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Circular coloring and fractional coloring in planar graphs

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Abstract

We study the following Steinberg-type problem on circular coloring: for an odd integer $k \geq 3$, what is the smallest number $f(k)$ such that every planar graph of girth k without cycles of length from $k+1$ to $f(k)$ admits a homomorphism to the odd cycle C_k (or equivalently, is circular $\left(k, \frac{k-1}{2}\right)$-colorable). Known results and counterexamples on Steinberg's Conjecture indicate that $f(3) \in \{6, 7\}$. In this paper, we show that $f(k)$ exists if and only if k is an odd prime. Moreover, we prove that for any prime $p \geq 5$,

$$p^2 - \frac{5}{2}p + \frac{3}{2} \leq f(p) \leq 2p^2 + 2p - 5.$$

We conjecture that $f(p) \leq p^2 - 2p$, and observe that the truth of this conjecture implies Jaeger's conjecture that every planar graph of girth $2p-2$ has a homomorphism to C_p for any prime $p \geq 5$. Supporting this conjecture, we prove a related fractional coloring result that every planar graph of girth k without cycles of length from $k+1$ to $\left\lfloor \frac{22k}{3} \right\rfloor$ is fractional $\left(k : \frac{k-1}{2}\right)$-colorable for any odd integer $k \geq 5$.

KEYWORDS

planar graphs, circular coloring, cycle length, fractional coloring, girth

1 | INTRODUCTION

The circular chromatic number of a graph is a natural generalization of the chromatic number of a graph, introduced by Vince [28]. For two positive integers k and d with $k \geq 2d$, a *circular* (k, d)*-coloring* of a graph G is a mapping $\varphi : V(G) \to \{0, 1, \ldots, k-1\}$ such that $d \leq |\varphi(u) - \varphi(v)| \leq k - d$ whenever $uv \in E(G)$. The circular chromatic number $\chi_c(G)$ of G is defined as the infimum of rational numbers $\frac{k}{d}$ for which G has a circular (k, d)-coloring. Notice that a circular $(k, 1)$-coloring of a graph G is just an ordinary proper k-coloring of G. We call $\chi_c(G)$ a refined measure of coloring because $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$ for every graph G, as proved in [4,28], where $\chi(G)$ is the chromatic number of G. Perhaps one of the most intriguing problems concerning circular coloring of planar graphs is the following conjecture, motivated from the dual of Jaeger's circular flow conjecture [19].

Conjecture 1.1. *For any planar graph G of girth at least $2t$, $\chi_c(G) \leq 2 + \frac{2}{t}$.*

The $t = 1$ case of this conjecture is the celebrated Four Color Theorem proved by Appel and Haken [2] in 1976; the $t = 2$ case is the classical Grötzsch's theorem [15] from 1959 that every triangle-free planar graph is 3-colorable. Conjecture 1.1 remains open for each $t \geq 3$. A result of Hell and Zhu [16] shows that Conjecture 1.1 is true for K_4-minor-free graphs, a subclass of planar graphs.

When $t = 2s$ is even, it is not hard to observe that a graph G is circular $(2t + 2, t)$-colorable if and only if G admits a homomorphism to the odd cycle C_{2s+1}. Indeed, $\chi_c(C_{2s+1}) = \frac{2s+1}{s}$ and each color class contains exactly one vertex of C_{2s+1} under a circular $(2s + 1, s)$-coloring; thus a circular $(2s + 1, s)$-coloring is also called a C_{2s+1}*-coloring* for convenience. For partial results of Conjecture 1.1, Dvořák and Postle [13] showed that every planar graph of girth at least 10 is C_5-colorable. In [11], by duality from flow results, a simpler proof of Dvořák and Postle's result was obtained, and it was extended to the next case that every planar graph of girth at least 16 is C_7-colorable. Independently, Postle and Smith-Roberge [23] also proved that every planar graph of girth at least 16 is C_7-colorable through the density of C_7-critical graphs. The current best general result was due to Lovász, Thomassen, Wu, and Zhang [21], from the dual of their more general flow results, that for each even t, $\chi_c(G) \leq \frac{2t+2}{t}$ for every planar graph G of girth at least $3t$. For odd t, a recent flow results in [20] also showed that $\chi_c(G) \leq \frac{2t+2}{t}$ for every planar graph G of girth at least $3t + 1$.

