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CIRCULAR FLOWS IN PLANAR GRAPHS*

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Abstract. For integers $a \geq 2b > 0$, a *circular a/b-flow* is a flow that takes values from $\{\pm b, \pm(b+1), \ldots, \pm(a-b)\}$. The Planar Circular Flow Conjecture states that every $2k$-edge-connected planar graph admits a circular $(2+\frac{2}{k})$-flow. The cases $k=1$ and $k=2$ are equivalent to the Four Color Theorem and Grötzsch's 3-Color Theorem. For $k \geq 3$, the conjecture remains open. Here we make progress when $k=4$ and $k=6$. We prove that (i) *every 10-edge-connected planar graph admits a circular 5/2-flow* and (ii) *every 16-edge-connected planar graph admits a circular 7/3-flow.* The dual version of statement (i) on circular coloring was previously proved by Dvořák and Postle [*Combinatorica*, 37 (2017), pp. 863–886], but our proof has the advantages of being much shorter and avoiding the use of computers for case-checking. Further, it has new implications for antisymmetric flows. Statement (ii) is especially interesting because the counterexamples to Jaeger's original Circular Flow Conjecture are 12-edge-connected nonplanar graphs that admit no circular 7/3-flow. Thus, the planarity hypothesis of (ii) is essential.

Key words. circular flow, circular coloring, planar, modulo orientation, strongly $\mathbb{Z}_5$ connected, strongly $\mathbb{Z}_7$ connected

AMS subject classifications. 05C15, 05C21, 05C40

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1. Introduction.

1.1. Planar circular flow conjecture. For integers $a \geq 2b > 0$, a *circular a/b-flow*[1] is a flow that takes values from $\{\pm b, \pm(b+1), \ldots, \pm(a-b)\}$. In this paper we study the following conjecture, which arises from Jaeger's Circular Flow Conjecture [9].

CONJECTURE 1.1 (Planar Circular Flow Conjecture). *Every $2k$-edge-connected planar graph admits a circular $(2+\frac{2}{k})$-flow.*

When $k=1$ this conjecture is the flow version of the Four Color Theorem. It is true for planar graphs (by the Four Color Theorem) but false for nonplanar graphs because of the Petersen graph and all other snarks. Tutte's 4-Flow Conjecture, from 1966, claims that Conjecture 1.1 extends to every graph with no Petersen minor. When $k=2$, Conjecture 1.1 is the dual of Grötzsch's 3-Color Theorem. Tutte's 3-Flow Conjecture, from 1972, asserts that it extends to all graphs (both planar and nonplanar). In 1981 Jaeger further extended Tutte's Flow Conjectures by proposing a general Circular Flow Conjecture: *for each even integer $k \geq 2$, every $2k$-edge-connected graph admits a circular $(2+\frac{2}{k})$-flow.* That is, he believed Conjecture 1.1 extends to all graphs for all even k. A weaker version of Jaeger's conjecture was

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[1] Jaeger [9] showed that if $p, q, r, s \in \mathbb{Z}^+$ and $p/q = r/s$, then each graph G has a circular p/q-flow if and only if it has a circular r/s-flow. (See [7] for more details.) We use this result implicitly in the present paper.

proved by Thomassen [19] for graphs with edge connectivity at least $2k^2+k$. This edge connectivity condition was substantially improved by Lovász, Thomassen, Wu, and Zhang [13].

THEOREM 1.2 (Lovász et al. [13]). *For each even integer $k \geq 2$, every $3k$-edge-connected graph admits a circular $(2+\frac{2}{k})$-flow.*

In contrast, Jaeger's Circular Flow Conjecture was recently disproved for all $k \geq 6$. In [8], for each even integer $k \geq 6$, the authors construct a $2k$-edge-connected nonplanar graph admitting no circular $(2+\frac{2}{k})$-flow. And for large odd integers k, we can also modify the construction in [8] to get $2k$-edge-connected nonplanar graphs admitting no circular $(2+\frac{2}{k})$-flow. Thus, the planarity hypothesis of Conjecture 1.1 seems essential. The case $k=4$ of Jaeger's Circular Flow Conjecture, which remains open, is particularly important, since Jaeger [9] observed that if every 9-edge-connected graph admits a circular $5/2$-flow, then Tutte's celebrated 5-Flow Conjecture follows.

Our main theorems improve on Theorem 1.2, restricted to planar graphs, when $k \in \{4,6\}$.

THEOREM 1.3. *Every 10-edge-connected planar graph admits a circular $5/2$-flow.*

THEOREM 1.4. *Every 16-edge-connected planar graph admits a circular $7/3$-flow.*

The dual version of Theorem 1.3, on circular coloring, was proved by Dvořák and Postle [5] (improving on earlier work of Borodin et al.). In fact, their coloring result holds for a larger class of graphs that includes some sparse nonplanar graphs, as well as all planar graphs with girth at least 10. However, our proof is much shorter and avoids using computers for case-checking. Our proof also has new implications for antisymmetric flows (see Theorem 2.4 below). Theorem 1.4 is especially interesting because the counterexamples in [8] to Jaeger's original circular flow conjecture are 12-edge-connected nonplanar graphs that admit no circular $7/3$-flow. After submitting this paper, we learned that Postle and Smith-Roberge [17] independently proved Theorem 1.4 ([18] is an extended abstract).

1.2. Circular flows and modulo orientations. Graphs in this paper are finite and can have multiple edges but no loops. Our notation is mainly standard. For a graph G, we write $|G|$ for $|V(G)|$ and write $\|G\|$ for $|E(G)|$. Let $\delta(G)$ denote the minimum degree in a graph G. A k-vertex is a vertex of degree k. For disjoint vertex subsets X and Y, let $[X,Y]_G$ denote the set of edges in G with one endpoint in each of X and Y. Let $X^c = V(G) \setminus X$, and let $d(X) = |[X, X^c]|$. For vertices v and w, let $\mu(vw) = |[\{v\},\{w\}]_G|$ and $\mu(G) = \max_{v,w \in V(G)} \mu(vw)$.

To *lift* a pair of edges w_1v, vw_2 incident to a vertex v in a graph G means to delete w_1v and vw_2 and create a new edge w_1w_2. To *contract* an edge e in G means to identify its two endpoints and then delete the resulting loop. For a subgraph H of G, we write G/H to denote the graph formed from G by successively contracting the edges of $E(H)$. The lifting and contraction operations are used frequently in this paper.

An orientation D of a graph G is a *modulo $(2p+1)$-orientation* if $d_D^+(v) - d_D^-(v) \equiv 0 \pmod{2p+1}$ for each $v \in V(G)$. By the following lemma of Jaeger [9], this problem is equivalent to finding circular flows (for a short proof, see [21, Theorem 9.2.3]).

LEMMA 1.5 (Jaeger [9]). *A graph admits a circular $(2+\frac{1}{p})$-flow if and only if it has a modulo $(2p+1)$-orientation.*

To prove our results, we study modulo orientations. Let G be a graph. A function $\beta : V(G) \mapsto \mathbb{Z}_{2p+1}$ is a *$\mathbb{Z}_{2p+1}$-boundary* if $\sum_{v \in V(G)} \beta(v) \equiv 0 \pmod{2p+1}$. Given a $\mathbb{Z}_{2p+1}$-boundary β, a *$(\mathbb{Z}_{2p+1}, \beta)$-orientation* is an orientation D such that $d_D^+(v) - d_D^-(v) \equiv \beta(v) \pmod{2p+1}$ for each $v \in V(G)$. When such an orientation exists, we say that the boundary β is *achievable*. If $\beta(v) = 0$ for all $v \in V(G)$, then a $(\mathbb{Z}_{2p+1}, \beta)$-orientation is simply a modulo $(2p+1)$-orientation. As defined in [10, 11], a graph G is *strongly* $\mathbb{Z}_{2p+1}$*-connected* if for any $\mathbb{Z}_{2p+1}$-boundary β, the graph G admits a $(\mathbb{Z}_{2p+1}, \beta)$-orientation. When the context is clear, we may simply write β-*orientation* for $(\mathbb{Z}_{2p+1}, \beta)$-orientation. Suppose we are given a graph G, an integer p, a $\mathbb{Z}_{2p+1}$-boundary β for G, and a connected subgraph $H \subsetneq G$. We form G' from G by contracting H; that is, $G' = G/H$. Let w denote the new vertex in G', formed by contracting $E(H)$. Define β' for G' by $\beta'(v) = \beta(v)$, for each $v \in V(G') \setminus \{w\}$, and $\beta'(w) = \sum_{v \in V(H)} \beta(v) \pmod{2p+1}$. Note that β' is a $\mathbb{Z}_{2p+1}$-boundary for G'. The motivation for generalizing modulo orientations is the following observation of Lai [10], which is also applied in Thomassen et al. [19, 13].

LEMMA 1.6 (Lai [10]). *Let G be a graph with a subgraph H, and let $G' = G/H$. Let β and β' be $\mathbb{Z}_{2p+1}$ boundaries (respectively) of G and G', as defined above. If H is strongly $\mathbb{Z}_{2p+1}$-connected, then every β'-orientation of G' can be extended to a β-orientation of G. In particular, each of the following holds.*

(i) *If H is strongly $\mathbb{Z}_{2p+1}$-connected and G/H has a modulo $(2p+1)$-orientation, then G has a modulo $(2p+1)$-orientation.*
(ii) *If H and G/H are strongly $\mathbb{Z}_{2p+1}$-connected, then G is also strongly $\mathbb{Z}_{2p+1}$-connected.*

Proof. We prove the first statement, since it implies (i) and (ii). Fix a β'-orientation of G'. This yields an orientation D of the subgraph $G - E(G[V(H)])$. By orienting arbitrarily each edge in $E(G[V(H)]) \setminus E(H)$, we obtain a β''-orientation D_1 of $G - E(H)$, for some β''. For each $v \in V(H)$, let $\gamma(v) = \beta(v) - \beta''(v)$. It is easy to check that γ is a $\mathbb{Z}_{2p+1}$-boundary of H. Since H is strongly $\mathbb{Z}_{2p+1}$-connected, H has a γ-orientation D_2. Hence $D_1 \cup D_2$ is a β-orientation of G. ∎

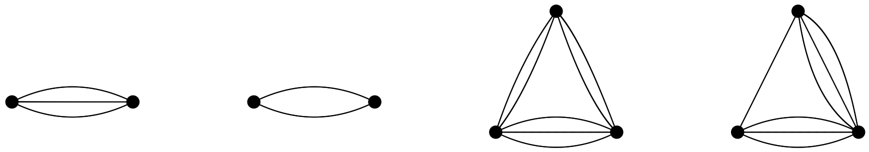
Proof outline for main results. To prove Theorems 1.3 and 1.4, we actually establish two stronger, more technical results on orientations; namely, we prove Theorems 2.2 and 3.3. Lemma 1.6 shows that strongly $\mathbb{Z}_{2p+1}$-connected graphs are contractible configurations when we are looking for modulo orientations. To prove Theorems 2.2 and 3.3, we use lifting and contraction operations to find many more reducible configurations. These configurations eventually facilitate a discharging proof. The proofs of Theorems 1.3 and 1.4 are similar, though the latter is harder. In the next section we just discuss Theorem 1.3, but most of the key ideas are reused in the proof of Theorem 1.4.

2. Circular 5/2-flows: Proof of Theorem 1.3.

In this section, we focus on modulo 5-orientations and prove Theorem 1.3.

2.1. Modulo 5-orientations and antisymmetric $\mathbb{Z}_5$-flows.

To prove Theorem 1.3, we will first present a more technical result, Theorem 2.2, which yields Theorem 1.3 as an easy corollary (as we show below in Theorem 2.5). The hypothesis in Theorem 2.2 uses a weight function w, which is motivated by the following Spanning Tree Packing Theorem of Nash-Williams [14] and Tutte [20]: *a graph G has k edge-disjoint spanning trees if and only if every partition $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ satisfies $\sum_{i=1}^{t} d(P_i) - 2k(t-1) \geq 0$.* This condition is necessary, since in a partition with t parts, each spanning tree has at least $t-1$ edges between parts. It is shown in [12, Proposition 3.9] that if G is strongly $\mathbb{Z}_{2p+1}$-connected, then it contains $2p$ edge-disjoint spanning trees (although this necessary condition is not always sufficient). To capture this idea, we define the following weight function.



$3K_2$ $2K_2$ $T_{2,2,3}$ $T_{1,3,3}$

FIG. 1. *The graphs* $3K_2, 2K_2, T_{2,2,3}, T_{1,3,3}$.

DEFINITION 2.1. *Let* $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ *be a partition of* $V(G)$*. Let*

$$w_G(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19$$

and

$$w(G) = \min\{w_G(\mathcal{P}) : \mathcal{P} \textit{ is a partition of } V(G)\}.$$

Let $T_{a,b,c}$ denote a 3-vertex graph (triangle) with its pairs of vertices joined by a, b, and c parallel edges; let aH denote the graph formed from H by replacing each edge with a parallel edges. For example, $w(3K_2) = 3$, $w(2K_2) = 1$, $w(T_{2,2,3}) = w(T_{1,3,3}) = 0$; see Figure 1. For each of these four graphs the minimum in the definition of $w(G)$ is attained only by the partition with each vertex in its own part. We typically assume $V(T_{a,b,c}) = \{v_1, v_2, v_3\}$ and $d(v_1) \le d(v_2) \le d(v_3)$.

