcal{F}_0$ with the maximum number of edges such that $|G^*| \sim \binom{n}{s-1}$ and there is a set L of $t-1$ vertices of $\mathcal{F}_0$ such that $L \cap V(G^*) = \emptyset$ and $e \cup \{v\} \in \mathcal{F}_0$ for any $(s-1)$-edge $e \in G^*$ and any $v \in L$. Moreover, $|\mathcal{F}_0 - L| = o(n^{s-1})$. Denote $L = \{v_1, v_2, \ldots, v_{t-1}\}$.

An s-edge e is called a *missing-edge* if e contains vertices of L and $e \notin \mathcal{F}_0$. Let M be the set of all the missing-edges, and let $m = |M|$ denote the number of missing-edges.

Since $|\mathcal{F}_0| - |\mathcal{F}_0 - L| + m = \binom{n}{s} - \binom{n-t+1}{s}$, we have

$$m = \begin{cases} |\mathcal{F}_0 - L| - 2 + 2(k-1) & \text{if } k = 2t, \\ |\mathcal{F}_0 - L| - \binom{n-t-1}{s-2} - 2 + 2(k-1) & \text{if } k = 2t+1. \end{cases} \tag{6.1}$$

So it follows from $|\mathcal{F}_0 - L| = o(n^{s-1})$ that

$$m = o(n^{s-1}). \tag{6.2}$$

We divide the remaining proof into two parts depending on the value of m. We will derive contradictions whenever $m \le \binom{n-8s-t+1}{s-2} - 1$ or $m > \binom{n-8s-t+1}{s-2} - 1$.

6.1. The case when m is small: $m \le \binom{n-8s-t+1}{s-2} - 1$. In this subsection, we assume that $m \le \binom{n-8s-t+1}{s-2} - 1$. The proof applies similar ideas as the proof of Theorem 1.2, where we manage to find a certain rainbow path of large length obtained from Lemma 2.7 and then extend to a rainbow $\mathbb{C}_k$ by selecting some specific edges in $\mathcal{G} - L$. However, the differences with Theorem 1.2 is big enough in many details, leading us to rewrite a complete proof of this case.

We start to prove claims below similar to Claims 4.3, 4.6, and 4.7.

Claim 6.2. *Every vertex $v \in V \setminus L$ belongs to G^*. Moreover, for any vertex subset S of V with $|S| \le 8s$ and $v \notin S$, there is an $(s-1)$-edge $g \in G^*$ such that $v \in g$ and g is disjoint with S.*

Proof. If there is a vertex v such that $v \in V \setminus L$ but $v \notin G^*$, then we have at least $\binom{n-(t-1)-1}{s-2}$ edges meeting L but not belonging to $\mathcal{F}_0$. This implies the number of missing-edges is at least $\binom{n-(t-1)-1}{s-2} > \binom{n-8s-t+1}{s-2} - 1$, a contradiction to our assumption that $m \le \binom{n-8s-t+1}{s-2} - 1$.

For the "moreover" part, if in G^* every $(s-1)$-set containing v meets S, then there are at least $\binom{n-|S|-t+1}{s-2} > m$ missing-edges, a contradiction. Therefore, there must exist an $(s-1)$-edge g in G^* containing v and disjoint with S. ∎

Claim 6.3. (a) *If $k = 2t \ge 8$, then there are no two edges e, f in $\mathcal{G} - L$ such that $|e \cap f| = 1$ or $e \cap f = \emptyset$.*

(b) *If $k = 2t + 1 \ge 11$, then there are no three edges e, f, h in $\mathcal{G} - L$ satisfying one of the following conditions:*

(i) *e, f, h form a $\mathbb{P}_3$;*

(ii) *e, f form a $\mathbb{P}_2$, and h is disjoint with $e \cup f$;*

(iii) *e, f, h are pairwise disjoint.*

Proof. (a) Let $k = 2t$. Suppose to the contrary that there exist two edges e, f in $\mathcal{G} - L$ such that $|e \cap f| = 1$. Let $u \in e \setminus f$, $v \in f \setminus e$. By Claim 6.2, $u, v \in V(G^*)$, and we can find an $(s-1)$-edge a_0 in G^* such that $u \in a_0$ and a_0 is disjoint with $e \cup f \setminus \{u\}$. Applying Claim 6.2 again, there is an $(s-1)$-edge b_0 in G^* such that $v \in b_0$ and b_0 is disjoint with $(e \cup f \cup a_0) \setminus \{v\}$. Let W be the vertex set of $e \cup f \cup a_0 \cup b_0$. By Lemma 2.7, we can find $(s-1)$-edge pairs $\{a_i, b_i\}$ disjoint from W for $i = 1, \ldots, t-1$ such that for every i, a_i and b_i have exactly one common vertex, and for any $j \ne i$, $\{a_i, b_i\}$ and $\{a_j, b_j\}$ are vertex disjoint. Then

$$P' = a_0 \bigoplus_{i=1}^{t-2} (\{v_i\} \oplus a_i \oplus b_i) \oplus \{v_{t-1}\} \oplus b_0$$

is a $\mathbb{P}_{k-2}$ in $\mathcal{F}_0$. Adding e, f to P', we obtain a rainbow $\mathbb{C}_k$, which is a contradiction.

Assume, by contradiction, that there are two edges e, f in $\mathcal{G} - L$ such that $e \cap f = \emptyset$. Select four distinct vertices x, y, z, w such that $x, y \in e$ and $z, w \in f$. By Claim 6.2, we can find an $(s-1)$-edge a in G^* such that $x \in a$ and a is disjoint with $(e \cup f) \setminus \{x\}$. Applying Claim 6.2 repeatedly, we can find $(s-1)$-edges b, a', b' one by one in G^* such that $y \in b$ and b is disjoint with $(e \setminus \{y\}) \cup f \cup a$, $z \in a'$ and a' is disjoint with $e \cup (f \setminus \{z\}) \cup a \cup b$, and $w \in b'$ and b' is disjoint with $e \cup (f \setminus \{w\}) \cup a \cup b \cup a'$. Let W be the vertex set of $e \cup f \cup a \cup b \cup a' \cup b'$. By Lemma 2.7, we can find $(s-1)$-edge pairs $\{a_i, b_i\}$ disjoint from W for $i = 1, \dots, t-1$. Then

$$P' = e \oplus b \oplus \{v_1\} \oplus a' \oplus f$$

is a $\mathbb{P}_4$ in $\mathcal{G}$, and

$$P'' = a \bigoplus_{i=2}^{t-2} (\{v_i\} \oplus a_{i-1} \oplus b_{i-1}) \oplus \{v_{t-1}\} \oplus b'$$

is a $\mathbb{P}_{k-4}$ in $\mathcal{F}_0$. Furthermore, $P' \cup P''$ forms a rainbow $\mathbb{C}_k$, a contradiction. This proves (a).

(b) Let $k = 2t+1$. We shall derive a contradiction by assuming that one of the conditions (i), (ii), (iii) holds.

(i) Assume that there is a linear path P_1 with three consecutive edges e, f, h in $\mathcal{G} - L$. Take two $free(P_1)$ vertices u, v such that $u \in e$ and $v \in h$. By Claim 6.2, $u, v \in V(G^*)$, and we can find an $(s-1)$-edge a_0 in G^* such that $u \in a_0$ and a_0 is disjoint with $(e \setminus \{u\}) \cup f \cup h$. Also, there is an $(s-1)$-edge b_0 in G^* such that $v \in b_0$ and b_0 is disjoint with $e \cup f \cup (h \setminus \{v\}) \cup a_0$. Let W be the vertex set of $e \cup f \cup h \cup a_0 \cup b_0$. By Lemma 2.7, we can find $(s-1)$-edge pairs $\{a_i, b_i\}$ disjoint from W for $i = 1, \dots, t-1$. Then

$$a_0 \bigoplus_{i=1}^{t-2} (\{v_i\} \oplus a_i \oplus b_i) \oplus \{v_{t-1}\} \oplus b_0$$

is a $\mathbb{P}_{k-3}$ in $\mathcal{F}_0$. Adding e, f, h to that $\mathbb{P}_{k-3}$ results in a rainbow $\mathbb{C}_k$, a contradiction.