Another influential coloring problem on planar graphs is Steinberg's Conjecture (see [26]) from 1976, which asserts that every planar graph without cycles of length 4 or 5 is C_3-colorable. We ask the following generalization on C_k-coloring.

Question 1.2. For any integer $k \geq 3$, what is the smallest number $f(k)$ such that every planar graph of girth k without cycles of length from $k + 1$ to $f(k)$ is C_k-colorable?

As an approach to Steinberg's Conjecture, Erdős (see [26]) asked to bound and determine $f(3)$. Abbott and Zhou [1] first established that $f(3) \leq 11$. The bounds are progressively improved to $f(3) \leq 9$ by Borodin [5] and by Sanders and Zhao [24] independently, and to $f(3) \leq 7$ by Borodin, Glebov, Raspaud, and Salavatipour [6], that is, every planar graph without cycles of length from 4 to 7 is 3-colorable. However, Steinberg's Conjecture has been disproved by Cohen-Addad, Hebdige, Král', Li, and Salgado [10], that is, there exists a planar graph without cycles of length 4 or 5 that is not 3-colorable. Those results imply that $f(3) \in \{6, 7\}$.

Our first main result of this paper describes the existence of $f(k)$ for all k.

Theorem 1.3. *The value $f(k)$ exists as a finite number if and only if k is an odd prime. Moreover, for any prime $p \geq 5$,*

$$p^2 - \frac{5p-3}{2} \leq f(p) \leq 2p^2 + 2p - 5.$$

We suspect that the lower bound in Theorem 1.3 is close to the exact value of $f(p)$, and propose the following conjecture for upper bound.

Conjecture 1.4. *For any prime $p \geq 5$, $f(p) \leq p(p-2)$. That is, every planar graph of girth p without cycles of length from $p+1$ to $p(p-2)$ is C_p-colorable.*

The following connection between Conjectures 1.1 and 1.4 is observed.

Proposition 1.5. *Let $p \geq 5$ be a prime. The truth of Conjecture 1.4 implies the validity of Conjecture 1.1 for $t = p-1$. That is, Conjecture 1.4 implies that every planar graph of girth at least $2p-2$ is C_p-colorable.*

Proposition 1.5 indicates that proving Conjecture 1.4 may be difficult. But on the other hand, it also suggests that Conjecture 1.4 may provide a possible new approach to solve Conjecture 1.1 for $t = p-1$ with odd prime p. Particularly, the $p = 5$ case of Conjecture 1.4 not only implies that every planar graph of girth 8 is C_5-colorable, but also implies the Five Coloring Theorem as shown in Observation 2.6.

The fractional chromatic number of a graph is another well-known variation of the chromatic number. For positive integers a and b with $a \geq b$, a *fractional* $(a:b)$*-coloring* φ of a graph G is a set coloring such that each vertex assigns a b-element subset of $\{1, \ldots, a\}$ satisfying $\varphi(u) \cap \varphi(v) = \emptyset$ whenever $uv \in E(G)$. The *fractional chromatic number* of G, denoted by $\chi_f(G)$, is the infimum of the fractions $\frac{a}{b}$ such that G admits a fractional $(a:b)$-coloring. Notice that a fractional $(a:1)$-coloring of a graph G coincides with an ordinary proper a-coloring of G. The fractional coloring was first introduced by Hilton, Rado, and Scott [17] in 1973 to seek for a proof of the Four Color Problem. Since then, it has been the focus of many intensive research efforts, see [25]. For a graph G, let $\omega(G)$ and $\alpha(G)$ denote the clique number and the independence number of G, respectively. It is well known (cf. [29,30]) that

$$\max\left\{\omega(G), \frac{|V(G)|}{\alpha(G)}\right\} \leq \chi_f(G) \leq \chi_c(G) \leq \lceil \chi_c(G) \rceil = \chi(G).$$