Let $\mathcal{T} = \{2K_2, 3K_2, T_{2,2,3}, T_{1,3,3}\}$. Each graph $G \in \mathcal{T}$ (see Figure 1) is not strongly $\mathbb{Z}_5$-connected, since there exists some $\mathbb{Z}_5$-boundary β for which G has no β-orientation. A short case analysis shows that none of the following boundaries are achievable. For $3K_2$, let $\beta(v_1) = \beta(v_2) = 0$. For $2K_2$, let $\beta(v_1) = 1$ and $\beta(v_2) = 4$. For $T_{2,2,3}$, let $\beta(v_1) = 1$ and $\beta(v_2) = \beta(v_3) = 2$. For $T_{1,3,3}$, let $\beta(v_1) = \beta(v_2) = 1$ and $\beta(v_3) = 3$.

Now suppose that G has a partition $\mathcal{P}$ such that $G/\mathcal{P} \in \mathcal{T}$, where the vertices in each P_i are identified to form v_i. To construct a $\mathbb{Z}_5$-boundary γ for which G has no γ-orientation, we assign boundary γ so that $\sum_{v \in P_i} \gamma(v) \equiv \beta(v_i)$. Hence G has no γ-orientation precisely because $G/\mathcal{P}$ has no β-orientation. We call a partition $\mathcal{P}$ *troublesome* if $G/\mathcal{P} \in \mathcal{T} = \{2K_2, 3K_2, T_{2,2,3}, T_{1,3,3}\}$. The main result of section 2 is Theorem 2.2.

THEOREM 2.2. *Let G be a planar graph and β be a $\mathbb{Z}_5$-boundary of G. If $w(G) \ge 0$, then G admits a $(\mathbb{Z}_5, \beta)$-orientation, unless G has a troublesome partition.*

Before proving Theorem 1.3, we prove a slightly weaker result, assuming the truth of Theorem 2.2.

THEOREM 2.3. *If G is an 11-edge-connected planar graph, then G is strongly $\mathbb{Z}_5$-connected.*

Proof. Let G be an 11-edge-connected planar graph. Fix a partition $\mathcal{P}$. Since G is 11-edge-connected, $d(P_i) \ge 11$ for each i, which implies $w_G(\mathcal{P}) \ge 19$. Thus $w(G) \ge 19$. Since it is easy to see each troublesome partition $\mathcal{P}$ has $w(G/\mathcal{P}) \le 3$, we obtain that G has no partition $\mathcal{P}$ such that $G/\mathcal{P}$ is troublesome. Now Theorem 2.2 implies that G is strongly $\mathbb{Z}_5$-connected. ∎

An *antisymmetric $\mathbb{Z}_5$-flow* in a directed graph $D = D(G)$ is a $\mathbb{Z}_5$-flow such that no two edges have flow values summing to 0. One example is any $\mathbb{Z}_5$-flow that uses only values 1 and 2. Esperet, de Verclos, Le, and Thomassé [6] proved that if a graph G is strongly $\mathbb{Z}_5$-connected, then every orientation $D(G)$ of G admits an antisymmetric $\mathbb{Z}_5$-flow. Together with work of Lovász et al. [13], this implies that every directed 12-edge-connected graph admits an antisymmetric $\mathbb{Z}_5$-flow. Esperet et al. [6] conjectured the stronger result that *every directed* 8*-edge-connected graph admits an antisymmetric* $\mathbb{Z}_5$*-flow.* The concept of antisymmetric flows and its dual, homomorphisms to oriented graphs, were introduced by Nešetřil and Raspaud [16]. In [15], Nešetřil, Raspaud, and Sopena showed that every orientation of a planar graph of girth at least 16 has a homomorphism to an oriented simple graph on at most 5 vertices. The girth condition is reduced to 14 in [4], to 13 in [3], and finally to 12 in [2]. By duality, the results of [16], [6], and [13] combine to imply that girth 12 suffices. After the girth 12 result of Borodin, Ivanova, and Kostochka [2] in 2007, Esperet et al. [6] remarked that "it is not known whether the same holds for planar graphs of girth at least 11." Note that the result of Dvořák and Postle [5] does not seem to apply to homomorphisms to oriented graphs. By Theorem 2.3, we improve this girth bound for planar graphs.

THEOREM 2.4. *Every directed* 11*-edge-connected planar graph admits an antisymmetric* $\mathbb{Z}_5$*-flow. Dually, every orientation of a planar graph of girth at least* 11 *has a homomorphism to an oriented simple graph on at most* 5 *vertices.*

A graph G has *odd edge-connectivity* t if the smallest edge cut of odd size has size t. Our strongest result on modulo 5-orientations is the following, which includes Theorem 1.3 as a special case.

THEOREM 2.5. *Every odd-*11*-edge-connected planar graph admits a modulo* 5*-orientation. In particular, every* 10*-edge-connected planar graph admits a modulo* 5*-orientation (and thus a circular* 5/2*-flow).*

Proof. The second statement follows from the first, by Lemma 1.5. To prove the first, suppose the theorem is false, and let G be a counterexample minimizing $\|G\|$. By Zhang's Splitting Lemma[2] for odd edge-connectivity [22], we know $\delta(G) \geq 11$. If G is 11-edge-connected, then we are done by Theorem 2.3; so assume it is not. Choose a smallest set $W \subset V(G)$ such that $d(W) < 11$. Note that $|W| \geq 2$, and every proper subset $W' \subsetneq W$ satisfies $d(W') \geq 11$. Let $H = G[W]$. For any partition $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ of H with $t \geq 2$, we know that $d_G(P_i) \geq 11$ by the minimality of W, since $P_i \subsetneq W$. This implies

$$\begin{aligned} w_H(\mathcal{P}) &= \sum_{i=1}^{t} d_H(P_i) - 11t + 19 \\ &= \sum_{i=1}^{t} d_G(P_i) - d_G(W^c) - 11t + 19 \\ &> 11t - 11 - 11t + 19 \geq 8. \end{aligned}$$

Thus $w(H) \geq 9$, which implies H is strongly $\mathbb{Z}_5$-connected by Theorem 2.2. By the minimality of G, the graph G/H has a modulo 5-orientation. By Lemma 1.6, this extends to a modulo 5-orientation of G, which completes the proof. ∎

[2]This says that if G has a vertex v with $d(v) \notin \{2, 11\}$, then we can lift a pair of edges incident to v that are successive in the circular order around v, and the resulting graph is still planar and odd-11-edge-connected. For example, if $d(v) = 10$, then all edges incident to v will be lifted in pairs, so the boundary value at v in the resulting orientation will be 0. This is why the proof yields a modulo 5-orientation but does not show that G is strongly $\mathbb{Z}_5$-connected.

2.2. Reducible configurations and partitions. To prove Theorem 2.2, we assume the result is false and study a minimal counterexample. In the next subsection we prove many structural results about the minimal counterexample, which ultimately imply it cannot exist. In this subsection we prove that a few small graphs cannot appear as subgraphs of the minimal counterexample. We call such a forbidden subgraph *reducible*. By Lemma 1.6, to show that H is reducible it suffices to show H is strongly $\mathbb{Z}_5$-connected.

Let G be a graph. We often lift a pair of edges w_1v, vw_2 incident to a vertex v in G to form a new graph G'. That is, we delete w_1v and vw_2 and create a new edge w_1w_2. If G' is strongly $\mathbb{Z}_k$-connected, then so is G, since from any β-orientation of G' we delete the edge w_1w_2 and add the directed edges w_1v and vw_2 to obtain a β-orientation of G. To prove G is strongly $\mathbb{Z}_k$-connected, we use lifting in two similar ways.

First, we lift some edge pairs to create a G' that contains a strongly $\mathbb{Z}_k$-connected subgraph H. If G'/H is strongly $\mathbb{Z}_k$-connected, then so is G' by Lemma 1.6. As discussed in the previous paragraph, so is G. Second, given a $\mathbb{Z}_k$-boundary β, we orient some edges incident to a vertex v to achieve $\beta(v)$. For each edge vw that we orient, we increase or decrease by 1 the value of $\beta(w)$. Now we delete v and all oriented edges and lift the remaining edges incident to v (in pairs). Call the resulting graph and boundary G' and β'. If G' has a β'-orientation, then G has a β-orientation. We call these *lifting reductions of the first and second type*, respectively. In this paper whenever we lift an edge pair vw, wx we require that edge vx already exists. Thus, our lifting reductions always preserve planarity.

LEMMA 2.6. *Each of the graphs* $4K_2, T_{2,3,3}$, $2K_4$, *and* $3C_4$, *shown in Figure* 2, *is strongly* $\mathbb{Z}_5$*-connected.*

Proof. Proving the lemma amounts to checking a finite list of cases. So our goal is to make this as painless as possible. Throughout we fix a $\mathbb{Z}_5$-boundary β and construct an orientation that achieves β.

Let $G = 4K_2$ and $V(G) = \{v_1, v_2\}$. To achieve $\beta(v_1) \in \{0, 1, 2, 3, 4\}$ the number of edges we orient out of v_1 is (respectively) 2, 0, 3, 1, 4.

Let $G = T_{2,3,3}$ and $V(G) = \{v_1, v_2, v_3\}$ with $d(v_1) = d(v_2) = 5$ and $d(v_3) = 6$. If $\beta(v_1) \neq 0$, then we achieve β by orienting 3 edges incident to v_1 and lifting a pair of unused, nonparallel edges incident to v_1 to create a fourth edge v_2v_3. Since $4K_2$ is strongly $\mathbb{Z}_5$-connected, we can use the resulting 4 edges to achieve $\beta(v_2)$ and $\beta(v_3)$. (This is a lifting reduction of the second type. In what follows, we are less explicit about such descriptions.) So we assume that $\beta(v_1) = 0$ and, by symmetry, $\beta(v_2) = 0$. This implies $\beta(v_3) = 0$. Now we orient all edges from v_1 to v_3, from v_1 to v_2, and from v_3 to v_2.

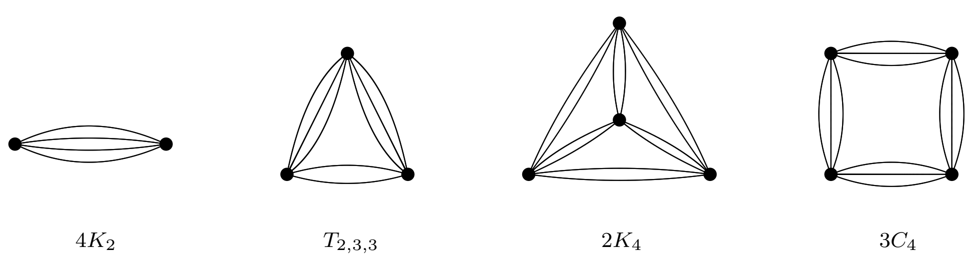


FIG. 2. *The graphs* $4K_2, T_{2,3,3}, 2K_4, 3C_4$.

Let $G = 2K_4$ and $V(G) = \{v_1, v_2, v_3, v_4\}$. If $\beta(v_1) \in \{0, 2, 3\}$, then we achieve $\beta(v_1)$ by orienting two nonparallel edges incident to v_1. Now we lift two pairs of unused edges incident to v_1 to get a $T_{2,3,3}$. Since $T_{2,3,3}$ is strongly $\mathbb{Z}_5$-connected, we are done by Lemma 1.6. So assume $\beta(v_1) \notin \{0, 2, 3\}$. By symmetry, we assume $\beta(v_i) \in \{1, 4\}$ for all i. Since β is a $\mathbb{Z}_5$-boundary, we further assume $\beta(v_i) = 1$ when $i \in \{1, 2\}$ and $\beta(v_j) = 4$ when $j \in \{3, 4\}$. Let $V_1 = \{v_1, v_2\}$ and $V_2 = \{v_3, v_4\}$. Orient all edges from V_2 to V_1. For each pair of parallel edges within V_1 or V_2, orient one edge in each direction. This achieves β.