(ii) Suppose there are three edges e, f, h in $\mathcal{G} - L$, satisfying that e, f form a $\mathbb{P}_2$ and h is disjoint with $e \cup f$. Take four distinct vertices x, y, z, w such that $x \in e \setminus f$, $y \in f \setminus e$, and $z, w \in h$. By Claim 6.2, we can find an $(s-1)$-edge a in G^* such that $x \in a$ and a is disjoint with $(e \setminus \{x\}) \cup f \cup h$. Applying Claim 6.2 repeatedly, we can find $(s-1)$-edges b, a', b' in G^* such that $y \in b$ and b is disjoint with $e \cup (f \setminus \{y\}) \cup a \cup h$, $z \in a'$ and a' is disjoint with $e \cup f \cup (h \setminus \{z\}) \cup a \cup b$, and $w \in b'$ and b' is disjoint with $e \cup f \cup (h \setminus \{w\}) \cup a \cup b \cup a'$. Let W be the vertex set of $e \cup f \cup h \cup a \cup b \cup a' \cup b'$. By Lemma 2.7, we can find $(s-1)$-edge pairs $\{a_i, b_i\}$ disjoint from W for $i = 1, \dots, t-1$. Then

$$P' = e \oplus f \oplus b \oplus \{v_1\} \oplus a' \oplus h$$

is a $\mathbb{P}_5$ in $\mathcal{G}$, and

$$P'' = a \bigoplus_{i=2}^{t-2} (\{v_i\} \oplus a_{i-1} \oplus b_{i-1}) \oplus \{v_{t-1}\} \oplus b'$$

is a $\mathbb{P}_{k-5}$ in $\mathcal{F}_0$. Thus, $P' \cup P''$ is a rainbow $\mathbb{C}_k$, a contradiction.

(iii) Suppose that there are three pairwise disjoint edges e, f, h in $\mathcal{G} - L$. Take distinct vertices x, y, z, w, u, v such that $x, y \in e$, $z, w \in f$, and $u, v \in h$. By applying Claim 6.2 repeatedly, we can find $(s-1)$-edges a, b, a', b', a'', b'' in G^* such that $x \in a$ and a is disjoint with $(e \setminus \{x\}) \cup f \cup h$, $y \in b$, and b is disjoint with $(e \setminus \{y\}) \cup f \cup h \cup a$, $z \in a'$ and a' is disjoint with $e \cup (f \setminus \{z\}) \cup h \cup a \cup b$, $w \in b'$ and b' is disjoint with $e \cup (f \setminus \{w\}) \cup h \cup a \cup b \cup a'$, $u \in a''$ and a'' is disjoint with $e \cup f \cup (h \setminus \{u\}) \cup a \cup b \cup a' \cup b'$, and $v \in b''$ and b'' is disjoint with $e \cup f \cup (h \setminus \{v\}) \cup a \cup b \cup a' \cup b' \cup a''$. Note that the size of vertex set S in applying Claim 6.2 is at most $8s$ as required. Let W be the vertex set of $e \cup f \cup h \cup a \cup b \cup a' \cup b' \cup a'' \cup b''$. By Lemma 2.7, we can find $(s-1)$-edges $\{a_i, b_i\}$ disjoint from W for $i = 1, \ldots, t-1$. Then

$$P' = e \oplus b \oplus \{v_1\} \oplus a' \oplus f \oplus b' \oplus \{v_2\} \oplus a'' \oplus h$$

is a $\mathbb{P}_7$ in $\mathcal{G}$, and

$$P'' = a \bigoplus_{i=3}^{t-2} (\{v_i\} \oplus a_{i-1} \oplus b_{i-1}) \oplus \{v_{t-1}\} \oplus b''$$

is a $\mathbb{P}_{k-7}$ in $\mathcal{F}_0$. Therefore, $P' \cup P''$ is a rainbow $\mathbb{C}_k$, a contradiction. This completes the proof of Claim 6.3. ∎

With the aid of Theorem 1.2, it is time to finish the proof of this case. Recall that P is a linear path of length k in $\mathcal{H}$. Since there are at most $2(t-1)$ edges meeting L in P, we have that there are at least two edges in $P - L \subset \mathcal{G} - L$ when $k = 2t$ and at least three edges in $P - L$ when $k = 2t+1$. Hence, we obtain edges satisfying the conditions of Claim 6.3, which is a contradiction.

6.2. The case when m is relatively large: $m > \binom{n-8s-t+1}{s-2} - 1$. In this subsection, we assume that $m > \binom{n-8s-t+1}{s-2} - 1$ to complete the proof. In this case, the edges meeting L in $\mathcal{F}_0$ may not be enough to make similar arguments as Claim 6.2, and it seems that we cannot use Lemma 2.7 directly to find a rainbow $\mathbb{C}_k$ as before. Our new strategy is to search a dense structure playing a similar role as Lemma 2.7, which is motivated by some ideas in [33]. The key ingredient is to find some $(s-2)$-sets of V such that each of them can form rainbow edges with every vertex in L and a large number of other vertices. We shall eventually use these substructures to establish certain desired paths or cycles.

For any vertex set Z of a hypergraph H, the *degree of Z in H*, denoted by $d_H(Z)$, is the number of edges containing the entire set Z in H.

In the following, we denote $K = 2s + k - 3$ for convenience.

Claim 6.4. *There are K pairwise disjoint $(s-2)$-sets T_i $(i = 1, \ldots, K)$ in $V \setminus L$ such that for every $i \in [K]$ and $j \in [t-1]$, we have*

$$d_{\mathcal{F}_0}(T_i \cup \{v_j\}) \geq n - s + 1 - \frac{k(s-1)m}{\binom{n-t+1}{s-2}}.$$

Proof. For any $i \in [K]$ and $j \in [t-1]$, let $d_{\overline{\mathcal{F}_0}}(T_i \cup \{v_j\})$ denote the number of s-sets e such that $T_i \cup \{v_j\} \subset e$, but $e \notin \mathcal{F}_0$. That is the number of missing-edges containing $T_i \cup \{v_j\}$. Thus, we have $d_{\mathcal{F}_0}(T_i \cup \{v_j\}) + d_{\overline{\mathcal{F}_0}}(T_i \cup \{v_j\}) = n - s + 1$.

Consider an $(s-2)$-set R of $V \setminus L$ selected uniformly randomly from all $(s-2)$-sets of $V \setminus L$. Let $X_i = d_{\overline{\mathcal{F}_0}}(R \cup \{v_i\})$. For every s-set $e \notin \mathcal{F}_0$, let

$$X_i(e) = \begin{cases} 1 & \text{if } R \cup \{v_i\} \subset e, \\ 0 & \text{if } R \cup \{v_i\} \not\subseteq e. \end{cases}$$

Then the expectation of X_i is

$$\mathbb{E}(X_i) = \sum_{e \notin \mathcal{F}_0} \mathbb{E}\left(X_i\left(e\right)\right) = \sum_{e \in M} \mathbb{E}\left(X_i\left(e\right)\right) \leq \frac{m\binom{s-1}{s-2}}{\binom{n-t+1}{s-2}} = \frac{(s-1)m}{\binom{n-t+1}{s-2}}.$$