One may also consider the fractional coloring variations of Conjecture 1.1 and Question 1.2. Analogous to Conjecture 1.1, Naserasr [22] conjectured that every planar graph of girth at least $2s+2$ is fractional $(2s+1:s)$-colorable. It is proved for K_4-minor-free graphs in [3,14] that every K_4-minor-free graph of girth at least $2s$ is fractional $(2s+1:s)$-colorable.

Our second main result provides a fractional coloring result of Question 1.2, which particularly confirms the fractional coloring version of Conjecture 1.4 for prime $p \geq 11$ in a strong sense.

Theorem 1.6. *For any odd integer $k \geq 5$, every planar graph of girth k without cycles of length from $k+1$ to $\left\lfloor \frac{22k}{3} \right\rfloor$ is fractional $\left(k : \frac{k-1}{2}\right)$-colorable.*

In a follow-up work [18], we also prove the remaining cases ($p = 5, 7$) of the fractional coloring version of Conjecture 1.4 with some refined arguments and additional efforts.

The rest of this paper is organized as follows. We introduce some preliminaries and prove Proposition 1.5 in Section 2. The proof of Theorem 1.3 is presented in Section 3 and the proof of Theorem 1.6 is completed in Section 4. We end this paper with a few remarks in Section 5.

2 | PRELIMINARIES

We start with some basic notation and terminologies. Let $G = (V(G), E(G))$ be a simple finite graph. For a vertex $v \in V(G)$, the neighborhood $N_G(v)$ of a vertex v is the set of vertices adjacent to v, and denotes $d_G(v) = |N_G(v)|$. The distance between two vertices u and v, denoted by $d_G(u, v)$, is the length of a shortest path from u to v in G. The subscript G is often omitted if the graph G is clear from the context. For $S \subseteq V(G)$, $G - S$ denotes the graph obtained from G by deleting all the vertices of S together with all the edges incident to at least one vertex in S. For a positive integer i, let $[i] = \{1, 2, \ldots, i\}$. We use i^+ to denote a number equal or greater than i. An i-vertex (i^+-vertex, resp.) is a vertex of degree i (at least i, resp.). Similarly, in a plane graph, an i-face (i^+-face, resp.) is a face of degree i (or at least i, resp.). In the rest of this paper, we usually assume $k \geq 5$ is an odd integer and $p \geq 5$ is a prime implicitly.

A common method in graph coloring is to study certain coloring properties of typical graphs under given precoloring. This usually provides some reducible subgraphs and facilitates a discharging proof. We shall define precoloring properties for circular coloring and fractional coloring, respectively. Let H be a graph with a vertex subset $S \subset V(H)$. A *precoloring* ω assigns colors in $[k]$ to vertices in S such that $H[S]$ is properly C_k-colored. The graph H is called (ω, S)-colorable if the precoloring ω of S can be extended to $V(H)$ to obtain a C_k-coloring of H. Similarly, a *precoloring* φ of S assigns colors in $\binom{[k]}{\frac{k-1}{2}}$ to vertices in S such that $H[S]$ is properly fractional $\left(k : \frac{k-1}{2}\right)$-colored. We say that H is φ_S*-colorable* if the precoloring φ of S can be extended to all vertices of H to obtain a fractional $\left(k : \frac{k-1}{2}\right)$-coloring.

We first observe the following fact on precoloring of k-cycle for C_k-coloring, which will be useful.