Let $G = 3C_4$ and $V(G) = \{v_1, v_2, v_3, v_4\}$ with $v_1, v_3 \in N(v_2) \cap N(v_4)$. If $\beta(v_1) \in \{0, 2, 3\}$, then we achieve $\beta(v_1)$ by orienting two nonparallel edges incident to v_1 and lifting two pairs of edges incident to v_1. The resulting unoriented graph is $T_{2,3,3}$, so we are done by Lemma 1.6. Assume instead, by symmetry, that $\beta(v_i) \in \{1, 4\}$ for all i. Since β is a $\mathbb{Z}_5$-boundary, two vertices v_i have $\beta(v_i) = 1$, and two vertices v_j have $\beta(v_j) = 4$. By symmetry, assume $\beta(v_1) = 1$. If $\beta(v_3) = 1$, then orient all edges out from v_1 and v_3. Assume instead, by symmetry, that $\beta(v_2) = 1$; now reverse one edge v_3v_2 from the previous orientation. ∎

DEFINITION 2.7. *For partitions $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ and $\mathcal{P}' = \{P'_1, P'_2, \ldots, P'_s\}$, we say that $\mathcal{P}'$ is a* refinement *of $\mathcal{P}$, denoted by $\mathcal{P}' \preceq \mathcal{P}$, if $\mathcal{P}'$ is obtained from $\mathcal{P}$ by further partitioning P_i into smaller sets for some P_i's in $\mathcal{P}$. More formally, we require that for every $P'_j \in \mathcal{P}'$, there exists $P_i \in \mathcal{P}$ such that $P'_j \subseteq P_i$.*

Since partitions are central to our theorems and proofs, we name a few common types of them. A partition $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ is trivial *if each part P_i is a singleton, i.e., $V(G)$ is partitioned into $|G|$ parts; otherwise $\mathcal{P}$ is* nontrivial*. A trivial partition is minimal under the relation $\prec$. A partition $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ is* almost trivial *if $t = |G| - 1$ and there is a unique part P_i with $|P_i| = 2$. A partition $\mathcal{P}$ is called* normal *if it is neither trivial nor almost trivial and $\mathcal{P} \neq \{V(G)\}$.*

Given a partition $\mathcal{P}$ of $V(G)$ and a partition $\mathcal{Q}$ of $G[P_1]$, the following lemma relates the weights of $\mathcal{P}$, $\mathcal{Q}$, and the refinement $\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})$.

LEMMA 2.8. *Let $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ be a partition of $V(G)$ with $|P_1| > 1$. Let $H = G[P_1]$, and let $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_s\}$ be a partition of $V(H)$. Now $\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})$ is a refinement of $\mathcal{P}$ satisfying*

$$w_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) = w_H(\mathcal{Q}) + w_G(\mathcal{P}) - 8. \tag{2.1}$$

Proof. Clearly, $\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})$ is a refinement of $\mathcal{P}$, and it follows from Definition 2.1 that

$$\begin{aligned}
w_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) &= \sum_{i=1}^{s} d_G(Q_i) + \sum_{j=2}^{t} d_G(P_j) - 11(s + t - 1) + 19 \\
&= \left[\sum_{i=1}^{s} d_G(Q_i) - d_G(P_1) - 11s + 19\right] + \left[\sum_{j=1}^{t} d_G(P_j) - 11(t - 1)\right] \\
&= \left[\sum_{i=1}^{s} d_H(Q_i) - 11s + 19\right] + \left[\sum_{j=1}^{t} d_G(P_j) - 11t + 19\right] - (19 - 11) \\
&= w_H(\mathcal{Q}) + w_G(\mathcal{P}) - (19 - 11).
\end{aligned}$$ ∎

2.3. Properties of a minimal counterexample to Theorem 2.2. Let G be a counterexample to Theorem 2.2 that minimizes $|G|+\|G\|$. Thus Theorem 2.2 holds for all graphs smaller than G. This implies the following lemma, which we will use frequently.

LEMMA 2.9. *If H is a planar graph with $w(H) \geq 0$ and $|H|+\|H\| < |G|+\|G\|$, then each of the following holds.*

(a) *If $w_H(\mathcal{P}) \geq 4$ for every nontrivial partition $\mathcal{P}$, then H is strongly $\mathbb{Z}_5$-connected unless $H \in \{2K_2, 3K_2, T_{1,3,3}, T_{2,2,3}\}$.*
(b) *If $w(H) \geq 1$ and H is 4-edge-connected, then H is strongly $\mathbb{Z}_5$-connected.*
(c) *If $w(H) \geq 4$, then H is strongly $\mathbb{Z}_5$-connected.*

Proof. To prove each part, we fix a $\mathbb{Z}_5$-boundary β and apply Theorem 2.2 to H. Notice that each troublesome partition $\mathcal{P}$ satisfies $w(G/\mathcal{P}) \leq 3$. So for (a), only the trivial partition can be troublesome. Thus, H is strongly $\mathbb{Z}_5$-connected unless $H \in \{2K_2, 3K_2, T_{1,3,3}, T_{2,2,3}\}$. For (b), G has no partition $\mathcal{P}$ with $G/\mathcal{P} \in \{2K_2, 3K_2\}$ since G is 4-edge-connected. And G has no partition $\mathcal{P}$ with $G/\mathcal{P} \in \{T_{1,3,3}, T_{2,2,3}\}$ since $w(H) \geq 1$. So H is again strongly $\mathbb{Z}_5$-connected, by Theorem 2.2. Finally, (c) follows from (b), since if H has an edge cut $[X, X^c]$ of size at most 3, then $w_H(\{X, X^c\}) \leq 2(3) - 11(2) + 19 = 3$, which contradicts our assumption that $w(H) \geq 4$. ∎

The main idea of our proof is to show that the value of the weight function $w_G(\mathcal{P})$ is relatively large for each nontrivial partition $\mathcal{P}$. This enables us to slightly modify certain proper subgraphs and still apply Lemma 2.9 to the resulting graph H. This added flexibility (to slightly modify the subgraph) helps us to prove that more subgraphs are reducible. In the next section, these forbidden subgraphs facilitate a discharging proof that shows that our minimal counterexample G cannot exist.

CLAIM 1. *G has no strongly $\mathbb{Z}_5$-connected subgraph H with $|H| > 1$. In particular,*
(a) *G has no copy of $4K_2$, $T_{2,3,3}$, $2K_4$, or $3C_4$ (by Lemma 2.6), and*
(b) *$|G| \geq 4$.*

Proof. Suppose to the contrary that H is a strongly $\mathbb{Z}_5$-connected subgraph of G with $|H| > 1$, and let $G' = G/H$. Since G is a minimal counterexample, G' is strongly $\mathbb{Z}_5$-connected, by Theorem 2.2. So Lemma 1.6 implies G is strongly $\mathbb{Z}_5$-connected, which is a contradiction. This proves both the first statement and (a). For (b), clearly $|G| \geq 3$, since $w(G) \geq 0$ and $G \notin \{2K_2, 3K_2\}$ and G contains no $4K_2$. So assume $|G| = 3$. Since $w(G/\mathcal{P}) \geq 0$ for the trivial partition $\mathcal{P}$, we know that $\|G\| \geq 8$. Since $G \notin \{T_{1,3,3}, T_{2,2,3}\}$, either G contains $4K_2$ or G contains $T_{2,3,3}$. Each case contradicts (a). ∎

CLAIM 2. *Let $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ be a nontrivial partition of $V(G)$. Now*
(a) *$w_G(\mathcal{P}) \geq 5$ and*
(b) *$w_G(\mathcal{P}) \geq 8$ if $\mathcal{P}$ is normal.*

Proof. Our proof is by contradiction. For an almost trivial partition $\mathcal{P}$, we have $w_G(\mathcal{P}) \geq w_G(V(G)) - 2(3) + 11 \geq 5$, since G does not contain $4K_2$ by Claim 1(a). If $\mathcal{P} = \{V(G)\}$, then $w_G(\mathcal{P}) = 0 - 11 + 19 = 8$. By definition, all other nontrivial partitions are normal.

Let $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ be a normal partition of $V(G)$. By symmetry we assume $|P_1| > 1$ and let $H = G[P_1]$. For any partition $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_s\}$ of $V(H)$, by (2.1) the refinement $\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})$ of $\mathcal{P}$ satisfies

$$w_H(\mathcal{Q}) = w_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) - w_G(\mathcal{P}) + 8. \tag{2.2}$$

(a) We first show that $w_G(\mathcal{P}) \geq 5$. If $w_G(\mathcal{P}) \leq 4$, then (2.2) implies $w_H(\mathcal{Q}) \geq 4$ for any partition $\mathcal{Q}$ of H, since $w_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) \geq 0$. Hence $w(H) \geq 4$ and H is strongly $\mathbb{Z}_5$-connected by Lemma 2.9(c), which contradicts Claim 1. This proves (a).

(b) We now show that $w_G(\mathcal{P}) \geq 8$. Suppose to the contrary that $w_G(\mathcal{P}) \leq 7$. If $\mathcal{P}$ contains at least two nontrivial parts, say, $|P_2| > 1$, then (a) implies $w_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) \geq 5$ for any partition $\mathcal{Q}$ of H. Hence $w(H) \geq 6$ by (2.2), and so H is strongly $\mathbb{Z}_5$-connected by Lemma 2.9(c), which contradicts Claim 1. So assume instead that $\mathcal{P}$ contains a unique nontrivial part P_1 and $|P_1| \geq 3$. For any nontrivial partition $\mathcal{Q}$ of H, the refinement $\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})$ of $\mathcal{P}$ is a nontrivial partition of G, and so $w_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) \geq 5$ by (a). Thus $w_H(\mathcal{Q}) \geq 6$ for any nontrivial partition $\mathcal{Q}$ of H by (2.2). For the trivial partition $\mathcal{Q}^*$ of H, since $w_G(\mathcal{P}) \leq 7$, (2.2) implies $w_H(\mathcal{Q}^*) \geq 1$. Since $|H| = |P_1| \geq 3$, we know $H \notin \{2K_2, 3K_2\}$. Since $w(H) \geq 1$, we know $H \ncong T_{a,b,c}$ with $a+b+c \leq 7$. So Lemma 2.9(a) implies that H is strongly $\mathbb{Z}_5$-connected, which contradicts Claim 1. ∎

The next two claims are consequences of Claim 2; they give lower bounds on the edge-connectivity of G.

Claim 3. *For a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$,*
(a) *if $|P_1| \geq 2$ and $|P_2| \geq 2$, then $w_G(\mathcal{P}) \geq 10$;*
(b) *if $|P_1| \geq 2$ and $|P_2| \geq 3$, then $w_G(\mathcal{P}) \geq 13$.*

Proof. Let $H = G[P_1]$ and $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_s\}$ be a partition of H. Let $\mathcal{P}' = \mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})$. Note that if $|P_2| \geq 2$, then the refinement $\mathcal{P}'$ is nontrivial, and if $|P_2| \geq 3$, then $\mathcal{P}'$ is normal. By (2.1),

$$w_G(\mathcal{P}') = w_H(\mathcal{Q}) + w_G(\mathcal{P}) - 8. \tag{2.3}$$

By Claim 1, H is not strongly $\mathbb{Z}_5$-connected. So, by Lemma 2.9(c), we can choose $\mathcal{Q}$ such that $w_H(\mathcal{Q}) \leq 3$. Substituting into (2.3) above yields $w_G(\mathcal{P}) = w_G(\mathcal{P}') + 8 - w_H(\mathcal{Q}) \geq w_G(P')+5$. (a) Since $\mathcal{P}'$ is nontrivial, Claim 2(a) implies $w_G(\mathcal{P}') \geq 5$, which gives $w_G(\mathcal{P}) \geq 5+5 = 10$. (b) Since $\mathcal{P}'$ is normal, Claim 2(b) implies $w_G(\mathcal{P}') \geq 8$, which gives $w_G(\mathcal{P}) \geq 8+5 = 13$. ∎

Claim 4. *Let $[X, X^c]$ be an edge cut of G.*
(a) *Now $|[X, X^c]| \geq 6$. That is, G is 6-edge-connected.*
(b) *If $|X| \geq 2$ and $|X^c| \geq 3$, then $|[X, X^c]| \geq 8$.*

Proof. If $[X, X^c]$ is an edge cut of G, then $\mathcal{P} = \{X, X^c\}$ is a partition of $V(G)$. (a) Clearly $\mathcal{P}$ is normal, since $|G| \geq 4$ by Claim 1(b). Now Claim 2(b) implies $8 \leq w_G(\mathcal{P}) = 2|[X, X^c]| - 22 + 19$, which yields $|[X, X^c]| \geq 6$. (b) If $|X| \geq 2$ and $|X^c| \geq 3$, then $w(\mathcal{P}) \geq 13$ by Claim 3(b). So $13 \leq w_G(\mathcal{P}) = 2|[X, X^c]| - 22 + 19$, which implies $|[X, X^c]| \geq 8$. ∎

Next we show that G contains no copy of any graph in Figure 3 below. We write H° to denote the graph formed from H by subdividing one copy of an edge of maximum multiplicity. So, for example, $4K_2^\circ = T_{1,1,3}$. We write $H^{\circ\circ}$ to denote $(H^\circ)^\circ$. (The reader may think of the $\circ$ as representing the new 2-vertex.)

Claim 5. *G has no copy of $T_{1,1,3}$.*

Proof. Suppose G contains a copy of $T_{1,1,3}$ with vertices x, y, z and $\mu(xy) = 3$. We lift xz, zy to become a new edge xy and then contract the corresponding $4K_2$ (contract xy). Let G' denote the resulting graph. The trivial partition $\mathcal{Q}^*$ of G' satisfies $w_{G'}(\mathcal{Q}^*) \geq w(G) - 2(5) + 11 \geq 1$. If $\mathcal{Q}' = \{V(G')\}$, then $\rho_{G'}(\mathcal{Q}') = 0 - 17 + 31 = 14$.