By Markov's inequality, we have

$$\Pr\left[X_i > k\frac{(s-1)m}{\binom{n-t+1}{s-2}}\right] < \frac{1}{k}.$$

Hence,

$$\Pr\left[\exists j \text{ such that } X_j > k\frac{(s-1)m}{\binom{n-t+1}{s-2}}\right] \leq \sum_{i=1}^{t-1}\Pr\left[X_i > k\frac{(s-1)m}{\binom{n-t+1}{s-2}}\right] < \frac{t-1}{k} < \frac{1}{2},$$

which implies that there are at least $\frac{1}{2}\binom{n-t+1}{s-2}$ such $(s-2)$-sets R's satisfying $d_{\mathcal{F}_0}(R\cup\{v_i\}) = n-s+1-d_{\overline{\mathcal{F}_0}}(R\cup\{v_i\}) \geq n-s+1-k\frac{(s-1)m}{\binom{n-t+1}{s-2}}$ for all $i \in [t-1]$. Among those $\frac{1}{2}\binom{n-t+1}{s-2}$ R's, we pick pairwise disjoint $(s-2)$-sets greedily as many as possible. Let ℓ be the largest number that we can pick pairwise disjoint $R_1, R_2, \ldots, R_\ell$. We show that $\ell \geq K$. In fact, if $\ell < K$, then the number of $(s-2)$-sets meeting $\bigcup_{j=1}^{\ell} R_j$ is at most

$$\sum_{r=1}^{s-2}\binom{\ell(s-2)}{r}\binom{n-t+1-\ell(s-2)}{s-2-r} < \frac{1}{2}\binom{n-t+1}{s-2}.$$

So we can select $R_{\ell+1}$ from the remained R's such that $R_{\ell+1}$ is disjoint with $\bigcup_{j=1}^{\ell} R_j$, a contradiction. Hence, $\ell \geq K$, and we can find K $(s-2)$-sets described in Claim 6.4. ∎

Let $T = \bigcup_{j=1}^{K} T_j$, and let U denote the vertex subset of $V \setminus (L \cup T)$ such that for every $u \in U$,

the edge $T_i \cup \{v_j\} \cup \{u\}$ is belonging to $\mathcal{F}_0$

for all $i \in [K]$, $j \in [t-1]$.

Claim 6.5. *We have*

$$|U| \geq n - K(s-2) - (t-1) - \frac{K(t-1)(s-1)km}{\binom{n-t+1}{s-2}}.$$

Proof. By Claim 6.4, for every v_j and T_i, the number of vertex x such that the edge $\{\{x\}\cup\{v_j\}\cup T_i\} \notin \mathcal{F}_0$ is bounded by $n-s+1-d_{\mathcal{F}_0}(T_i\cup\{v_j\}) \leq k\frac{(s-1)m}{\binom{n-t+1}{s-2}}$. So we have

$$|U| \geq n - K(s-2) - (t-1) - K(t-1)\frac{(s-1)km}{\binom{n-t+1}{s-2}}$$

as required. ∎

The following Claim 6.6 plays the role of Claim 6.3 in the previous subsection. The difference is that, in Claim 6.3, we assemble s-edges with $(s-1)$-sets in G^* and vertices in L, but now we use T_i ($i \in [K]$) and some vertices in L and U to form desired rainbow s-edges. Since $\mathcal{F}_0 \subseteq \mathcal{G} - (\bigcup_{i=1}^{k-1} e_i)$, the colors appearing in $\mathcal{F}_0$ are distinct with colors appearing in $\bigcup_{i=1}^{k-1} e_i$. Thus, we can also use edges which have colors appearing in $\bigcup_{i=1}^{k-1} e_i$, along with edges in $\mathcal{F}_0$, to build rainbow paths and cycles.

Let J denote the set of edges in $E(\mathcal{H}-L) \setminus E(P)$ which are received colors appearing in $\bigcup_{i=1}^{k-1} e_i$.

Claim 6.6. (1) *If $k = 2t$, then there are no two edges $e, f \in (\mathcal{F}_0 - L) \cup J$ such that e, f form a rainbow $\mathbb{P}_2$ with $|(f\setminus e)\cap U| \geq 1$ and $|(e\setminus f)\cap U| \geq 1$.*

(2) *If $k = 2t+1$, then there are no two edges $e, f \in \mathcal{F}_0 - L$ such that e, f form a $\mathbb{P}_2$, with $|(f\setminus e)\cap U| \geq 1$ or $|(e\setminus f)\cap U| \geq 1$.*

Proof. (1) For $k = 2t$, if there are two edges $e, f \in (\mathcal{F}_0 - L) \cup J$ such that e, f form a rainbow $\mathbb{P}_2$ with $|(f \setminus e) \cap U| \geq 1$ and $|(e \setminus f) \cap U| \geq 1$, let $x \in (f \setminus e) \cap U$ and $y \in (e \setminus f) \cap U$. Since T is disjoint with U, there are at most $(2s-3)$ such T_i's that contain vertices of e or f. Hence, there are at least $K - (2s-3) = k > k-2$ such T_i's that are disjoint with e and f. Suppose, without loss of generality, that T_i is disjoint with e and f for each $i \in [k-2]$. Then by the definitions of T and U, there is a rainbow $\mathbb{P}_{k-2}$ in $\mathcal{F}_0$ with edges $h_1, h_2, \ldots, h_{k-2}$ such that

$$\begin{aligned}&h_1 = \{x\} \cup T_1 \cup \{v_1\}, h_2 = \{v_1\} \cup T_2 \cup \{u_1\}, h_3 = \{u_1\} \cup T_3 \cup \{v_2\},\\ &\ldots, h_{k-3} = \{u_{t-2}\} \cup T_{k-3} \cup \{v_{t-1}\}, h_{k-2} = \{v_{t-1}\} \cup T_{k-2} \cup \{y\},\end{aligned}$$

where $u_1, \ldots, u_{t-2}$ are distinct vertices selected in $U \setminus (e \cup f)$. Adding edges e, f to that $\mathbb{P}_{k-2}$, we obtain a rainbow $\mathbb{C}_k$, a contradiction.

(2) For $k = 2t+1$, suppose to the contrary that there are edges $e, f \in \mathcal{F}_0 - L$ such that $|e \cap f| = 1$ and $x \in (f \setminus e) \cap U$. As there are at most $(2s-2)$ T_i's containing vertices of e or f, we obtain that there are at least $K - (2s-2) = k-1 > k-3$ such T_i's that are disjoint with e and f. Without loss of generality, assume that for $i \in [k-3]$, T_i is disjoint with e and f. Then there is a $\mathbb{P}_{k-3}$ with the first edge containing x, and all the $k-3$ edges are of the form $T_i \cup \{v_j, u_\ell\}$ similar to the above, where $i \in [k-3], j \in [t-1]$ and $u_\ell \in U$. Adding edges e, f to that $\mathbb{P}_{k-3}$, we obtain a $\mathbb{P}_{k-1}$ in $\mathcal{F}_0$, a contradiction to Claim 6.1 that $\mathcal{F}_0$ is $\mathbb{P}_{k-1}$-free for $k = 2t+1$. ∎

Now we treat the edges of $\mathcal{F}_0 - L$ in detail. For $0 \leq i \leq s$, let

$$B(i) = \{e \in \mathcal{F}_0 - L : \ |e \cap U| = i\},$$

and let $B(2^+) = B(2) \cup B(3) \cup \cdots \cup B(s)$.

By (6.2), we have $m < \epsilon(n-t-s)^{s-1} < \epsilon n^{s-1}$ for every fixed positive constant ϵ when n is sufficiently large. Let $\delta = 8(t+1)\left[2Kk(t-1)\cdot(s-1)!\right]^s \epsilon^{s-2}$. Here we set the constant δ to satisfy $0.5 < \delta < 1$ by selecting an appropriate small constant $\epsilon > 0$.