Lemma 2.1. *Let $G = v_0v_1 \ldots v_{k-1}v_0$ be an odd cycle of length k. Let ω be a precoloring of $\{v_i, v_j\} \subseteq V(G)$. Then G is $(\omega, \{v_i, v_j\})$-colorable if and only if*

$$\omega(v_i) - \omega(v_j) \equiv \frac{k-1}{2} \cdot (i-j) \quad \text{or} \quad \frac{k+1}{2} \cdot (i-j) \pmod{k}. \tag{1}$$

Proof. If ω can be extended to a C_k-coloring $\tilde{\omega}$ of G, then the C_k-coloring $\tilde{\omega} : V(G) \mapsto \{0, 1, \ldots, k-1\}$ provides a coloring of G such that

$$\text{either} \quad \tilde{\omega}(v_t) \equiv \frac{k-1}{2} \cdot t + \tilde{\omega}(v_0) \pmod{k} \quad \text{for each} \quad 0 \leq t \leq k-1,$$

$$\text{or} \quad \tilde{\omega}(v_t) \equiv \frac{k+1}{2} \cdot t + \tilde{\omega}(v_0) \pmod{k} \quad \text{for each} \quad 0 \leq t \leq k-1.$$

Hence, for $v_i, v_j \in V(G)$ we have Equation (1).

Conversely, if Equation (1) holds, then we can properly define a C_k-coloring of G as above. This proves the lemma. □

In a graph G, a d-C_k-*replacement* operation on a given edge $e = xy \in E(G)$ is to replace the edge e with a k-cycle $C_k = v_0v_1 \ldots v_{k-1}v_0$ by identifying x with v_0 and identifying y with v_d. When d is not explicitly stated, we just call it a C_k-replacement operation on the edge $e \in E(G)$. Lemma 2.1 implies the following relation between C_k-coloring and d-C_k-replacement operation.

Proposition 2.2. *Let G be a graph, and let $G(d, k)$ be a graph obtained from G by applying d-C_k-replacement operation on each edge of G. Assume that d and k are coprime, that is, $gcd(d, k) = 1$. Then G is C_k-colorable if and only if $G(d, k)$ is C_k-colorable.*

Proof. Let φ be a C_k-coloring of G. Define a precoloring ω of $G(d, k)$ by coloring each vertex $u \in V(G) \subset V(G(d, k))$ with $\omega(u) \equiv d\varphi(u) \pmod{k}$. Since $\varphi(u) - \varphi(v) \in \left\{\frac{k-1}{2}, \frac{k+1}{2}\right\}$ for each edge $uv \in E(G)$, we have, in the graph $G(d, k)$,

$$\omega(u) - \omega(v) \equiv d\varphi(u) - d\varphi(v) \equiv \frac{k-1}{2} \cdot d \quad \text{or} \quad \frac{k+1}{2} \cdot d \pmod{k}.$$

It follows from Lemma 2.1 that ω can be extended to a C_k-coloring of $G(d, k)$ by coloring each k-cycle of $G(d, k)$ properly.