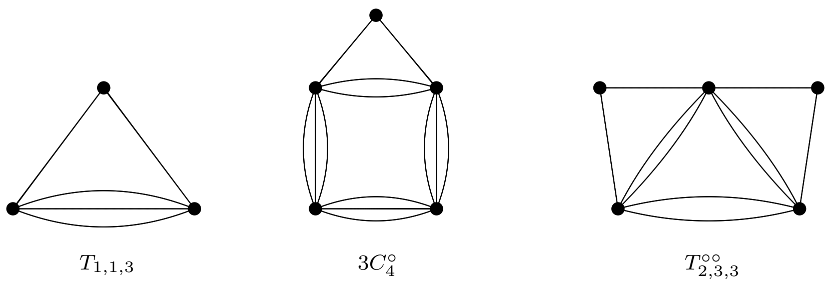


FIG. 3. *The graphs* $T_{1,1,3}, 3C_4^\circ, T_{2,3,3}^{\circ\circ}$.

Every other nontrivial partition $\mathcal{Q}'$ of G' corresponds to a normal partition $\mathcal{Q}$ of G in which the contracted vertex is replaced by $\{x, y\}$. Since xz, zy are the only two edges possibly counted in $w_G(\mathcal{Q})$ but not in $w_{G'}(\mathcal{Q}')$, we have $w_{G'}(\mathcal{Q}') \geq w_G(\mathcal{Q}) - 4 \geq 4$, by Claim 2(b). Thus $w(G') \geq 1$. By Claim 4, G is 6-edge-connected, so G' is 4-edge-connected. Thus G' is strongly $\mathbb{Z}_5$-connected, by Lemma 2.9(b). This is a lifting reduction of the first type, so G is strongly $\mathbb{Z}_5$-connected, which is a contradiction. ◻

CLAIM 6. *G has no copy of* $3C_4^\circ$.

Proof. Suppose G contains a copy of $3C_4^\circ$ with vertices v_1, v_2, v_3, v_4, z, where z is a 2-vertex with $N(z) = \{v_1, v_2\}$. We lift v_1z, zv_2 to become a new edge v_1v_2 and then contract the corresponding $3C_4$ to obtain the graph G'. For the trivial partition $\mathcal{Q}^*$ of G', we have $w_{G'}(\mathcal{Q}^*) \geq w(G) - 2(13) + 3(11) \geq 7$. For every nontrivial partition $\mathcal{Q}'$ of G', we have $w_{G'}(\mathcal{Q}') \geq w_G(\mathcal{Q}) - 4 \geq 4$ for the same reason as in the previous claim. Thus $w(G') \geq 4$, so G' is strongly $\mathbb{Z}_5$-connected by Lemma 2.9(c). This is a lifting reduction of the first type. Hence G is strongly $\mathbb{Z}_5$-connected, which contradicts Claim 1. ◻

Now we can slightly strengthen Claim 2(b).

CLAIM 7. *Every normal partition* $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ *satisfies*

$$w(\mathcal{P}) \geq 9.$$

Proof. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ be a normal partition of G with $|P_1| > 1$. Suppose to the contrary that $w(\mathcal{P}) = 8$, by Claim 2(b). Now $|P_1| \geq 3$ and $|P_2| = \cdots = |P_t| = 1$, by Claim 3(a). As in Claim 2, let $H = G[P_1]$, let $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_s\}$ be a partition of H, and let $\mathcal{P}' = \mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})$ be a refinement of $\mathcal{P}$. Equation (2.1) implies

$$w_H(\mathcal{Q}) = w_G(\mathcal{P}') - w_G(\mathcal{P}) + 8 = w_G(\mathcal{P}').$$

If $\mathcal{Q}$ is a nontrivial partition of H, then $\mathcal{P}'$ is nontrivial in G, so $w_H(\mathcal{Q}) = w_G(\mathcal{P}') \geq 5$, by Claim 2(a). If $\mathcal{Q}$ is the trivial partition of H, then $w_H(\mathcal{Q}) = w_G(\mathcal{P}') \geq 0$. Since $|H| = |P_1| \geq 3$, we know $H \notin \{2K_2, 3K_2\}$. And since G has no copy of $T_{1,1,3}$, by Claim 5, we know $H \notin \{T_{1,3,3}, T_{2,2,3}\}$. Now Lemma 2.9(a) implies that H is strongly $\mathbb{Z}_5$-connected, which contradicts Claim 1. ◻

Claim 7 allows us to also prove that the third graph in Figure 3 is reducible.

CLAIM 8. *G has no copy of* $T_{2,3,3}^{\circ\circ}$.

Proof. Suppose G contains a copy of $T^{\circ\circ}_{2,3,3}$ with vertices w, x, y, z_1, z_2, where z_1 and z_2 are 2-vertices with $N(z_1) = \{w, x\}$ and $N(z_2) = \{x, y\}$. We lift wz_1, z_1x to become a new edge wx and lift xz_2, z_2y to become a new edge xy. Now $\{w, x, y\}$ induces a copy of $T_{2,3,3}$, so we contract $\{w, x, y\}$ to form a graph G'. Since $\delta(G) \geq 6$ by Claim 4(a), we have $\delta(G') \geq 4$. The size of each edge cut decreases at most 4 from G to G', and it decreases at least 3 only if that edge cut has at least two vertices on each side. In that case Claim 4(b) shows the original edge cut in G has size at least 8. Since G is 6-edge-connected by Claim 4, each edge cut in G' has size at least 4, so G' is 4-edge-connected.

The trivial partition $\mathcal{Q}^*$ of G' satisfies $w_{G'}(\mathcal{Q}^*) \geq w(G) - 2(10) + 11(2) \geq 2$. If $\mathcal{Q}' = \{V(G')\}$, then $\rho_{G'}(\mathcal{Q}') = 0 - 17 + 31 = 14$. Every other nontrivial partition $\mathcal{Q}'$ of G' corresponds to a normal partition $\mathcal{Q}$ of G in which the contracted vertex is replaced by $\{w, x, y\}$. So $w_{G'}(\mathcal{Q}') \geq w_G(\mathcal{Q}) - 2(4) \geq 1$, by Claim 7. Thus, G' is 4-edge-connected and $w(G') \geq 1$. By Lemma 2.9(b), G' is strongly $\mathbb{Z}_5$-connected. This is a lifting reduction of the first type. Since $T_{2,3,3}$ is strongly $\mathbb{Z}_5$-connected by Lemma 2.6, graph G is strongly $\mathbb{Z}_5$-connected, which contradicts Claim 1. ∎

2.4. The final step: Discharging. Now we use discharging to show that some subgraph in Figure 2 or 3 must appear in G. This contradicts one of the claims in the previous section and thus finishes the proof.

Fix a plane embedding of G. (We assume that all parallel edges between two vertices v and w are embedded consecutively, in the cyclic orders, around both v and w.) Let $F(G)$ denote the set of all faces of G. For each face $f \in F(G)$, we write $\ell(f)$ for its length. A face f is a *k-face*, *k^+-face*, or *k^--face* if (respectively) $\ell(f) = k$, $\ell(f) \geq k$, or $\ell(f) \leq k$. A sequence of faces $f_1 f_2 \ldots f_s$ is called a *face chain* if, for each $i \in \{1, \ldots, s-1\}$, faces f_i and f_{i+1} are adjacent, i.e., their boundaries share a common edge. The *length* of this chain is $s + 1$. Two faces f and f' are *weakly adjacent* if there is a face chain $f f_1 \ldots f_s f'$ such that that f_i is a 2-face for each $i \in \{1, \ldots, s\}$. We allow s to be 0, meaning f and f' are adjacent. A *string* is a maximal face chain such that each of its faces is a 2-face. The boundary of a string consists of two edges, each of which is incident to a 3^+-face. A k-face is called a $(t_1, t_2, \ldots, t_k)$-face if its boundary edges are contained in strings with lengths $t_1, t_2, \ldots, t_k$. Here t_i is allowed to be 1, meaning the corresponding edge is not contained in a string.

Since $w(G) \geq 0$, we have $2\|G\| - 11|G| + 19 \geq 0$. By Euler's formula, $|G| + |F(G)| - \|G\| = 2$. We solve for $|G|$ in the equation and substitute into the inequality, which gives

$$\sum_{f \in F(G)} \ell(f) = 2\|G\| \leq \frac{22}{9}|F(G)| - \frac{2}{3}. \tag{2.4}$$

We assign to each face f initial charge $\ell(f)$. So the total charge is strictly less than $22|F(G)|/9$. To redistribute charge, we use the following three discharging rules.

(R1) Each 2-face receives charge $\frac{2}{9}$ from each weakly adjacent 3^+-face.

(R2) Each $(2,2,2)$-face receives charge $\frac{1}{9}$ from each weakly adjacent 4^+-face and $(2,1,1)$-face.

(R3) Each $(2,2,2)$-face receives charge $\frac{1}{18}$ from each weakly adjacent $(2,2,1)$-face.

If two faces are weakly adjacent through multiple edges or strings, then the discharging rules apply for each edge and string. After applying these rules, we claim that every face has charge at least $\frac{22}{9}$, which contradicts (2.4).

Each 2-face ends with $2 + 2(\frac{2}{9}) = \frac{22}{9}$. Since G contains no $4K_2$ and no $T_{1,1,3}$, the charge each face sends across each boundary edge is at most $2(\frac{2}{9})$. Thus, when $k \geq 5$

each k-face ends with at least $k - k(2(\frac{2}{9})) = \frac{5k}{9} \geq \frac{25}{9}$. Since G contains no $3C_4$ and no $3C_4^\circ$, each 4-face ends with at least $4 - 7(\frac{2}{9}) = \frac{22}{9}$. It is straightforward to check that each $(1,1,1)$-face ends with 3, each $(2,1,1)$-face ends with at least $3 - \frac{2}{9} - \frac{1}{9} = \frac{24}{9}$, and each $(2,2,1)$-face ends with at least $3 - 2(\frac{2}{9}) - 2(\frac{1}{18}) = \frac{22}{9}$. It remains to check $(2,2,2)$-faces.

Suppose to the contrary that a $(2,2,2)$-face xyz ends with less than $\frac{22}{9}$. After (R1), face xyz has $3 - 3(\frac{2}{9}) = \frac{21}{9}$. Since xyz ends with less than $\frac{22}{9}$, it receives at most $\frac{1}{18}$ by (R2) and (R3). So xyz must be adjacent to three 3-faces, and at most one of these is a $(2,2,1)$-face, while the others are $(2,2,2)$-faces. By Claim 8, G contains no $T^{\circ\circ}_{2,3,3}$, so the three 3-faces adjacent to xyz must share a new common vertex, say w. If one of wx, wy, wz is not contained in a string, then xyz is adjacent to two $(2,2,1)$-faces and so receives at least $2(\frac{1}{18})$ by (R3), contradicting our assumption above. Thus we assume $\mu(wx) = \mu(wy) = \mu(wz) = 2$. So $G[\{x,y,z,w\}]$ contains a $2K_4$, contradicting Claim 1(a). This shows that each $(2,2,2)$-face ends with at least $\frac{22}{9}$, which completes the proof.

3. Circular 7/3-flows: Proof of Theorem 1.4.

In this section we prove Theorem 1.4. As in the previous section, this theorem is implied by the more technical result, Theorem 3.3. The proof of Theorem 3.3 is similar to that of Theorem 2.2, but with more reducible configurations and more details.

3.1. Preliminaries on modulo 7-orientations.

We define a weight function ρ as follows (which is similar to w in Definition 2.1).

DEFINITION 3.1. *Let $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ be a partition of $V(G)$. Let*

$$\rho_G(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 17t + 31$$

and $\rho(G) = \min\{\rho_G(\mathcal{P}) : \mathcal{P}$ is a partition of $V(G)\}$.

Analogous to Lemma 2.8, we have the following.

LEMMA 3.2. *Let $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ be a partition of $V(G)$ with $|P_1| > 1$. Let $H = G[P_1]$, and let $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_s\}$ be a partition of $V(H)$. Now $\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})$ is a refinement of $\mathcal{P}$ satisfying*

$$\rho_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) = \rho_H(\mathcal{Q}) + \rho_G(\mathcal{P}) - (31 - 17). \tag{3.1}$$

Proof. The proof is identical to that of Lemma 2.8, with 17 in place of 11 and with 31 in place of 19. ∎

We typically assume that each edge has multiplicity at most 5 (since $6K_2$ is strongly $\mathbb{Z}_7$-connected and so cannot appear in a minimal counterexample to Theorem 3.3, as we prove in Claim 9, below). Now $\rho(aK_2) = 2a - 3$, $\rho(T_{a,b,c}) = 2a + 2b + 2c - 20$, and $\rho(3K_4) = -1$; see Figure 4. In each case, the minimum in the definition of ρ is achieved uniquely by the partition with each vertex in its own part.