As a consequence of Claim 6.5, we show the following inequality holds:

$$(t+1)\binom{n-|U|}{s-1} < \frac{\delta}{4}m < \frac{m}{4}. \tag{6.3}$$

In fact, it follows from Claim 6.5 that

$$\begin{aligned}(t+1)\binom{n-|U|}{s-1} &\leq \frac{2(t+1)}{(s-1)!}\left(K(s-2) + (t-1) + \frac{K(t-1)(s-1)km}{\binom{n-t+1}{s-2}}\right)^{s-1}\\ &\leq \frac{2^s(t+1)}{(s-1)!}\left[(K(s-2)+(t-1))^{s-1} + \left(\frac{K(t-1)(s-1)km}{\binom{n-t+1}{s-2}}\right)^{s-1}\right]\\ &< \frac{\delta}{8}m + \frac{2^s(t+1)}{(s-1)!}\left(\frac{K(t-1)(s-1)km}{\frac{(n-t-s)^{s-2}}{(s-2)!}}\right)^{s-1}\\ &< \frac{\delta}{8}m + (t+1)\left[2Kk(t-1)\cdot(s-1)!\right]^s\left(\frac{m}{(n-t-s)^{s-1}}\right)^{s-2} m\\ &< \frac{\delta}{8}m + \frac{\delta}{8}m = \frac{\delta}{4}m < \frac{m}{4},\end{aligned}$$

where the third line of the inequality holds since the constant

$$\frac{2^s(t+1)}{(s-1)!}\left(K(s-2)+(t-1)\right)^{s-1} < \frac{1}{16}\binom{n-8s-t+1}{s-2} \le \frac{\delta}{8}m$$

for sufficiently large n.

Therefore, (6.3) holds, and we will use it to bound the size of $B(0), B(1), B(2^+)$ below.

Claim 6.7. (a) $|B(0)| < \frac{\delta}{4}m < \frac{m}{4}$. *In particular, if* $m = O(n^{s-2})$, *then* $|B(0)| \le O(1)$.

(b) $|B(1)| < \frac{\delta}{2}m < \frac{m}{2}$. *In particular, if* $m = O(n^{s-2})$, *then* $|B(1)| \le O(n)$.

Proof. (a) Since the edges in $B(0)$ cannot form a $\mathbb{C}_k$, by (6.3), we have

$$|B(0)| \le ex(n-|U|, s, \mathbb{C}_k) < (t+1)\binom{n-|U|}{s-1} < \frac{\delta}{4}m < \frac{m}{4}.$$

If $m = O(n^{s-2})$, then by Claim 6.5, there exists a positive real number C such that $|U| \ge n - C$. Thus, we have

$$|B(0)| \le \binom{n-|U|}{s} \le \binom{C}{s} = O(1).$$

(b) Let Q be the collection of $(s-1)$-sets $h \in V \setminus (U \cup L)$ such that there exists $e \in B(1)$ and $h \subset e$. We divide Q into two sets Q_1 and Q_2 such that for every $h \in Q_1$, there is only one vertex $u \in U$ satisfying $h \cup \{u\} \in B(1)$, and $Q_2 = Q \setminus Q_1$. Hence, we have

$$|B(1)| < |Q_1| + |Q_2|n.$$

Clearly, $|Q_1| \le \binom{n-|U|}{s-1} < \frac{\delta}{4}m < \frac{m}{4}$ by (6.3).

To bound $|Q_2|$, notice that there are no $h_1, h_2 \in Q_2$ such that $|h_1 \cap h_2| = 1$. Otherwise, we obtain two s-edges $e, f \in \mathcal{F}_0 - L$ containing h_1, h_2, respectively, where e, f form a rainbow $\mathbb{P}_2$ with $|(f \setminus e) \cap U| = 1$ and $|(e \setminus f) \cap U| = 1$. This contradicts Claim 6.6. Thus, we have

$$|Q_2| < ex(n-|U|, s-1, \mathbb{P}_2) \le \binom{n-|U|}{s-3}.$$

Therefore, by (6.3),

$$\begin{aligned} |B(1)| &< |Q_1| + |Q_2|n < \frac{\delta}{4}m + \binom{n-|U|}{s-3}n \\ &\le \frac{\delta}{4}m + (t+1)\binom{n-|U|}{s-1} \\ &< \frac{\delta}{4}m + \frac{\delta}{4}m = \frac{\delta}{2}m < \frac{m}{2}. \end{aligned}$$ ∎

In particular, if $m = O(n^{s-2})$, then by Claim 6.5, we have $n \ge |U| \ge n - C$ for a positive constant number C, and so

$$|B(1)| \le \binom{n-|U|}{s-1}\binom{|U|}{1} = O(n).$$

Note that

$$|B(2^+)| = |B(2) \cup B(3) \cup \cdots \cup B(s)| = |\mathcal{F}_0 - L| - |B(0)| - |B(1)|. \tag{6.4}$$

Next, we explore properties of $|B(2^+)|$ and m.

CLAIM 6.8. (a) *We have* $|B(2^+)| \le ex(n-t+1, s, \mathbb{P}_2) = \binom{n-t-1}{s-2}$.

(b) *We must have* $k = 2t$ *and* $m = O(n^{s-2})$. *In fact, we have* $m < \frac{4}{(s-2)!}n^{s-2}$.

Proof. (a) If $|B(2^+)| > ex(n-t+1, s, \mathbb{P}_2)$, then there is a $\mathbb{P}_2$ with two edges $e, f \in B(2^+)$. Moreover, by definition of $B(2^+)$, we have $|(f \setminus e) \cap U| \ge 1$ and $|(e \setminus f) \cap U| \ge 1$, a contradiction to Claim 6.6.

(b) If $k = 2t+1$, then it follows from (6.1), (6.4), and Claim 6.7 that

$$|B(2^+)| > |\mathcal{F}_0 - L| - \frac{3}{4}m = \frac{m}{4} + \binom{n-t-1}{s-2} + 2 - 2(k-1) > ex(n-t+1, s, \mathbb{P}_2),$$

a contradiction to Claim 6.8 (a).

So we must have $k = 2t$. Similar to the inequality above, by (6.1) (6.4) and Claims 6.7 and 6.8 (a) for $k = 2t$, we have

$$ex(n-t+1, s, \mathbb{P}_2) \ge |B(2^+)| > |\mathcal{F}_0 - L| - \frac{3}{4}m = \frac{m}{4} + 2 - 2(k-1),$$

which shows that $m < \frac{4n^{s-2}}{(s-2)!}$ as required. ∎

Note that applying (6.1) and (6.4) again, Claims 6.7 and 6.8 provide a further estimation of $|B(2^+)|$ as follows:

$$\binom{n-t-1}{s-2} \ge |B(2^+)| \ge m - O(n) - O(1) + 2 - 2(k-1). \tag{6.5}$$

Now we are preparing to find certain edges aiming to lead a contradiction to Claim 6.6.

CLAIM 6.9. (i) *For any* $j \in \{1, k-1\}$, *there are at least two* $free(P)$ *vertices in* e_j *which are not belonging to* L.

(ii) *There exist a* $free(P)$ *vertex* $x \in e_1$ *and a* $free(P)$ *vertex* $y \in e_{k-1}$ *such that* $x, y \notin L$ *and not all edges in* $B(2^+)$ *contain both* x *and* y.

Proof. (i) In fact, for any $free(P)$ vertex $v \in e_j \cap L$ and any $free(P)$ vertex $u \in e_{j'}$, where $j, j' \in \{1, k-1\}$ and $j \ne j'$, the edge g consisting of u, v and $s-2$ vertices in $V \setminus V(P)$ must be colored with a color appearing in $\bigcup_{i=1}^{k-1} e_i$; otherwise, we have a rainbow $\mathbb{C}_k$. This indicates that g is a missing-edge. Hence, if Claim 6.9 (i) does not hold, then we count the number of missing-edges as

$$m \ge (s-2)(s-3)\binom{n-|V(P)|}{s-2},$$

violating (6.5). Hence, Claim 6.9 (i) holds.