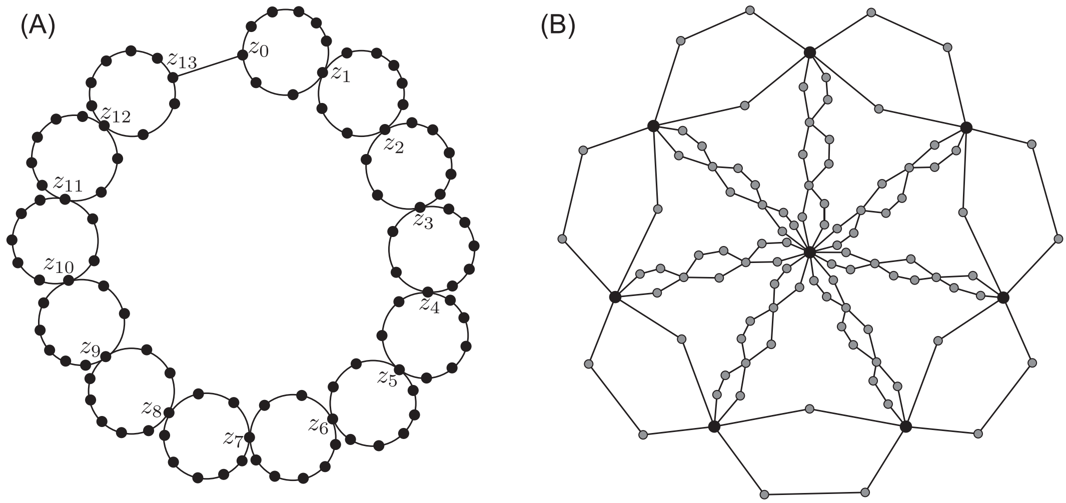
Conversely, assume that $G(d, k)$ admits a C_k-coloring ω. Then for each edge $uv \in E(G)$, we have $\omega(u) - \omega(v) \in \left\{\frac{k-1}{2} \cdot d, \frac{k+1}{2} \cdot d\right\} \pmod{k}$ by Lemma 2.1. Define $\varphi = d^{-1}\omega \pmod{k}$. (Note that d^{-1} exists in $\mathbb{Z}_k$ since $gcd(d, k) = 1$.) Then $\varphi(u) - \varphi(v) \in \left\{\frac{k-1}{2}, \frac{k+1}{2}\right\}$ for each edge $uv \in E(G)$. That is, φ restricted to $V(G)$ provides a proper C_k-coloring of G. □

Applying Lemma 2.1, we also show that $f(k)$ does not exist for nonprime k by construction using d-C_k-replacement operations.

Proposition 2.3. *Let $k > 0$ be an odd nonprime integer. Then $f(k)$ does not exist. That is, for any integer $m > k$ there exist planar graphs of girth k without cycles of length from $k + 1$ to m admitting no C_k-coloring.*

Proof. Denote $k = st$, where s, t are positive integers with $t \geq s > 1$. Take an $(m + 1)$-cycle $z_0z_1z_2 \ldots z_mz_0$. For each $0 \leq i \leq m - 1$, apply s-C_k-replacement operation on the edge z_iz_{i+1}. Let G be the resulting graph. Then G is a planar graph of girth k without cycles of length from $k + 1$ to ms. See Figure 1A for the construction of G when $k = 9$ and $m = 13$.

It is routine to check that G is not C_k-colorable. To see this, suppose for a contradiction that $\omega : V(G) \mapsto \{0, 1, \ldots, k - 1\}$ is a C_k-coloring of G. By Lemma 2.1, for each $0 \leq i \leq m - 1$, we have



Construction of G for $k = 9$ and $m = 13$. Construction of H_p for $p = 5$.

FIGURE 1 Constructions in Propositions 2.3 and 2.4. (A) Construction of G for $k = 9$ and $m = 13$, and (B) construction of H_p for $p = 5$

$$\omega(z_{i+1}) - \omega(z_i) \equiv \frac{k-1}{2} \cdot s \quad \text{or} \quad \frac{k+1}{2} \cdot s \pmod{k}.$$

Thus $\omega(z_{i+1}) - \omega(z_i)$ is a multiple of s since $k = st$. This implies that

$$\omega(z_m) - \omega(z_0) = \sum_{i=0}^{m-1} (\omega(z_{i+1}) - \omega(z_i)) \quad \text{is a multiple of } s.$$

On the other hand, as $z_m z_0$ is an edge in $E(G)$, we must have $|\omega(z_m) - \omega(z_0)| \in \left\{\frac{k-1}{2}, \frac{k+1}{2}\right\}$. But as $k = st$, neither $\frac{k-1}{2}$ nor $\frac{k+1}{2}$ is a multiple of s, a contradiction. This completes the proof. □

In contrast, we will show below in Theorem 3.4 that $f(p)$ exists as a quadratic function of p for odd prime p. Now we give a low bound of $f(p)$ with similar arguments as Proposition 2.3.