Let $\mathcal{F} = \{aK_2 : 2 \leq a \leq 5\} \cup \{T_{a,b,c} : 10 \leq a + b + c \leq 11$ and $T_{a,b,c}$ is 6-edge-connected$\}$. It is straightforward[3] to check that neither $3K_4$ nor any graph in $\mathcal{F}$ is

[3]The graph $3K_4$ cannot achieve the boundary $\beta(v) = 0$ for all v. In such an orientation D each vertex v must have $|d^+_D(v) - d^-_D(v)| = 7$. But now some two adjacent vertices must either both have indegree 8 or both have outdegree 8, and we cannot orient the three edges between them to

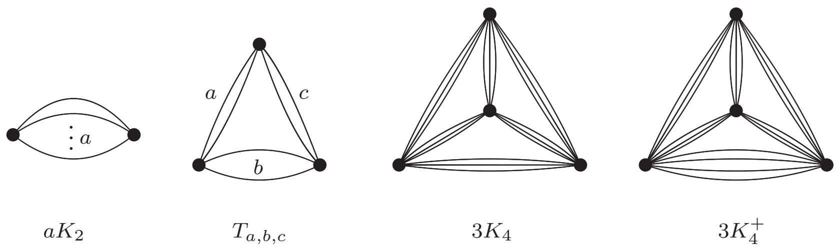


FIG. 4. *The graphs* $aK_2, T_{a,b,c}, 3K_4, 3K_4^+$.

strongly $\mathbb{Z}_7$-connected. Further, if $T_{a,b,c}$ is 8-edge-connected, then $\|G\| \geq 3\delta(G)/2 \geq 12$. Thus, no graph in $\mathcal{F}$ is 8-edge-connected. The following theorem is the main result of section 3. We call a partition $\mathcal{P}$ *problematic* if $G/\mathcal{P} \in \mathcal{F}$.

THEOREM 3.3. *Let G be a planar graph and β be a $\mathbb{Z}_7$-boundary of G. If $\rho(G) \geq 0$, then G admits a $(\mathbb{Z}_7, \beta)$-orientation, unless G has a problematic partition.*

As easy corollaries of Theorem 3.3 we get the following two results.

THEOREM 3.4. *Every* 17*-edge-connected planar graph is strongly $\mathbb{Z}_7$-connected.*

THEOREM 3.5. *Every odd-*17*-edge-connected planar graph admits a modulo* 7*-orientation. In particular, every* 16*-edge-connected planar graph admits a modulo* 7*-orientation (and thus a circular* 7/3*-flow).*

The proofs of Theorems 3.4 and 3.5 are identical to those of Theorems 2.3 and 2.5, but with 17 in place of 11 and with 31 in place of 19. Note that Theorem 3.5 includes Theorem 1.4 as a special case.

For the proof of Theorem 3.3, we need the following two lemmas. Their proofs are more tedious than enlightening, so we postpone them to the appendix. When a graph H is edge-transitive, we write H^+ or H^- to denote the graph formed by adding or removing a single copy of one edge.

LEMMA 3.6. *Each of the following graphs is strongly $\mathbb{Z}_7$-connected: $6K_2$, $3K_4^+$, and every* 6*-edge-connected graph $T_{a,b,c}$ where $a + b + c = 12$.*

Let $5C_4^=$ denote the graph formed from $5C_4$ by deleting a perfect matching.

LEMMA 3.7. *The graph $5C_4^=$ is strongly $\mathbb{Z}_7$-connected. Further, if G is a graph with $|G| = 4$, $\|G\| = 19$, $\mu(G) \leq 5$, and $\delta(G) \geq 8$, then G is strongly $\mathbb{Z}_7$-connected.*

3.2. Properties of a minimal counterexample in Theorem 3.3. Let G be a counterexample to Theorem 3.3 that minimizes $|G| + \|G\|$. Thus Theorem 3.3 holds for all graphs smaller than G. This implies the following lemma, which we will use frequently.

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achieve this. When $a \leq 5$, the graph aK_2 has seven $\mathbb{Z}_7$-boundaries and at most 6 orientations, so at least one boundary is not achievable. For $T_{a,b,c}$, it suffices to consider the case $a + b + c = 11$. Let $V(G) = \{v_1, v_2, v_3\}$. By symmetry, we assume $d(v_1) \leq d(v_2) \leq d(v_3)$. For $T_{1,5,5}$, we cannot achieve $\beta(v_1) = \beta(v_2) = 1$ and $\beta(v_3) = 5$, since v_1 and v_2 must each have all incident edges oriented in. For $T_{2,4,5}$, we cannot achieve $\beta(v_1) = 1$, $\beta(v_2) = 2$, and $\beta(v_3) = 4$, since v_1 must have all incident edges oriented in, and v_2 must have all but one edges oriented in. For $T_{3,3,5}$, we cannot achieve $\beta(v_1) = 1$ and $\beta(v_2) = \beta(v_3) = 3$, since v_1 must have all incident edges oriented in. For $T_{3,4,4}$, we cannot achieve $\beta(v_1) = \beta(v_2) = 2$ and $\beta(v_3) = 3$, since v_1 and v_2 must each have all but one incident edge oriented in.

Lemma 3.8. *If H is a planar graph with $\rho(H) \geq 0$ and $|H| + \|H\| < |G| + \|G\|$, then each of the following holds.*

(a) *If $\rho_H(\mathcal{P}) \geq 8$ for every nontrivial partition $\mathcal{P}$, then H is strongly $\mathbb{Z}_7$-connected unless $H \in \mathcal{F}$.*

(b) *If $\rho(H) \geq 8$, then H is strongly $\mathbb{Z}_7$-connected.*

(c) *Assume that H is 6-edge-connected.*

  (c-i) *If $\rho_H(\mathcal{P}) \geq 3$ for every nontrivial partition $\mathcal{P}$, then H is strongly $\mathbb{Z}_7$-connected unless $H \cong T_{a,b,c}$ with $a + b + c \in \{10, 11\}$.*

  (c-ii) *If $\rho(H) \geq 3$, then H is strongly $\mathbb{Z}_7$-connected.*

  (c-iii) *If H is 8-edge-connected, then H is strongly $\mathbb{Z}_7$-connected.*

Proof. We apply Theorem 3.3 to H. (a) For each $J \in \mathcal{F}$, the trivial partition $\mathcal{Q}^*$ satisfies $\rho_J(\mathcal{Q}^*) \leq \max\{2(5)-2(17)+31, 2(11)-3(17)+31\} = 7$. Since $\rho_H(\mathcal{P}) \geq 8$ for every nontrivial partition $\mathcal{P}$, we know that $H/\mathcal{P} \notin \mathcal{F}$. Part (b) follows immediately from (a). Consider (c). Since H is 6-edge-connected, there does not exist $\mathcal{P}$ such that $|H/\mathcal{P}| = 2$ and $\|H/\mathcal{P}\| \leq 5$. For (c-i), suppose there is a nontrivial partition $\mathcal{P}$ such that $H/\mathcal{P} \cong T_{a,b,c}$ with $a + b + c \in \{10, 11\}$. Now $\rho_H(\mathcal{P}) = 2(11) - 3(17) + 31 = 2$, which contradicts the hypothesis. Note that (c-ii) follows directly from (c-i). Finally, we prove (c-iii). Since G is 8-edge-connected, so is $G/\mathcal{P}$, for each partition $\mathcal{P}$. Recall that each element of $\mathcal{F}$ has edge-connectivity at most 7. Thus, $G/\mathcal{P} \notin \mathcal{F}$. ∎

As in section 2, the main idea of the proof is to show that $\rho_G(\mathcal{P})$ is relatively large for each nontrivial partition $\mathcal{P}$. This gives us the ability to apply Lemma 3.8 to subgraphs of G even after modifying them slightly, which yields more power when proving subgraphs are reducible.

Claim 9. *G has no strongly $\mathbb{Z}_7$-connected subgraph H with $|H| > 1$. In particular,*

(a) *G has no copy of $6K_2$, $3K_4^+$, or a 6-edge-connected graph $T_{a,b,c}$ with $a+b+c = 12$; and*

(b) $|G| \geq 4$.

Proof. The proof of the first statement is identical to that of Claim 1, with $\mathbb{Z}_7$ in place of $\mathbb{Z}_5$. Note that (a) follows from the first statement and Lemma 3.6.

Now we prove (b). Clearly $|G| \geq 2$, so first suppose $|G| = 2$. Since $\rho(G) \geq 0$, we know $\|G\| \geq 2$. Since G has no problematic partition, we know $\|G\| \geq 6$. But now G contains $6K_2$, which contradicts (a). So assume $|G| = 3$, that is, $G = T_{a,b,c}$. Since $\rho(G) \geq 0$, we know $a + b + c \geq 10$. Since G has no problematic partition, G is 6-edge-connected. By the definition of $\mathcal{F}$, this implies that $a+b+c \geq 12$. Recall that G contains no $6K_2$ by (a); thus $\max\{a, b, c\} \leq 5$. A short case analysis shows that G contains as a subgraph one of $T_{2,5,5}$, $T_{3,4,5}$, or $T_{4,4,4}$. Each of these has 12 edges and is 6-edge-connected, which contradicts (a). ∎

Claim 10. *If $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ is a nontrivial partition of $V(G)$, then*

(a) *$\rho_G(\mathcal{P}) \geq 7$ and*

(b) *$\rho_G(\mathcal{P}) \geq 12$ if $\mathcal{P}$ is normal.*

Proof. We argue by contradiction. For an almost trivial partition $\mathcal{P}$, we have $\rho_G(\mathcal{P}) \geq \rho_G(V(G)) - 2(5) + 17 \geq 7$, since G does not contain $6K_2$ by Claim 9. If $\mathcal{P} = \{V(G)\}$, then $w_G(\mathcal{P}) = 0 - 17 + 31 = 14$. We now only need to consider the weight of normal partitions.

Let $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ be a normal partition of $V(G)$. We may assume $|P_1| > 1$ and let $H = G[P_1]$. For any partition $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_s\}$ of $V(H)$, by (3.1) the refinement $\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})$ of $\mathcal{P}$ satisfies

$$\rho_H(\mathcal{Q}) = \rho_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) - \rho_G(\mathcal{P}) + 14. \tag{3.2}$$

(a) We first show that $\rho_G(\mathcal{P}) \geq 7$. If $\rho_G(\mathcal{P}) \leq 6$, then (3.2) implies that $\rho_H(\mathcal{Q}) \geq 8$ for any partition $\mathcal{Q}$ of H, since $\rho_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) \geq 0$. Hence $\rho(H) \geq 8$ and H is strongly $\mathbb{Z}_7$-connected by Lemma 3.8(b), which contradicts Claim 9. This proves (a).

(b) We now show that $\rho_G(\mathcal{P}) \geq 12$. Suppose, to the contrary, that $\rho_G(\mathcal{P}) \leq 11$. If $\mathcal{P}$ contains at least two nontrivial parts, say, $|P_2| > 1$, then (a) implies $\rho_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) \geq 7$ for any partition $\mathcal{Q}$ of H. Hence $\rho(H) \geq 10$ by (3.2), and so H is strongly $\mathbb{Z}_7$-connected by Lemma 3.8(b), which contradicts Claim 9. Assume instead that $\mathcal{P}$ contains a unique nontrivial part P_1 and $|P_1| \geq 3$. For any nontrivial partition $\mathcal{Q}$ of H, the refinement $\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})$ of $\mathcal{P}$ is a nontrivial partition of G, and so $\rho_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) \geq 7$ by (a). Thus $\rho_H(\mathcal{Q}) \geq 10$ for any nontrivial partition $\mathcal{Q}$ of H by (3.2). For the trivial partition $\mathcal{Q}^*$ of H, since $\rho_G(\mathcal{P}) \leq 11$, (3.2) implies $\rho_H(\mathcal{Q}^*) \geq 3$. Since $|H| = |P_1| \geq 3$, we know $H \ncong aK_2$. Since $\rho(H) \geq 3$, we know $H \ncong T_{a,b,c}$ with $a + b + c \leq 11$. So Lemma 3.8(a) implies that H is strongly $\mathbb{Z}_7$-connected, which contradicts Claim 9. ∎

The next two claims follow from Claim 10. They give lower bounds on the edge-connectivity of G.

Claim 11. *For a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$,*
(a) *if $|P_1| \geq 2$ and $|P_2| \geq 2$, then $\rho_G(\mathcal{P}) \geq 14$, and*
(b) *if $|P_1| \geq 2$ and $|P_2| \geq 3$, then $\rho_G(\mathcal{P}) \geq 19$.*

Proof. Let $H = G[P_1]$ and $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_s\}$ be a partition of H. By (3.1),

$$\rho_H(\mathcal{Q}) = \rho_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) - \rho_G(\mathcal{P}) + 14. \tag{3.3}$$

By Claim 9, H is not strongly $\mathbb{Z}_7$-connected. So, by Lemma 3.8(b), we can choose $\mathcal{Q}$ such that $w_H(\mathcal{Q}) \leq 7$. Substituting into (3.3) above yields $w_G(\mathcal{P}) = w_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) + 14 - w_H(\mathcal{Q}) \geq w_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) + 7$. (a) Since $\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})$ is nontrivial, Claim 10(a) implies $w_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) \geq 7$, which gives $w_G(\mathcal{P}) \geq 7 + 7 = 14$. (b) Since $\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})$ is normal, Claim 10(b) implies $w_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) \geq 12$, which gives $w_G(\mathcal{P}) \geq 12 + 7 = 19$. ∎

Claim 12. *Let $[X, X^c]$ be an edge cut of G.*
(a) *Now $|[X, X^c]| \geq 8$. That is, G is 8-edge-connected.*
(b) *If $|X| \geq 2$ and $|X^c| \geq 3$, then $|[X, X^c]| \geq 11$.*

Proof. (a) Let $\mathcal{P} = \{X, X^c\}$. Since $|G| \geq 4$ by Claim 9(b), the partition $\mathcal{P}$ is normal. Now Claim 10(b) gives $12 \leq \rho_G(\mathcal{P}) = 2|[X, X^c]| - 34 + 31$, which implies $|[X, X^c]| \geq 8$.