(ii) By Claim 6.9 (i), assume that x, z are $free(P)$ vertices in $e_1 \setminus L$ and y is a $free(P)$ vertex in $e_{k-1} \setminus L$. By contradiction, suppose that all the edges of $B(2^+)$ contain the vertex pair $\{x, y\}$ and all the edges of $B(2^+)$ contain the vertex pair $\{z, y\}$ as well. Then all the edges in $B(2^+)$ contain $\{x, y, z\}$, and so $|B(2^+)| \le \binom{n-t+1-3}{s-3}$, a contradiction to (6.5). This proves Claim 6.9 (ii). ∎

Finally, we are ready to complete the proof.

By Claim 6.9 (ii), there exist a $free(P)$ vertex $x \in e_1$ and a $free(P)$ vertex $y \in e_{k-1}$ such that $x, y \notin L$ and not all edges in $B(2^+)$ contain both x and y. Let $e^* \in B(2^+)$ be such an edge that does not contain both x and y.

Assume that $x, y \notin e^*$. Since $e^* \in B(2^+)$, we select a vertex $u \in e^* \cap U$, and so $u \notin \{x, y\}$. Consider an s-edge f consisting of x, y, u and $s-3$ vertices disjoint with P, L, and e^* such that $|f \cap U| \geq 2$. Then we have $f \in J$; otherwise, $f \cup P$ forms a rainbow $\mathbb{C}_k$. Thus, f and e^* form a rainbow $\mathbb{P}_2$ with $|(f \setminus e^*) \cap U| \geq 1$ and $|(e^* \setminus f) \cap U| \geq 1$, which contradicts Claim 6.6 (1).

Assume instead that one of x, y belongs to e^*. Without loss of generality, suppose that $\{x\} = e^* \cap \{x, y\}$. Consider the edge g consisting of x, y and $s-2$ vertices disjoint with P, L, and e^* such that $|(g \setminus \{x\}) \cap U| \geq 1$. Then we have $g \in J$ with the same reason as above. Moreover, g and e^* form a rainbow $\mathbb{P}_2$ with $|(e^* \setminus f) \cap U| \geq 1$, again contradicting Claim 6.6 (1).

Therefore, we establish the upper bound and complete the proof of Theorem 1.4. ∎

7. Loose cycle—Proof of Theorem 1.5. Since the proof of Theorem 1.5 is similar to Theorem 1.4, in this section, we omit some details and pay more attention to the difference between the proofs of Theorems 1.4 and 1.5.

Let $\mathcal{H}$ be a complete s-uniform hypergraph on n vertices. Denote by V the vertex set of $\mathcal{H}$. The lower bound in Theorem 1.5 follows from a similar construction as Theorem 1.4 by applying the extreme s-graphs without $\mathcal{P}_{k-1}$ obtained from Theorem 2.1.

For the upper bound, when $k = 2t$, since a loose cycle is also a linear cycle, we have $ar(n, s, \mathcal{C}_k) \leq ar(n, s, \mathbb{C}_k) = \binom{n}{s} - \binom{n-t+1}{s} + 2$, and we are done.

For $k = 2t + 1$, we shall show below how to modify the proof of the upper bound for the anti-Ramsey number of linear cycles to obtain the upper bound for the anti-Ramsey number of loose cycles. For loose cycles, we again consider, by contradiction, a coloring of $\mathcal{H}$ using $\binom{n}{s} - \binom{n-t+1}{s} + 3$ colors yielding no rainbow $\mathcal{C}_k$. Since $ar(n, s, \mathcal{P}_k) = \binom{n}{s} - \binom{n-t+1}{s} + 3$ by Theorem 1.3, there is a rainbow loose path P of length k in $\mathcal{H}$. As before, let $\mathcal{G}$ be a subgraph of $\mathcal{H}$ with $|\mathcal{G}| = \binom{n}{s} - \binom{n-t+1}{s} + 3$ such that $P \subset \mathcal{G}$ and each color appears on exactly one edge of $\mathcal{G}$. Denote by $e_1, e_2, \ldots, e_k$ the edges of P, and let $\mathcal{F} = \mathcal{G} - \bigcup_{i=i}^{k-1} e_i$.

With similar argument as the proof of Claim 6.1, we have the following claim.

Claim 7.1. *If $\mathcal{F}$ contains a linear path P_1 of length $k-1$, then $\mathcal{F} - E(P_1)$ contains no $\mathbb{P}_{k-1}$.*

If there is a linear path P_1 of length $k-1$ in $\mathcal{F}$, then we let $\mathcal{F}_0 = \mathcal{F} - E(P_1)$; if there is no linear path of length $k-1$ in $\mathcal{F}$, we delete any $k-1$ edges of $\mathcal{F}$ and denote the subgraph remained by $\mathcal{F}_0$. So we have

$$|\mathcal{F}_0| = |\mathcal{F}| - 2(k-1) = \binom{n}{s} - \binom{n-t+1}{s} + 3 - 2(k-1),$$

and $\mathcal{F}_0$ is $\mathbb{P}_{k-1}$-free.

Note that $|\mathcal{F}_0| \sim (t-1)\binom{n}{s-1}$. By Theorem 2.5, we can find an $(s-1)$-graph $G^* \subset \partial\mathcal{F}_0$ with $|G^*| \sim \binom{n}{s-1}$ and a set L of $t-1$ vertices of $\mathcal{F}_0$ such that $L \cap V(G^*) = \emptyset$ and $e \cup \{v\} \in \mathcal{F}_0$ for any $(s-1)$-edge $e \in G^*$ and any $v \in L$. Moreover, $|\mathcal{F}_0 - L| = o(n^{s-1})$. Select a G^* with the maximum number of $(s-1)$-edges. Denote $L = \{v_1, v_2, \ldots, v_{t-1}\}$ as before.

We still call an s-edge e a missing-edge if e contains vertices of L and $e \notin \mathcal{F}_0$. Let M be the set of all the missing-edges, and let $m = |M|$. We have $|\mathcal{F}_0| - |\mathcal{F}_0 - L| + m = \binom{n}{s} - \binom{n-t+1}{s}$, and so

$$m = |\mathcal{F}_0 - L| - 3 + 2(k-1).$$

If $m \leq \binom{n-8s-t+1}{s-2} - 1$, then Claim 6.2 still holds. Instead of Claim 6.3, we have the following similar claim.

CLAIM 7.2. *When $k = 2t+1 \geq 11$, there are no three edges e, f, h in $\mathcal{G} - L$ satisfying one of the following conditions:*
(i) *e, f, h form a $\mathcal{P}_3$;*
(ii) *e, f form a $\mathcal{P}_2$, and h is disjoint with $e \cup f$;*
(iii) *e, f, h are pairwise disjoint.*

The only difference between Claims 6.3 and 7.2 is to construct a $\mathcal{C}_k$ rather than $\mathbb{C}_k$ to obtain a contradiction with essentially the same argument (see also Claim 5.4 for details).

Then, as P has length k, we can derive that $P - L$ must contain edges satisfying one of the conditions in Claim 7.2, giving the finial contradiction in the case $m \leq \binom{n-8s-t+1}{s-2} - 1$.

If $m > \binom{n-8s-t+1}{s-2} - 1$, with the arguments that are identical to the linear cycles, Claims 6.4 and 6.5 hold. By replacing $\mathcal{P}_2$ with $\mathbb{P}_2$, we obtain the following result similar to Claim 6.6(2).