Proposition 2.4. *For any prime $p \geq 5$, there exist planar graphs of girth p without cycles of length from $p+1$ to $p^2 - \frac{5p-1}{2}$ admitting no C_p-coloring. That is, $f(p) \geq p^2 - \frac{5}{2}p + \frac{3}{2}$.*

Proof. Construct a graph W_p from a $(2p-3)$-cycle $z_0 z_1 \ldots z_{2p-4} z_0$ by adding a new center vertex x connecting each z_i with a new path of length $p-2$ for $0 \leq i \leq 2p-4$. This graph W_p was constructed by DeVos (see [8]) to show the tightness of Conjecture 1.1, that is, W_p is a planar graph of girth $2p-3$ without C_p-coloring. To see that W_p is not C_p-colorable, we prove by contradiction. Suppose to the contrary that ω is a C_p-coloring of W_p. If $\omega(x) = \omega(z_i)$ for some i, then after identifying x and z_i in the path of length $p-2$ between x and z_i, we obtain a C_p-coloring of $(p-2)$-cycle, a contradiction. So $\omega(x) \neq \omega(z_i)$ for each $0 \leq i \leq 2p-4$. Hence the $(2p-3)$-cycle $z_0 z_1 \ldots z_{2p-4} z_0$ admits a

C_p-coloring with colors $\{0, 1, ..., p-1\} \setminus \{\omega(x)\}$. This provides a homomorphism from the $(2p-3)$-cycle to a path of length $p-2$; in particular, it indicates that the $(2p-3)$-cycle is 2-colorable, a contradiction.

Construct a graph H_p from W_p by applying $\left(\frac{p-1}{2}\right)$-$C_p$-replacement operation on each edge of W_p. See Figure 1B for the construction of H_5. Since W_p is not C_p-colorable, we obtain that H_p is not C_p-colorable by Proposition 2.2. As W_p has girth $2p-3$, H_p is of girth p and without cycles of length from $p+1$ to $(2p-3)\frac{p-1}{2}-1$. □

Next, we shall prove Proposition 1.5 using analogous approaches.

Proposition 2.5 (Restatement of Proposition 1.5). *Let $p \geq 5$ be a prime. If $f(p) \leq p(p-2)$, then every planar graph of girth at least $2p-2$ is C_p-colorable.*

Proof. Assume that $f(p) \leq p(p-2)$. That is, every planar graph of girth p without cycles of length from $p+1$ to $p(p-2)$ is C_p-colorable. Let G be a planar graph of girth at least $2p-2$. Apply the $\left(\frac{p-1}{2}\right)$-$C_p$-replacement operation on each edge of G to obtain a graph $G\left(\frac{p-1}{2}, p\right)$. Then $G\left(\frac{p-1}{2}, p\right)$ is a planar graph of girth p without cycles of length from $p+1$ to $p(p-2)$. Since $f(p) \leq p(p-2)$, we know that $G\left(\frac{p-1}{2}, p\right)$ is C_p-colorable. Hence G is C_p-colorable as well by Proposition 2.2. □

Similar arguments also show that the $p=5$ case of Conjecture 1.4 is stronger than the Five Color Theorem.

Observation 2.6. The truth of $f(5) \leq 17$ implies that every planar graph is 5-colorable.

Proof. Assume that $f(5) \leq 17$, that is, every planar graph of girth 5 without cycles of length from 6 to 17 is C_5-colorable. Let G be a planar graph, and H be the graph obtained from G by replacing each edge with a path of length 3. Let F be the graph obtained from H by applying 2-C_5-replacement operation on each edge of H. Then by construction F is a planar graph of girth 5 without cycles of length from 6 to 17, and hence F is C_5-colorable by $f(5) \leq 17$. Now, by Proposition 2.2 and Lemma 2.1, the C_5-coloring ω of F induces a proper 5-coloring of G, since $\omega(u) \neq \omega(v)$ whenever $uv \in E(G)$. □

At the end of this section, we define some graphs, serving for reducible configurations in later proofs.