(b) If $|X| \geq 2$ and $|X^c| \geq 3$, then $\rho_G(\mathcal{P}) \geq 19$ by Claim 11(b). So $19 \leq \rho_G(\mathcal{P}) = 2|[X, X^c]| - 34 + 31$, which implies $|[X, X^c]| \geq 11$. ∎

Let $T^{\bullet}_{1,1,5}$ denote the graph formed from $T_{1,1,5}$ by subdividing an edge of multiplicity 1. We now show that G contains none of the folllowing (shown in Figure 5) as subgraphs: $T_{1,1,5}$, $T^{\circ}_{1,1,5}$, $T^{\bullet}_{1,1,5}$, and $T_{2,2,4}$.

Claim 13. *G has no copy of $T_{1,1,5}$.*

Proof. Suppose G contains a copy of $T_{1,1,5}$ with vertices x, y, z and $\mu(xy) = 5$. We lift xz, zy to become a new edge xy and contract the resulting $6K_2$ induced by $\{x, y\}$. Let G' denote the resulting graph. The trivial partition $\mathcal{Q}^*$ of G' satisfies $\rho_{G'}(\mathcal{Q}^*) \geq \rho(G) - 2(7) + 17 \geq 3$. If $Q' = \{V(G')\}$, then $\rho_{G'}(Q') = 0 - 17 + 31 = 14$.

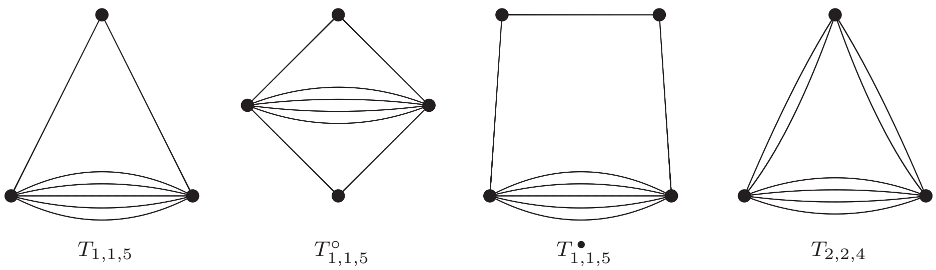

FIG. 5. *The graphs* $T_{1,1,5}, T^{\bullet}_{1,1,5}, T^{\circ}_{1,1,5}, T_{2,2,4}$.

Every other nontrivial partition $\mathcal{Q}'$ of G' corresponds to a normal partition $\mathcal{Q}$ of G in which the contracted vertex is replaced by $\{x, y\}$. Since xz, zy are the only two edges possibly counted in $\rho_G(\mathcal{Q})$ but not in $\rho_{G'}(\mathcal{Q}')$, we have $\rho_{G'}(\mathcal{Q}') \geq \rho_G(\mathcal{Q}) - 2(2) \geq 8$, by Claim 10(b). So $\rho(G') \geq 3$. Since G is 8-edge-connected by Claim 12, graph G' is 6-edge-connected, and so G' is strongly $\mathbb{Z}_7$-connected by Lemma 3.8(c-ii). This is a lifting reduction of the first type. It shows that G is strongly $\mathbb{Z}_7$-connected, which contradicts Claim 9. ∎

CLAIM 14. $|G| \geq 5$.

Proof. Suppose the claim is false. Claim 9(b) implies $|G| = 4$. Since $\rho(G) \geq 0$, the trivial partition shows that $\|G\| \geq 19$. First suppose $\|G\| > 19$, and let $G' = G - e$ for some arbitrary edge e. Since $\|G'\| < \|G\|$, we will apply Lemma 3.8(c-i) to prove G' is strongly $\mathbb{Z}_7$-connected. Since $|G'| = 4$, we know $G' \notin \mathcal{F}$. So it suffices to show that G' is 6-edge-connected and $\rho_{G'}(\mathcal{P}) \geq 3$ for every nontrivial partition $\mathcal{P}$. The first condition holds because G is 8-edge-connected, by Claim 12(a). The second holds because $\rho_{G'}(\mathcal{P}) \geq \rho_G(\mathcal{P}) - 2 \geq 5$, by Claim 10(a). So G' is strongly $\mathbb{Z}_7$-connected by Lemma 3.8(c-i), which contradicts Claim 9.

Instead assume $\|G\| = 19$. Claim 12(a) implies $\delta(G) \geq 8$. Since G contains no $6K_2$ by Claim 9(a), we know $\mu(G) \leq 5$. Now Lemma 3.7 shows that G is strongly $\mathbb{Z}_7$-connected. Thus, G is not a counterexample, which proves the claim. ∎

CLAIM 15. *G has no copy of* $T^{\circ}_{1,1,5}$.

Proof. Suppose G contains a copy of $T_{1,1,5}$ with vertices w, x, y, z and $\mu(xy) = 4$. We lift xz, zy to become a new edge xy, and lift xw, wy to become another new edge xy, and then contract the resulting $6K_2$ to form a new graph G'. The trivial partition $\mathcal{Q}^*$ of G' satisfies $\rho_{G'}(\mathcal{Q}^*) \geq \rho(G) - 2(8) + 17 \geq 1$. If $Q' = \{V(G')\}$, then $\rho_{G'}(Q') = 0 - 17 + 31 = 14$. Every other nontrivial partition $\mathcal{Q}'$ of G' corresponds to a normal partition $\mathcal{Q}$ of G in which the contracted vertex is replaced by $\{x, y\}$. Since xz, zy, xw, wy are the only edges possibly counted in $\rho_G(\mathcal{Q})$ but not in $\rho_{G'}(\mathcal{Q}')$, Claim 10(b) implies $\rho_{G'}(\mathcal{Q}') \geq \rho_G(\mathcal{Q}) - 2(4) \geq 4$. Since $w \neq z$, Claim 12(a,b) implies G' is 6-edge-connected. Because $|V(G')| = |V(G)| - 1 \geq 4$, we know $G' \ncong T_{a,b,c}$ with $a + b + c \in \{10, 11\}$. Hence G' is strongly $\mathbb{Z}_7$-connected by Lemma 3.8(c-i). This is a lifting reduction of the first type. So G is strongly $\mathbb{Z}_7$-connected, which is a contradiction. ∎

CLAIM 16. *G has minimum degree at least* 10. *So G is* 10*-edge-connected by Claim* 12.

Proof. The second statement follows from the first. To prove the first, suppose there exists $x \in V(G)$ with $8 \leq d(x) \leq 9$. Let x_1, x_2 be two neighbors of x. To form a graph G' from G, we lift x_1x, xx_2 to become a new edge x_1x_2, orient the remaining edges incident with x to achieve $\beta(x)$, and finally delete x. This is similar to achieving $\beta(v_1)$ in the proof of Lemma 2.6 (that G has no copy of $6K_2$). This is a lifting reduction of the second type. So, to show G has a β-orientation, it suffices to show that G' is strongly $\mathbb{Z}_7$-connected.

Observe that the trivial partition $\mathcal{Q}^*$ of G' satisfies $\rho_{G'}(\mathcal{Q}^*) \geq \rho(G) - 2(9-1) + 17 \geq 1$. Also, for an almost trivial partition $\mathcal{Q}'$ of G' with $|Q_1| = 2$, we have $\rho_{G'}(\mathcal{Q}') \geq \rho_{G'}(\mathcal{Q}^*) + 17 - 2(5) \geq 8$. Note that when $Q_1 = \{x_1, x_2\}$ we still have $\mu_{G'}(x_1x_2) \leq 5$ by Claim 13. If $Q' = \{V(G')\}$, then $\rho_{G'}(Q') = 0 - 17 + 31 = 14$. Moreover, for any other normal partition $\mathcal{Q}'$ of G' (besides $\{V(G')\}$), since $\mathcal{Q} = \mathcal{Q}' \cup \{x\}$ is a normal partition of G, we have $\rho_{G'}(\mathcal{Q}') \geq \rho_G(\mathcal{Q}) - 2(9) + 17 \geq 11$. Since $|G'| = |G| - 1 \geq 4$ and $\rho_{G'}(\mathcal{Q}') \geq 8$ for any nontrivial partition, Lemma 3.8(a) implies that G' is strongly $\mathbb{Z}_7$-connected. ∎

CLAIM 17. *G has no copy of $T^{\bullet}_{1,1,5}$.*

Proof. Suppose G has a copy of $T^{\bullet}_{1,1,5}$ with vertices v_1, v_2, v_3, v_4 (in order around a 4-cycle) and $\mu(v_1v_4) = 5$. We lift the edges v_1v_2, v_2v_3, v_3v_4 to become a new copy of edge v_1v_4 and contract the resulting $6K_2$; call this new graph G'. The trivial partition $\mathcal{Q}^*$ of G' satisfies $\rho_{G'}(\mathcal{Q}^*) \geq \rho(G) - 2(8) + 17 \geq 1$. If $\mathcal{Q}' = \{V(G')\}$, then $\rho_{G'}(\mathcal{Q}') = 0 - 17 + 31 = 14$. Every other nontrivial partition $\mathcal{Q}'$ of G' corresponds to a normal partition $\mathcal{Q}$ of G in which the contracted vertex is replaced by $\{v_1, v_4\}$. Since v_1v_2, v_2v_3, v_3v_4 are the only edges possibly counted in $\rho_G(\mathcal{Q})$ but not in $\rho_{G'}(\mathcal{Q}')$, we have $\rho_{G'}(\mathcal{Q}') \geq \rho_G(\mathcal{Q}) - 2(3) \geq 6$ by Claim 10(b). Claim 14 implies $|G'| = |G| - 1 \geq 4$, so $G' \notin \mathcal{F}$. Since G is 10-edge-connected by Claim 16, the graph G' is 6-edge-connected. So G' is strongly $\mathbb{Z}_7$-connected by Lemma 3.8(c-i). ∎

CLAIM 18. *G has no copy of $T_{2,2,4}$.*

Proof. Suppose G contains a copy of $T_{2,2,4}$ with vertices x, y, z and $\mu(xy) = 4$. To form a new graph G' from G, we delete two copies (each) of xz, zy and add two new parallel edges xy, and then contract the resulting $6K_2$ induced by $\{x, y\}$. Claim 16 shows G' is 6-edge-connected. Similar to the proof of Claim 15, the trivial partition $\mathcal{Q}^*$ of G' satisfies $\rho_{G'}(\mathcal{Q}^*) \geq \rho(G) - 2(8) + 17 \geq 1$, and every nontrivial partition $\mathcal{Q}'$ of G' satisfies $\rho_{G'}(\mathcal{Q}') \geq \rho_G(\mathcal{Q}) - 2(4) \geq 4$. Since $|G'| = |G| - 1 \geq 4$, Lemma 3.8(c-i) implies G' is strongly $\mathbb{Z}_7$-connected. This is a lifting reduction of the first type, which implies that G is strongly $\mathbb{Z}_7$-connected, and thus gives a contradiction. ∎

CLAIM 19. *For any normal partition $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ with $|P_1| \geq 3$, we have*

$$\rho_G(\mathcal{P}) \geq 14.$$

Proof. Suppose the claim is false, and let $\mathcal{P}$ be such a partition with $\rho_G(\mathcal{P}) \leq 13$. Let $H = G[P_1]$. Since G contains no copy of $T_{1,1,5}$ or $T_{2,2,4}$, we know $H \ncong T_{a,b,c}$ with $a + b + c \in \{10, 11\}$ (and $\min\{a, b, c\} \geq 1$). Thus, since $|H| = |P_1| \geq 3$, we know $H \notin \mathcal{F}$.

Let $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_s\}$ be a partition of H. Now $\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})$ is a partition of G, and (3.1) implies $\rho_H(\mathcal{Q}) = \rho_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) - \rho_G(\mathcal{P}) + 14 \geq \rho_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) + 1$. If $\mathcal{Q}$ is a nontrivial partition of H, then $\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})$ is a nontrivial partition of G, and so Claim 10(a) implies $\rho_H(\mathcal{Q}) \geq \rho_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) + 1 \geq 8$. If $\mathcal{Q}$ is the trivial partition of H, then $\rho_H(\mathcal{Q}) \geq \rho_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) + 1 \geq 1$. By Lemma 3.8(a), the subgraph H is strongly $\mathbb{Z}_7$-connected, which contradicts Claim 9. ∎

Now we can strengthen Claim 12(b).

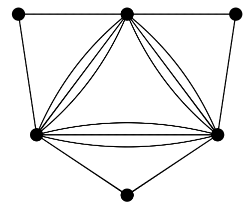

Fig. 6. *The graph* $T^{\circ\circ\circ}_{4,4,4}$.