CLAIM 7.3. *For $k = 2t+1$, there are no two edges $e, f \in \mathcal{F}_0 - L$ such that e, f form a $\mathcal{P}_2$, $|(f \setminus e) \cap U| \geq 1$ or $|(e \setminus f) \cap U| \geq 1$.*

We still let $B(i) = \{e \in \mathcal{F}_0 - L : |e \cap U| = i\}$ and $B(2^+) = B(2) \cup B(3) \cup \cdots \cup B(s)$. Then the counting arguments in Claim 6.7 still hold that $|B(0)| < \frac{m}{4}$ and $|B(1)| < \frac{m}{2}$. Hence, we have

$$\begin{aligned} |B(2^+)| &> |\mathcal{F}_0 - L| - \frac{m}{4} - \frac{m}{2} \\ &= m + 3 - 2(k-1) - \frac{3m}{4} \\ &= \frac{m}{4} + 3 - 2(k-1). \end{aligned} \tag{7.1}$$

By Claim 7.3, there are no two edges $f_1, f_2 \in B(2^+)$ such that $|f_1 \cap U| = i$ and $|f_2 \cap U| = j$ with $i \neq j$. Moreover, for fixed r, if $|f_1 \cap U| = |f_2 \cap U| = r$ for two edges $f_1, f_2 \in B(2^+)$, then $f_1 \cap U = f_2 \cap U$; i.e., all the edges in $B(2^+)$ contain exactly the same r vertices of U. Then it follows that

$$|B(2^+)| \leq \max_{2 \leq r \leq s} \binom{n-t+1-|U|}{s-r} = \binom{n-t+1-|U|}{s-2}.$$

By Claim 6.5 and a similar inequality as (6.3), we have

$$|B(2^+)| \leq \binom{n-t+1-|U|}{s-2} < \frac{\delta}{8} m < \frac{m}{4} + 3 - 2(k-2),$$

which contradicts (7.1). Hence, we obtain the final contradiction, which proves Theorem 1.5. ∎

8. Berge path and Berge cycle. We shall present the proofs of Theorem 1.6 and Proposition 1.7 on Berge paths and Berge cycles in this section.

8.1. Berge path—Proof of Theorem 1.6. For the lower bounds, we will prove that $ar(n, s, \mathcal{B}_k) \geq \frac{2n}{k}\binom{\lfloor k/2 \rfloor}{s}$ if $k > 2s+1$ and that $ar(n, s, \mathcal{B}_k) \geq \frac{n}{s+1}\lfloor \frac{k-2}{2} \rfloor$ if $3 < k \leq 2s+1$. For $k > 2s+1$, we partition the n vertices into sets of size $\lfloor k/2 \rfloor$ (possibly one of those sets has size smaller than $\lfloor k/2 \rfloor$). Denote by $S_1, S_2, \ldots, S_\ell$ those obtained sets of size $\lfloor k/2 \rfloor$. Then for each k-set S_i, color each edge contained in S_i with a distinct color. The rest of the edges are colored with one additional color. It is routine to check that there is no rainbow $\mathcal{B}_k$ in the above coloring. So we have $ar(n, s, \mathcal{B}_k) \geq \frac{2n}{k}\binom{\lfloor k/2 \rfloor}{s}$.

For $3 < k \leq 2s+1$, we partition the n vertices into sets of size $s+1$. Then we select $\lfloor k/2 \rfloor - 1$ edges in each $(s+1)$-set and color each of those edges with a different color. The rest edges are colored with one additional color. Similarly, this provides a $\frac{n}{s+1}\lfloor \frac{k-2}{2} \rfloor$-coloring without a rainbow $\mathcal{B}_k$. Hence, $ar(n, s, \mathcal{B}_k) \geq \frac{n}{s+1}\lfloor \frac{k-2}{2} \rfloor$.

For the upper bounds, we will show that if $k \geq s+2$, then for sufficiently large n, $ar(n, s, \mathcal{B}_k) \leq \frac{n}{k-1}\binom{k-1}{s}+1$, and if $k \leq s+1$, then $ar(n, s, \mathcal{B}_k) \leq \frac{(k-2)n}{s+1}$ for sufficiently large n.

(I) For $k \geq s+2$, let $\mathcal{H}$ be a complete s-uniform hypergraph on n vertices. Consider a coloring of $\mathcal{H}$ using $\frac{n}{k-1}\binom{k-1}{s}+1$ colors and yielding no rainbow $\mathcal{B}_k$. Let $\mathcal{G}$ be a subgraph of $\mathcal{H}$ with $\frac{n}{k-1}\binom{k-1}{s}+1$ edges such that each color appears on exactly one edge of $\mathcal{G}$. So the number of edges of $\mathcal{G}$ is $|\mathcal{G}| = \frac{n}{k-1}\binom{k-1}{s} + 1 > ex(n, s, \mathcal{B}_{k-1})$. Hence, there is a rainbow Berge path P of length $k-1$ in $\mathcal{G}$. Denote by $e_1, e_2, \ldots, e_{k-1}$ the edges of P with colors $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$, respectively. And there are k vertices $w_1, w_2, \ldots, w_k$ in P such that $w_i, w_{i+1} \in e_i$ for $i = 1, \ldots, k-1$. Let $\mathcal{F}$ be the hypergraph obtained by removing all the edges of P from $\mathcal{G}$. We have that $|\mathcal{F}| = \frac{n}{k-1}\binom{k-1}{s} + 1 - (k-1) = \frac{n}{k-1}\binom{k-1}{s} - k + 2$.

If there is a Berge path P^* of length $k-1$ in $\mathcal{F}$, denote by $g_1, g_2, \ldots, g_{k-1}$ the edges of P^*. And there are k vertices $z_1, z_2, \ldots, z_k$ in P^* such that $z_i, z_{i+1} \in g_i$ for $i = 1, \ldots, k-1$. Then either $w_1 \neq z_1$ or $w_1 \neq z_k$. Without loss of generality, suppose that $w_1 \neq z_1$. Consider the edge e consisting of w_1, z_1 and $s-2$ vertices in $V(\mathcal{F}) \setminus (V(P) \cup V(P^*))$. If e is colored with a color not in $\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}\}$, then $e \cup P$ is a rainbow $\mathcal{B}_k$. So e is colored with a color belonging to $\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}\}$; then $e \cup P^*$ is a rainbow $\mathcal{B}_k$. Therefore, we have showed that

$$\mathcal{F} \text{ contains no } \mathcal{B}_{k-1}. \tag{8.1}$$

We further claim that the minimum degree $\delta(\mathcal{F})$ of $\mathcal{F}$ satisfying

$$\delta(\mathcal{F}) \geq \frac{1}{k-1}\binom{k-1}{s} - k + 1. \tag{8.2}$$

Indeed, if there is a vertex v having degree $d_{\mathcal{F}}(v) < \frac{1}{k-1}\binom{k-1}{s} - k + 1$ in $\mathcal{F}$, then the number of edges in $\mathcal{F} - v$ is more than $|\mathcal{F}| - (\frac{1}{k-1}\binom{k-1}{s} - k + 1) = \frac{n-1}{k-1}\binom{k-1}{s} + 1 \geq ex(n-1, s, \mathcal{B}_{k-1}) + 1$ for sufficiently large n. So there is a $\mathcal{B}_{k-1}$ in $\mathcal{F} - v$, which contradicts (8.1). This proves (8.2).

Since $|\mathcal{F}| > ex(n, s, \mathcal{B}_{k-2})$ for sufficiently large n, there is a Berge path P' of length $k-2$ in $\mathcal{F}$. Denote by $f_1, f_2, \ldots, f_{k-2}$ the edges of P' with colors $\beta_1, \beta_2, \ldots, \beta_{k-2}$, respectively. And there are $k-1$ vertices $u_1, u_2, \ldots, u_{k-1}$ in P' such that $u_i, u_{i+1} \in f_i$ for $i = 1, \ldots, k-2$. Since $\mathcal{F}$ contains no $\mathcal{B}_{k-1}$ by (8.1), the neighbors of u_1 and u_{k-1} must belong to $\{u_1, u_2, \ldots, u_{k-1}\}$. In fact, we shall further show in the following claim that the neighbors of each vertex in $\{u_2, \dots, u_{k-2}\}$ also belong to $\{u_1, u_2, \dots, u_{k-1}\}$. Before that, we need the definition of Berge cycles to state the following claim. An s-uniform *Berge cycle* of length ℓ is a cyclic list of distinct s-sets $a_1, \dots, a_\ell$ and ℓ distinct vertices $v_1, \dots, v_\ell$ such that for each $i = 1, 2, \dots, \ell$, a_i contains v_i and v_{i+1} (where $v_{\ell+1} = v_1$).