Definition 2.7. Let G be a graph.

(i) A *thread* in G is a path whose internal vertices are 2-vertices in G. The end vertices of the path are called the end vertices of the thread. A thread with end vertices x, y is also called an (x, y)-thread, denoted by $T(x, y)$. An s-thread is a thread with s internal vertices. A $(k_1, k_2, ..., k_t)$*-thread* T_x in G is a subgraph consisting of distinct k_1-thread, k_2-thread, $\cdots$, k_t-thread which share a common end vertex x, where $t \geq 3$.

The common end vertex x is called a $(k_1, k_2, ..., k_t)$-*vertex*. Let y_i be the other end vertex of the k_i-thread, and define $\{y_1, y_2, ..., y_t\}$ to be the end vertices of T_x. If z is a 2-vertex of an (x, y)-thread, then we say x and z are *weakly adjacent*.

(ii) An s-*necklace* in G is a subgraph obtained from an s-thread by applying C_k-replacement operations on some edges. A vertex z is an end vertex of the s-necklace if and only if z is an end vertex of the s-thread. A necklace with end vertices x, y is also called an (x, y)-necklace, denoted by $N(x, y)$. A $(k_1, k_2, ..., k_t)$-*necklace*N_x is a subgraph obtained from a $(k_1, k_2, ..., k_t)$-thread T_x by applying C_k-replacement operations on some edges. The vertex x is called the center vertex of N_x. A vertex z is an end vertex of the $(k_1, k_2, ..., k_t)$-necklace if and only if z is an end vertex of the $(k_1, k_2, ..., k_t)$-thread. A $(k_1, k_2; k_3)$-*bull-necklace* is a subgraph obtained from a (k_1, k_2, k_3)-thread by applying C_k-replacement operations on some edges of the k_3-thread. A $(k_1, k_2, ..., k_t)$-*crown-necklace* is obtained from a $(k_1, k_2, ..., k_t)$-necklace by replacing the center vertex with a k-cycle. A vertex z is an end vertex of the $(k_1, k_2, ..., k_t)$-crown-necklace if and only if z is an end vertex of the $(k_1, k_2, ..., k_t)$-necklace. See Figure 2 for some examples.

3 | THE C_p-COLORING FOR PRIME p

This section is aiming to show $f(p) \leq 2p^2 + 2p - 5$ in Theorem 1.3. We first present some reducible configurations under precoloring in Section 3.1, and then complete the proof in Section 3.2 by a discharging method. Unlike some standard discharging arguments, our method mainly analyzes certain modified graphs obtained from the original graph, which benefits in handling some structures involving p-cycles.