Claim 20. *If* $[X, X^c]$ *is an edge cut with* $|X| \geq 2$ *and* $|X^c| \geq 3$*, then* $|[X, X^c]| \geq 12$.

Proof. Let X satisfy the hypotheses, and let $\mathcal{P} = \{X, X^c\}$. We will prove $\rho_G(\mathcal{P}) \geq 21$. Assume, to the contrary, that $\rho_G(\mathcal{P}) \leq 20$. Let $H = G[X]$, and let $\mathcal{Q} = \{Q_1, \ldots, Q_s\}$ be a partition of H. Let $\mathcal{P}' = \mathcal{Q} \cup \{X^c\}$. Equation (3.1) implies $\rho_H(\mathcal{Q}) = \rho_G(\mathcal{P}') - \rho_G(\mathcal{P}) + 14$. Since $|X^c| \geq 3$, Claim 19 implies $\rho_G(\mathcal{P}') \geq 14$. Thus $\rho_H(\mathcal{Q}) \geq 14 - 20 + 14 = 8$. By Lemma 3.8(b), subgraph H is strongly $\mathbb{Z}_7$-connected, which contradicts Claim 10(b). So $21 \leq \rho_G(\mathcal{P}) = 2|[X, X^c]| - 34 + 31$, which implies $|[X, X^c]| \geq 12$. ◻

The value of Claim 20 is that it allows us to lift three pairs of edges and know that the resulting graph G' is still 6-edge-connected. Thus, we will show that G' is strongly $\mathbb{Z}_7$-connected, since it satisfies the hypotheses of Lemma 3.8(c-i).

Claim 21. *G contains no copy of* $T^{\circ\circ\circ}_{4,4,4}$.

Proof. Suppose G contains a copy of $T^{\circ\circ\circ}_{4,4,4}$ with vertices $v_1, v_2, v_3, w_1, w_2, w_3$ and $d(v_i) = 8$ and $d(w_i) = 2$ for all i and $N(w_i) = \{v_1, v_2, v_3\} \setminus \{v_i\}$. See Figure 6. Form G' from G by lifting the pair of edges incident to each vertex w_i and contracting the resulting $T_{4,4,4}$. This is a lifting reduction of the first type. Since $T_{4,4,4}$ is strongly $\mathbb{Z}_7$-connected by Lemma 3.6, it suffices to show that G' is also strongly $\mathbb{Z}_7$-connected. Claims 20 and 16 imply that G' is 6-edge-connected. If $Q' = \{V(G')\}$, then $\rho_{G'}(Q') = 0 - 17 + 31 = 14$. Each other partition $\mathcal{P}'$ of G' corresponds to a normal partition $\mathcal{P}$ of G in which the contracted vertex is replaced by $\{v_1, v_2, v_3\}$. We show below that for such a partition we can strengthen Claim 19 to $\rho_G(\mathcal{P}) \geq 15$. Then we have $\rho_{G'}(\mathcal{P}') \geq \rho_G(\mathcal{P}) - 2(6) \geq 3$, since at most six edges are counted in $\rho_G(\mathcal{P})$ but not in $\rho_{G'}(\mathcal{P}')$. Thus, $\rho(G') \geq 3$, so Lemma 3.8(c-ii) implies that G' is strongly $\mathbb{Z}_7$-connected, which is a contradiction. Now it suffices to show that $\rho_G(\mathcal{P}) \geq 15$.

Suppose, to the contrary, that $\rho_G(\mathcal{P}) \leq 14$. Let P_1 be the part of $\mathcal{P}$ containing $\{v_1, v_2, v_3\}$, and let $H = G[P_1]$. We will show that H is strongly $\mathbb{Z}_7$-connected, which gives a contradiction. Let $\mathcal{Q} = \{Q_1, \ldots, Q_s\}$ be a partition of H. Let $\mathcal{P}'' = \mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})$. Equation (3.1) implies $\rho_H(\mathcal{Q}) = \rho_G(\mathcal{P}'') - \rho_G(\mathcal{P}) + 14 \geq \rho_G(\mathcal{P}'') \geq 0$. Further, if $\mathcal{Q}$ is a nontrivial partition of H, then $\mathcal{P}''$ is a nontrivial partition of G, so Claim 10 implies $\rho_H(\mathcal{Q}) \geq \rho_G(\mathcal{P}'') \geq 7$. Since H contains $T_{3,3,3}$ by construction, and G does not contain $T_{2,2,4}$, we know that $H \notin \mathcal{F}$. To apply Lemma 3.8(c-i), we show that H is 6-edge-connected. Consider a bipartition $\mathcal{Q} = \{Q_1, Q_2\}$ of H. Since $\mathcal{Q}$ is nontrivial, $7 \leq \rho_G(\mathcal{P}'') \leq \rho_H(\mathcal{Q}) = 2|[Q_1, Q_2]_H| - 2(17) + 31$, which implies $|[Q_1, Q_2]_H| \geq 5$. That is, H is 5-edge-connected. If H is 6-edge-connected, then Lemma 3.8(c-i) implies that H is strongly $\mathbb{Z}_7$-connected, which is a contradiction. So assume H has a bipartition $\mathcal{Q} = \{Q_1, Q_2\}$ with $|[Q_1, Q_2]_H| = 5$. By symmetry, we assume $|Q_1| \geq |Q_2|$. Since H contains $T_{3,3,3}$ and $T_{3,3,3}$ is 6-edge-connected, we know that $|Q_1| \geq 3$. Now $\rho_G(\mathcal{P}'') = \rho_G(\mathcal{P}) + 2(5) - 17 \leq 14 - 7 = 7$. Since $\mathcal{P}''$ is normal with $|Q_1| \geq 3$, this contradicts Claim 10. $\square$

3.3. Discharging. Fix a plane embedding of a planar graph G such that $\rho(G) \geq 0$. (We assume that all parallel edges between two vertices v and w are embedded consecutively, in the cyclic orders, around both v and w.) If G has a cut-vertex, then each block of G is strongly $\mathbb{Z}_7$-connected by minimality, so G is strongly $\mathbb{Z}_7$-connected by Lemma 1.6, which is a contradiction. Hence G is 2-connected. Since $\rho(G) \geq 0$, we have $2\|G\| - 17|G| + 31 \geq 0$. By Euler's formula, $|G| + |F(G)| - \|G\| = 2$. Now solving for $|G|$ and substituting into the inequality gives

$$\sum_{f \in F(G)} \ell(f) = 2\|G\| \leq \frac{34}{15}|F(G)| - \frac{2}{5}.$$

We assign to each face f initial charge $\ell(f)$. So the total charge is strictly less than $34|F(G)|/15$. To reach a contradiction, we redistribute charge so that each face ends with charge at least $34/15$. We use the following three discharging rules.

(R1) Each 2-face takes charge $2/15$ from each weakly adjacent 3^+-face.

(R2) Each 3-face takes charge $2/15$ from each weakly adjacent 4^+-face with which its parallel edge has multiplicity at most 3 and $1/30$ from each weakly adjacent 4^+-face with which its parallel edge has multiplicity 4.

(R3) After (R1) and (R2), each 3-face with more than $34/15$ splits its excess equally among weakly adjacent 3-faces with less than $34/15$.

Now we show that each face ends with charge at least $34/15$. By (R1) each 2-face ends with $2 + 2(2/15) = 34/15$. Consider a 5^+-face f. Since G contains no copy of $6K_2$, each edge of f has mutliplicity at most 5. So face f sends at most $4(2/15)$ across each of its edges by (R1) and (R2). Thus f ends with at least $\ell(f) - 4(2/15)\ell(f) = 7\ell(f)/15 \geq 35/15$. Consider a 4-face f. Since G contains no copy of $T^{\bullet}_{1,1,5}$, each edge of f has multiplicity at most 4. So f sends at most $3(2/15) + 1/30 = 13/30$ across each of its edges. Thus, f ends with at least $4 - 4(13/30) = 34/15$.

Let f be a 3-face $T_{a,b,c}$. If $a + b + c \leq 8$, then f ends (R2) with at least $3 - (8 - 3)(2/15) = 35/15$. So assume $a + b + c \geq 9$. Since G has no $T_{1,1,5}$, we know $\max\{a, b, c\} \leq 4$. Since G has no $T_{2,2,4}$, if $\max\{a, b, c\} = 4$, then $\min\{a, b, c\} = 1$. Thus, each 3-face $T_{a,b,c}$ finishes (R1) with excess charge at least $1/15$ unless $T_{a,b,c} \in \{T_{1,4,4}, T_{3,3,3}\}$. So we only need to consider $T_{1,4,4}$ and $T_{3,3,3}$. Suppose f is $T_{1,4,4}$. Each face adjacent to f across an edge of multiplicity 4 is not a 3-face, since G has no $T^{\circ}_{1,1,5}$. So f ends (R2) with at least $3 - (9 - 3)(2/15) + 2(1/30) = 34/15$. Hence, each 3-face f ends (R2) with at least $35/15$ unless $f \in \{T_{3,3,3}, T_{1,4,4}\}$, and if it is $T_{1,4,4}$ then it ends (R2) with at least $34/15$.

Finally, assume that f is $T_{3,3,3}$. If any weakly adjacent face is not a 3-face, then f ends (R2) with at least $3 - (9 - 3)(2/15) + 2/15 = 35/15$. So assume each adjacent face is a 3-face. If these three adjacent faces do not intersect outside f, then G contains a copy of $T^{\circ\circ\circ}_{4,4,4}$, a contradiction. If all three faces intersect outside f, then $|V(G)| = 4$, which contradicts Claim 14. So assume that exactly two faces adjacent to f intersect outside f. Let f_1 and f_2 denote the 3-faces adjacent to f that intersect outside f. Denote the boundaries of f, f_1, and f_2 by (respectively) vwx, vwy, and wxy. Recall that $\delta(G) \geq 10$, by Claim 16. Thus, $\mu(wy) \geq 4$. Hence $f_1, f_2 \notin \{T_{3,3,3}, T_{1,4,4}\}$, so each of f_1 and f_2 ends (R2) with at least $35/15$. Thus, by (R3) each gives f at least $(1/2)(1/15)$. As a result, f ends (R3) with at least $3 - (9 - 3)(2/15) + 2(1/2)(1/15) = 34/15$. This completes the proof.

Appendix: Proofs of Lemmas 3.6 and 3.7.

LEMMA 3.6. *Each of the following graphs is strongly $\mathbb{Z}_7$-connected: $6K_2$, $3K_4^+$, and every 6-edge-connected graph $T_{a,b,c}$ where $a+b+c=12$.*

Proof. Throughout we fix a $\mathbb{Z}_7$-boundary β and construct an orientation to achieve β.

Let $G = 6K_2$, with $V(G) = \{v_1, v_2\}$. To achieve $\beta(v_1) \in \{0, 1, 2, 3, 4, 5, 6\}$, the number of edges we orient out of v_1 is (respectively) 3, 0, 4, 1, 5, 2, 6.

Let $G = T_{a,b,c}$, with $a+b+c = 12$ and $\delta(G) \geq 6$. (We handle this before $3K_4^+$.) Let $V(G) = \{v_1, v_2, v_3\}$. If G contains a 6-vertex, say v_1, then $\mu(v_2v_3) = 6$. Since $G/v_2v_3 \cong 6K_2$ is strongly $\mathbb{Z}_7$-connected, G is strongly $\mathbb{Z}_7$-connected by Lemma 1.6(ii). So assume that $\delta(G) \geq 7$. If G contains a 7-vertex v_i and $\beta(v_i) \neq 0$, then we orient 5 edges incident to v_i to achieve $\beta(v_i)$ and lift the remaining pair of nonparallel edges to form a new edge. We are done, since $6K_2$ is strongly $\mathbb{Z}_7$-connected. If G contains an 8-vertex v_j and $\beta(v_j) \notin \{1, 6\}$, then we orient 4 edges incident to v_j to achieve $\beta(v_j)$ and lift two pairs of nonparallel edges to form new edges. Again we are done, since $6K_2$ is strongly $\mathbb{Z}_7$-connected. Since $\|G\| = 12$ and $\delta(G) \geq 7$, the possible degree sequences of G are (a) $\{7, 7, 10\}$, (b) $\{7, 8, 9\}$, and (c) $\{8, 8, 8\}$. The edge multiplicities of G are the three values $\|G\| - d(v_i)$. So G is (a) $T_{2,5,5}$, (b) $T_{3,4,5}$, or (c) $T_{4,4,4}$. In each case we assume $d(v_1) \leq d(v_2) \leq d(v_3)$. In (a) we may assume $\beta(v_1) = \beta(v_2) = 0$, which implies $\beta(v_3) = 0$. To achieve this boundary, orient all edges out of v_1 and all edges into v_2. In (b) we may assume $\beta(v_1) = 0$, $\beta(v_2) = 1$, and $\beta(v_3) = 6$. To achieve this boundary, orient all edges out of v_2 and all edge into v_1. (If instead $\beta(v_2) = 6$ and $\beta(v_3) = 1$, then we reverse the direction of all edges.) In (c) we assume $\beta(v_i) \in \{1, 6\}$ for all i. This yields a contradiction, since $\sum_{i=1}^{3} \beta(v_i) \equiv 0 \pmod 7$.