(8.3) If there is a Berge cycle of length $k-1$ and containing the vertices $u_1, u_2, \dots, u_{k-1}$, then $u_1, u_2, \dots, u_{k-1}$ constitute a component of $\mathcal{F}$.

Suppose that there is a Berge cycle C containing the vertices $u_1, u_2, \dots, u_{k-1}$. If an edge f in the C contains some vertex x other than $u_1, u_2, \dots, u_{k-1}$, then deleting f from C, we have a $\mathcal{B}_{k-2}$, which can be extended to a $\mathcal{B}_{k-1}$ with edge f, contradicting (8.1). Thus, every edge in the cycle must be contained within the vertices $u_1, u_2, \dots, u_{k-1}$. Moreover, for each vertex u_i in C, the neighbors of u_i must belong to $\{u_1, u_2, \dots, u_{k-1}\}$. Suppose to the contrary that u_i has a neighbor y other than $u_1, u_2, \dots, u_{k-1}$. Then the edge containing both u_i and y is not an edge of C, as shown in the argument above. Thus, removing an appropriate edge of C so that we get a path of length $k-2$ with u_i as an endpoint, we can extend this to a $\mathcal{B}_{k-1}$ with y as an endpoint, a contradiction to (8.1). This proves (8.3).

Now we show that one can always find a Berge cycle of length $k-1$ containing the vertices $u_1, u_2, \dots, u_{k-1}$. If there is an edge in $\mathcal{F}$ containing both u_1 and u_{k-1}, then we can obtain a Berge cycle of length $k-1$. If not, recall that by (8.2), we have $\delta(\mathcal{F}) \geq \frac{1}{k-1}\binom{k-1}{s} - k + 1 > \binom{\frac{k-1-2}{2}}{s-1}$. That implies that there exist edges f' and f'' in $\mathcal{F}$ such that for some i, $u_1, u_{i+1} \in f'$ and $u_i, u_{k-1} \in f''$. Thus, we have a Berge cycle of length $k-1$ on the vertices

$$u_1, u_{i+1}, u_{i+2}, u_{i+3}, \dots, u_{k-1}, u_i, u_{i-1}, u_{i-2}, \dots, u_1.$$

Hence, we can find a Berge cycle of length $k-1$ containing the vertices $u_1, u_2, \dots, u_{k-1}$ in $\mathcal{F}$. By (8.3), $u_1, u_2, \dots, u_{k-1}$ constitute a component of $\mathcal{F}$.

Let $\mathcal{R}$ denote the hypergraph obtained by deleting vertices $u_1, u_2, \dots, u_{k-1}$ from $\mathcal{F}$. Then $|\mathcal{R}| \geq \frac{n}{k-1}\binom{k-1}{s} - k + 2 - \binom{k-1}{s} > ex(n-(k-1), s, \mathcal{B}_{k-2})$ for sufficiently large n. Hence, there is a Berge path P'' of length $k-2$ in $\mathcal{R}$. Denote by h_1, $h_2, \dots, h_{k-2}$ the edges of P'' with colors $\gamma_1, \gamma_2, \dots, \gamma_{k-2}$, respectively. And there are $k-1$ vertices v_1, $v_2, \dots, v_{k-1}$ in P'' such that $v_i, v_{i+1} \in h_i$ for $i = 1, \dots, k-2$. Note that $\{u_1, u_2, \dots, u_{k-1}\} \cap \{v_1, v_2, \dots, v_{k-1}\} = \emptyset$. Since we have either $w_1 \notin \{u_1, v_1\}$ or $w_1 \notin \{u_{k-1}, v_{k-1}\}$, suppose, without loss of generality, that $w_1 \notin \{u_1, v_1\}$ holds. Consider the edge e' with w_1, u_1, v_1 and $s-3$ vertices in $V(\mathcal{H}) \setminus (V(P) \cup V(P') \cup V(P''))$. If $s > 3$, e' can only be colored with a color in $\{\alpha_1, \alpha_2, \dots, \alpha_{k-1}\}$, then $h_1 \cup e' \cup P'$ is a rainbow $\mathcal{B}_k$. If $s = 3$, then $e' = \{w_1, u_1, v_1\}$. If the color of e' is not belonging to $\{\beta_1, \beta_2, \dots, \beta_{k-2}\} \cup \{\gamma_1, \gamma_2, \dots, \gamma_{k-2}\}$, then $h_1 \cup e' \cup P'$ is a rainbow $\mathcal{B}_k$. If the color of e' is in $\{\beta_1, \beta_2, \dots, \beta_{k-2}\}$, let $\tilde{P} = e' \cup P''$; then $\tilde{P}$ is a rainbow $\mathcal{B}_{k-1}$ in $\mathcal{H}$. Consider an edge $e'' = \{w_1, u_1, x\}$, where $x \notin V(P) \cup V(P') \cup V(\tilde{P})$. To prevent extending P, the color of e'' must be in $\{\alpha_1, \alpha_2, \dots, \alpha_{k-1}\}$. However, to prevent extending $\tilde{P}$, e'' must be colored with a color from $\{\beta_1, \beta_2, \dots, \beta_{k-2}\} \cup \{\gamma_1, \gamma_2, \dots, \gamma_{k-2}\}$, a contradiction. By symmetry, if the color of e' is from $\{\gamma_1, \gamma_2, \dots, \gamma_{k-2}\}$, we can deduce a similar contradiction as well. In conclusion, any coloring of $\mathcal{H}$ using $\frac{n}{k-1}\binom{k-1}{s} + 1$ colors yields a rainbow $\mathcal{B}_k$.

(II) For $k \leq s+1$, let $\mathcal{H}$ be a complete s-uniform hypergraph on n vertices. Consider a coloring of $\mathcal{H}$ using $\frac{n(k-2)}{s+1} + 1$ colors and yielding no rainbow $\mathcal{B}_k$. Let $\mathcal{G}$

be a subgraph of $\mathcal{H}$ with $\frac{n(k-2)}{s+1}+1$ edges such that each color appears on exactly one edge of $\mathcal{G}$. So the number of edges of $\mathcal{G}$ is $|\mathcal{G}| = \frac{n(k-2)}{s+1}+1$. Denote by $\mathcal{C}_1$, $\mathcal{C}_2$, $\ldots$, $\mathcal{C}_t$ the components of $\mathcal{G}$ and by $n_1, n_2, \ldots, n_t$ the number of vertices of each component, respectively. Then there is a component $\mathcal{C}_i$ such that $|\mathcal{C}_i| > \frac{n_i(k-2)}{s+1} \geq ex(n_i, s, \mathcal{B}_{k-1})$. Hence, there is a rainbow Berge path P of length $k-1$ in $\mathcal{C}_i$. Denote by $e_1, e_2, \ldots, e_{k-1}$ the edges of P with colors $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$, respectively. And there are k vertices $w_1, w_2, \ldots, w_k$ in P such that $w_i, w_{i+1} \in e_i$ for $i = 1, \ldots, k-1$. Let $\mathcal{F}$ be the hypergraph obtained by removing all the edges of P from $\mathcal{G}$. We have that $|\mathcal{F}| = \frac{(k-2)}{s+1}n + 1 - (k-1) = \frac{n(k-2)}{s+1} - k + 2 > \frac{(k-3)}{s+1}n$.

We will make use of the following result given in [25].

PROPOSITION 8.1 (see [25]). *Fix ℓ and s such that $s \geq \ell > 2$. Let $\mathcal{H}$ be a connected s-uniform hypergraph with*

$$|\mathcal{H}| > \frac{\ell-1}{s+1}n$$

edges, where n is the number of vertices in $\mathcal{H}$. Then for each edge $e \in \mathcal{H}$, there is a Berge path of length ℓ in $\mathcal{H}$ starting with e.

Let the components of $\mathcal{F}$ be $\mathcal{C}_1^*$, $\mathcal{C}_2^*$, $\ldots$, $\mathcal{C}_\mu^*$ and $n_1^*, n_2^*, \ldots, n_\mu^*$ be the number of vertices of each component, respectively. Then there is a component $\mathcal{C}_j^*$ satisfying that $|\mathcal{C}_j^*| > \frac{k-3}{s+1}n_j^* \geq ex(n_j^*, s, \mathcal{B}_{k-2})$. Now we focus on finding a Berge path of length $k-2$ containing some new vertices in $\mathcal{C}_j^*$.

If there exists such a $\mathcal{C}_j^*$ satisfying that $|\mathcal{C}_j^*| > \frac{k-3}{s+1}n_j^*$ and $\mathcal{C}_j^* \cap \mathcal{C}_i = \emptyset$, then we can find a Berge path of length $k-2$ in $\mathcal{C}_j^*$, and its vertices are disjoint with P.

If every such $\mathcal{C}_j^*$ with $|\mathcal{C}_j^*| > \frac{k-3}{s+1}n_j^*$ satisfying that $\mathcal{C}_j^* \subseteq \mathcal{C}_i$, then we have that $n_j^* \geq s+1 \geq k$ since the number of vertices of a $\mathcal{B}_{k-2}$ is at least $s-1+2 = s+1$. Furthermore, we claim that

$$n_j^* \geq 2k. \tag{8.4}$$

In fact, if $n_j^* < 2k$, then $|\mathcal{C}_j^*| \leq \binom{n_j^*}{s} < \binom{2k}{s}$. Delete the component $\mathcal{C}_j^*$ from $\mathcal{F}$; we have $n - n_j^*$ vertices and more than $\frac{k-2}{s+1}n - k + 2 - \binom{2k}{s} > \frac{k-3}{s+1}(n-k) \geq \frac{k-3}{s+1}(n-n_j^*)$ edges. So there is a component $\mathcal{C}_t^*$ such that $|\mathcal{C}_t^*| > \frac{k-3}{s+1}n_t^* \geq ex(n_t^*, s, \mathcal{B}_{k-2})$ edges and $\mathcal{C}_t^* \cap \mathcal{C}_i = \emptyset$, a contradiction. So we have $n_j^* \geq 2k$, which proves (8.4). Hence, there is a vertex $u_1 \notin \{w_1, w_2, \ldots, w_k\}$ in $\mathcal{C}_j^*$. We take an edge e in $\mathcal{C}_j^*$ containing u_1, and by Proposition 8.1, there is a $\mathcal{B}_{k-2}$ starting with e.

In both cases above, we denote by P' the $\mathcal{B}_{k-2}$ we obtained and denote by f_1, $f_2, \ldots, f_{k-2}$ the edges of P' with colors $\beta_1, \beta_2, \ldots, \beta_{k-2}$, respectively. There are $k-1$ vertices $u_1, u_2, \ldots, u_{k-1}$ in P' such that $u_i, u_{i+1} \in f_i$ for $i = 1, \ldots, k-2$. Note that $u_1 \notin \{w_1, w_2, \ldots, w_k\}$. Let $\mathcal{R}$ denote the hypergraph obtained by deleting $f_1, f_2, \ldots, f_{k-2}$ from $\mathcal{F}$. Then $|\mathcal{R}| = \frac{k-2}{s+1}n - k + 2 - (k-2) > \frac{k-3}{s+1}n$. Let the components of $\mathcal{R}$ be $\mathcal{C}_1^{**}$, $\mathcal{C}_2^{**}$, $\ldots$, $\mathcal{C}_\tau^{**}$ and $n_1^{**}, n_2^{**}, \ldots, n_\tau^{**}$ be the number of vertices of each component, respectively. Then there is a component $\mathcal{C}_\ell^{**}$ such that $|\mathcal{C}_\ell^{**}| > \frac{k-3}{s+1}n_\ell^{**} \geq ex(n_\ell^{**}, s, \mathcal{B}_{k-2})$. If there exists such a $\mathcal{C}_\ell^{**}$ satisfying that $\mathcal{C}_\ell^{**} \cap \mathcal{C}_j^* \cap \mathcal{C}_i = \emptyset$, then we can find an edge e' containing a vertex $v_1 \notin \{u_1, u_2, \ldots, u_{k-1}, w_1, w_2, \ldots, u_k\}$, and the color of e' is different with $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \beta_1, \beta_2, \ldots, \beta_{k-2}$. Otherwise, $\mathcal{C}_\ell^{**} \subseteq \mathcal{C}_j^*$ or $\mathcal{C}_\ell^{**} \subseteq \mathcal{C}_i$. We have $n_\ell^{**} \geq s+1 \geq k$. With the argument similar to the proof of (8.4), we can obtain that

$$n_\ell^{**} \geq 2k.$$

Thus, there is a vertex not belonging to $\{u_1, u_2, \ldots, u_{k-1}, w_1, w_2, \ldots, u_k\}$. We still denote it by v_1. Take an edge e' in $\mathcal{C}_\ell^{**}$ containing v_1. Denote this edge by e'. So the color of e' is different with $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \beta_1, \beta_2, \ldots, \beta_{k-2}$. Consider an edge e'' containing w_1, u_1, v_1; it must be colored with a color from $\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}\}$ since otherwise P can be extended. But then $e' \cup e'' \cup P'$ is a rainbow $\mathcal{B}_k$, a contradiction. Therefore, we have proved the upper bound. ∎

8.2. Berge cycle—Proof of Proposition 1.7. Note that the lower bound in Proposition 1.7 is obvious, which follows from a similar observation as in (1.1) for hypergraphs. Now we prove the upper bound in Proposition 1.7. Let $\mathcal{H}$ be a complete s-uniform hypergraph on n vertices. By contradiction, consider a coloring of $\mathcal{H}$ using $ex(n, s, \mathcal{B}_{k-1}) + k$ colors yielding no rainbow $\mathcal{BC}_k$. Let $\mathcal{G}$ be a subgraph of $\mathcal{H}$ with $ex(n, s, \mathcal{B}_{k-1}) + k$ edges such that each color appears on exactly one edge of $\mathcal{G}$. So the number of edges of $\mathcal{G}$ is $|\mathcal{G}| = ex(n, s, \mathcal{B}_{k-1}) + k > ex(n, s, \mathcal{B}_{k-1})$. Hence, there is a rainbow Berge path P of length $k-1$ in $\mathcal{G}$. Denote by $e_1, e_2, \ldots, e_{k-1}$ the edges of P with colors $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$, respectively. And there are k vertices $w_1, w_2, \ldots, w_k$ in P such that $w_i, w_{i+1} \in e_i$ for $i = 1, \ldots, k-1$.

Let $\mathcal{F}$ be the hypergraph obtained from $\mathcal{G}$ by removing all the edges of P. Then we have that $|\mathcal{F}| = ex(n, s, \mathcal{B}_{k-1}) + 1$. Therefore, there is a Berge path P^* of length $k-1$ in $\mathcal{F}$. Denote by $g_1, g_2, \ldots, g_{k-1}$ the edges of P^*, where there are k vertices $z_1, z_2, \ldots, z_k$ in P^* such that $z_i, z_{i+1} \in g_i$ for $i = 1, \ldots, k-1$. Consider an s-edge e containing w_1, w_k, z_1, z_k. If e is colored with a color not in $\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}\}$, then $e \cup P$ is a rainbow $\mathcal{BC}_k$. So e is colored with a color belonging to $\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}\}$, but now $e \cup P^*$ is a rainbow $\mathcal{BC}_k$, a contradiction. Hence, $ar(n, s, \mathcal{BC}_{k-1}) \leq ex(n, s, \mathcal{B}_{k-1}) + k$ for any possible n. This proves Proposition 1.7. ∎

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