Let $G = 3K_4^+$ with $V(G) = \{v_1, v_2, v_3, v_4\}$ and $d(v_1) = d(v_2) = 9$ and $d(v_3) = d(v_4) = 10$. Similar to the previous paragraph, we may assume $\beta(v_1) = \beta(v_2) = 0$, $\beta(v_3) = 1$, and $\beta(v_4) = 6$. (If not, then we can lift some edges pairs at v_i and use the remaining edges incident to v_i to achieve $\beta(v_i)$.) To achieve this boundary, start by orienting all edges out of v_1, all edges into v_2, and all edges v_4v_3 out of v_4. Now reverse one copy of v_3v_2 and reverse one copy of v_1v_4. ∎

LEMMA 3.7. *The graph $5C_4^=$ is strongly $\mathbb{Z}_7$-connected. Further, if G is a graph with $|G| = 4$, $\|G\| = 19$, $\mu(G) \leq 5$, and $\delta(G) \geq 8$, then G is strongly $\mathbb{Z}_7$-connected.*

Proof. Assume G satisfies the hypotheses (either the first or second), and let $V(G) = \{v_1, v_2, v_3, v_4\}$. Our plan is to form a new graph G_i from G by lifting one, two, or three pairs of edges incident to v_i, using the remaining edges incident to v_i to achieve the desired boundary $\beta(v_i)$ at v_i. This is a lifting reduction of the second type. If $\|G_i\| \geq 12$ and G_i is 6-edge-connected, then G_i is strongly $\mathbb{Z}_7$-connected by Lemma 3.6, and so we can find an orientation to achieve the β boundary of G. We will show that in every case we can construct such a G_i and achieve $\beta(v_i)$ using edges incident to v_i that are not lifted to form G_i.

Denote $V(5C_4^=)$ by $\{v_1, v_2, v_3, v_4\}$, with $N(v_1) = N(v_3) = \{v_2, v_4\}$, and fix a $\mathbb{Z}_7$-boundary β. If $\beta(v_1) \in \{1, 3, 4, 6\}$, then we lift three pairs of edges incident to v_1 and use the remaining edges to achieve $\beta(v_1)$. Notice that the resulting graph G_1 satisfies $\|G_1\| = 12$, and we are done in this case. So, by symmetry, we assume $\beta(v_i) \in \{0, 2, 5\}$ for each i. The possible multisets of β values are $\{0, 0, 0, 0\}$, $\{0, 0, 2, 5\}$, and $\{2, 5, 2, 5\}$. Up to symmetry, we have five possible $\mathbb{Z}_7$-boundaries. Figure 7 shows orientations that achieve these.

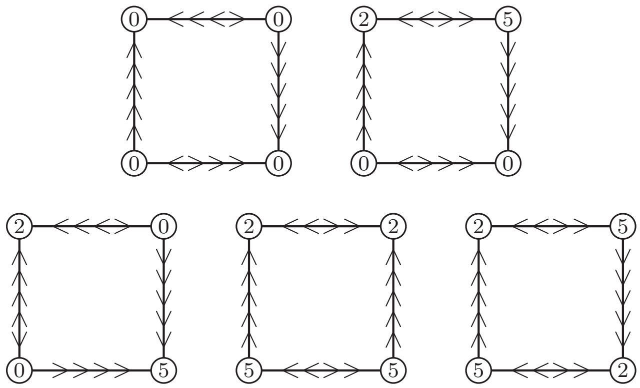

FIG. 7. *Orientations achieving the possible boundaries with $\beta(v_i) \in \{0, 2, 5\}$ for all i.*

Now we prove the second statement. Suppose G contains an 8-vertex v_i. To form G_i, we lift one (arbitrary, nonparallel) pair of edges incident to v_i. Now $\|G_i\| = 19 - 8 + 1 = 12$. If G_i contains a copy of $6K_2$, then we are done by Lemma 1.6, since $6K_2$ is strongly $\mathbb{Z}_7$-connected, and contracting this copy of $6K_2$ yields another $6K_2$. So instead we assume $\mu(G_i) \leq 5$. The edge-connectivity of G_i is $\delta(G_i) = \|G_i\| - \mu(G_i) \geq 12 - 5 = 7$. Since G_i is 6-edge-connected, we are done by Lemma 3.6. Hence, we assume that $\delta(G) \geq 9$ below.

Suppose some pair v_i, v_j of vertices has no edges joining it; that is, $\mu(v_iv_j) = 0$. By symmetry, we assume $i = 1$ and $j = 2$. Since $d(v_1) \geq 9$ and $d(v_2) \geq 9$, we get that $\mu(v_1v_3) + \mu(v_1v_4) \geq 9$ and $\mu(v_2v_3) + \mu(v_2v_4) \geq 9$. Since G has no $6K_2$, each edge of the 4-cycle $v_1v_3v_2v_4$ has multiplicity at least 4. Either $\mu(v_1v_3) = 5$ or $\mu(v_1v_4) = 5$; by symmetry we assume the latter. If $\mu(v_3v_4) = 1$, then we lift edge v_1v_3, v_3v_4 to form a new copy of v_1v_4. We contract the resulting $6K_2$ induced by $\{v_1, v_4\}$. The resulting graph G' is $T_{3,4,5}$, so we are done by Lemmas 1.6 and 3.6. Instead assume $\mu(v_1v_3) = 0$. Now $G = 5C_4^-$ (formed from $5C_4$ by deleting a single edge). Thus G contains $5C_4^=$ as a spanning subgraph, and so G is strongly $\mathbb{Z}_7$-connected by Lemma 3.6. Thus, we assume $\mu(v_iv_j) \geq 1$ for all distinct $i, j \in [4]$.

Suppose $\mu(v_iv_j) = 5$ for some distinct $i, j \in [4]$; by symmetry, say $\mu(v_1v_1) = 5$. Since $\mu(v_1v_3) \geq 1$ and $\mu(v_2v_3) \geq 1$, we lift one copy of each of v_1v_3 and v_3v_2 to form a new copy of v_1v_2 and then contract $\{v_1, v_2\}$ (calling the new vertex w). Denote this new graph by G'. We show that G' is strongly $\mathbb{Z}_7$-connected, which implies the result for G by Lemma 1.6, since $6K_2$ is strongly $\mathbb{Z}_7$-connected. We first show that G is 8-edge-connected. Each edge cut separating a single vertex v_i has size $d(v_i) \geq \delta(G) \geq 8$. If an edge cut S separates G into two parts of size 2, then $|S| \geq \|G\| - 2\mu(G) \geq 19 - 2(5) = 9$. Thus, G is 8-edge-connected, which implies that G' is 6-edge-connected. Since $\|G\| = 19$, we have $\|G'\| = 19 - 7 = 12$. So G' is strongly $\mathbb{Z}_7$-connected, by Lemma 3.6. Thus G is strongly $\mathbb{Z}_7$-connected by Lemma 1.6(ii). This implies that $\mu(v_iv_j) \leq 4$ for each pair $i, j \in [4]$.

Suppose that $\mu(v_iv_j) = 1$ for some pair $i, j \in [4]$, say $\mu(v_1v_2) = 1$. Since $d(v_1) \geq 9$ and $d(v_2) \geq 9$ and $\mu(G) \leq 4$, we have $\mu(v_1v_3) = \mu(v_1v_4) = \mu(v_2v_3) = \mu(v_2v_4) = 4$. Since $\|G\| = 19$, this implies $\mu(v_3v_4) = 2$; see Case 1 in Figure 8. By orienting 5 edges incident to a vertex v_i we can achieve any boundary value $\beta(v_i)$ other than 0. So if $\beta(v_1) \neq 0$ or $\beta(v_2) \neq 0$, then we achieve it by orienting 5 incident edges and lifting two pairs of incident edges to reduce to a 6-edge-connected subgraph G_i with $\|G_i\| = 12$.

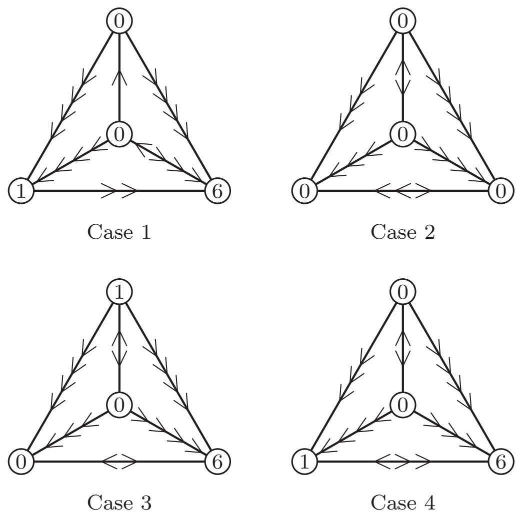

FIG. 8. *In each case v_1 is at top, v_2 center, v_3 left, and v_4 right.*

Similarly, by orienting 4 edges incident to a vertex v_i we can achieve any boundary value at v_i other than 1 or 6. So if $\beta(v_3) \notin \{1, 6\}$ or $\beta(v_4) \notin \{1, 6\}$, then we achieve $\beta(v_i)$ by orienting 4 edges incident to v_i and lifting 3 pairs of incident edges; we do this so that the three newly created edges in G_i are not all parallel. Since $\mu(G) \leq 4$ we have $\mu(G_i) \leq 6$. Now we can finish on G_i, by Lemma 3.6. Thus, by symmetry between v_3 and v_4, we assume $\beta(v_1) = \beta(v_2) = 0$, $\beta(v_3) = 1$, and $\beta(v_4) = 6$. Case 1 in Figure 8 shows an orientation achieving this boundary. So in what remains we assume that $\mu(v_iv_j) \geq 2$ for each pair $i, j \in [4]$.

Since $\|G\| = 19$ and $\delta(G) \geq 9$, the degree sequence is either $\{9, 9, 9, 11\}$ or $\{9, 9, 10, 10\}$. Suppose we are in the first case. By symmetry, we assume $d(v_4) = 11$, $\mu(v_1v_4) = \mu(v_2v_4) = 4$, and $\mu(v_3v_4) = 3$. Since $d(v_1) = d(v_2) = d(v_3) = 9$ and $\mu(v_1v_2) + \mu(v_1v_3) + \mu(v_2v_3) = 8$, we have $\mu(v_1v_2) = 2$ and $\mu(v_1v_3) = \mu(v_2v_3) = 3$. See Case 2 of Figure 8. If $\beta(v_i) \neq 0$ for any $i \in \{1, 2, 3\}$, then we achieve $\beta(v_i)$ by orienting 5 edges incident to v_i, and we lift two pairs of incident edges to form G_i, which is 6-edge-connected and has $\|G_i\| = 12$. So we assume $\beta(v_1) = \beta(v_2) = \beta(v_3) = 0$. This implies that also $\beta(v_4) = 0$. Case 2 in Figure 8 shows an orientation achieving this boundary.

Finally, assume the degree sequence is $\{9, 9, 10, 10\}$ and $\mu(v_iv_j) \geq 2$ for each pair $i, j \in [4]$. If $\mu(v_iv_j) \geq 3$ for each pair $i, j \in [4]$, then $G \cong 3K_4^+$, which contradicts Lemma 3.6. So assume by symmetry that $\mu(v_1v_2) = 2$. First suppose that $d(v_1) = 10$. This implies $\mu(v_1v_3) = \mu(v_1v_4) = 4$. Since each edge has multiplicity 2, 3, or 4, we cannot have $d(v_2) = 10$ (because otherwise $\mu(v_3v_4) = 1$). So $d(v_2) = 9$ and, by symmetry between v_3 and v_4, we assume $d(v_3) = 9$ and $d(v_4) = 10$. This implies that $\mu(v_2v_3) = 3$, $\mu(v_2v_4) = 4$, and $\mu(v_3v_4) = 3$; see Case 3 of Figure 8. As above, we can lift two or three pairs of incident edges if either $\beta(v_2) \neq 0$, $\beta(v_3) \neq 0$, $\beta(v_1) \notin \{1, 6\}$, or $\beta(v_4) \notin \{1, 6\}$. So we assume $\beta(v_2) = \beta(v_3) = 0$, $\beta(v_1) = 1$, and $\beta(v_4) = 6$. (If, instead, $\beta(v_2) = \beta(v_3) = 0$, $\beta(v_1) = 6$, and $\beta(v_4) = 1$, then we can achieve this by reversing every edge.) The desired orientation is shown in Case 3 of Figure 8.

Again assume the degree sequence is $\{9, 9, 10, 10\}$ and that $\mu(v_1v_2) = 2$. Rather than as above, we now assume $d(v_1) = d(v_2) = 9$. So $d(v_3) = d(v_4) = 10$. By symmetry between v_3 and v_4 (and also between v_1 and v_2) we assume $\mu(v_1v_3) = \mu(v_2v_4) = 3$, $\mu(v_1v_4) = \mu(v_2v_3) = 4$, and $\mu(v_3v_4) = 3$. For the same reasons as in the previous paragraph, we assume $\beta(v_1) = \beta(v_2) = 0$, $\beta(v_3) = 1$, and $\beta(v_4) = 6$. Now the desired orientation is shown in Case 4 of Figure 8. This completes the proof. $\square$

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