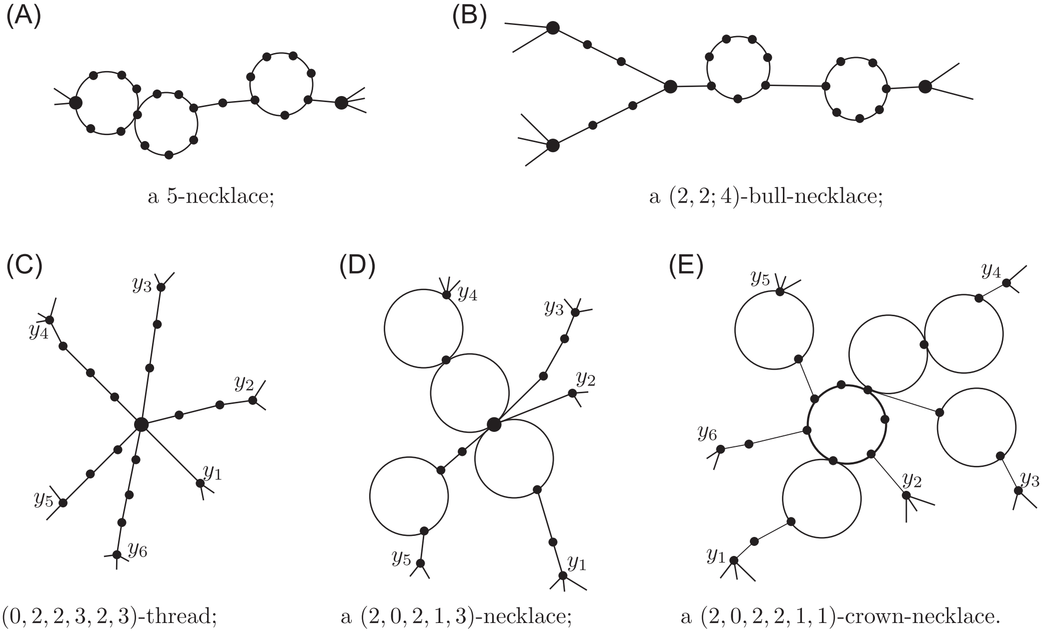


FIGURE 2 Examples for Definition 2.7

3.1 | Precoloring and reducible subgraphs for C_p-coloring

Let H be a thread, or a necklace, or a $(k_1, k_2, \ldots, k_t)$-thread, or a $(k_1, k_2, \ldots, k_t)$-necklace, or a $(k_1, k_2, \ldots, k_t)$-crown-necklace with W being the end vertex set of H. The graph H is called *reducible* if in any graph G containing H as a subgraph, any C_p-coloring of $G - (V(H)\setminus W)$ can be extended to a C_p-coloring of G. In other words, it is equivalent to say that H is (ω, W)-colorable for any precoloring ω of W. It is known from [8,23,29] that some threads and certain $(k_1, k_2, \ldots, k_t)$-threads are reducible configurations for C_k-coloring. Our main reducible configurations in this section are certain necklaces and crowns, generalizing from threads, for prime p.

We need the following well-known Cauchy–Davenport Theorem over prime field. For two sets A, B, define $A + B = \{a + b : a \in A, b \in B\}$.

Theorem 3.1 (Cauchy–Davenport Theorem, [9,12]). *Let p be a prime. If A and B are two nonempty subsets of $\mathbb{Z}_p$, then we have*

$$|A + B| \geq \min\{p, |A| + |B| - 1\}.$$

Lemma 3.2. *Let $N(x_0, x_{s+1})$ be an s-necklace, where for each $0 \leq i \leq s$, there is either an edge $x_i x_{i+1}$ or a p-cycle between x_i and x_{i+1} consisting of a k_i-thread and a $(p - 2 - k_i)$-thread. Let ω be a precoloring of x_0, and let $B(x_i)$ be the set of available colors of x_i from a coloring of x_{i-1} for each $i \in [s + 1]$, where $B(x_0) = \{\omega(x_0)\}$. Then each of the following holds.*

(i) We have $|B(x_i)| \geq \min\{i + 1, p\}$ for each $i \in [s + 1]$.
(ii) If $s \geq p - 2$, then an s-necklace is reducible for C_p-coloring.

Proof. (i) For any $s \geq i \geq 0$, we shall count the number of colors $\omega(x_{i+1})$ that can be extended from a color $\omega(x_i)$ of x_i. Note that x_0 receives a fixed coloring $\omega(x_0)$. If $x_0 x_1$ is an edge in G, then we have $\omega(x_1) \in \left\{\omega(x_0) + \frac{p-1}{2}, \omega(x_0) - \frac{p-1}{2}\right\}$, that is, $B(x_1) = \left\{\omega(x_0) + \frac{p-1}{2}, \omega(x_0) - \frac{p-1}{2}\right\}$. The arithmetic operations here and below are taken modulo p. If there is a p-cycle between x_0 and x_1 which consists of a k_0-thread and a $(p - 2 - k_0)$-thread, then by Lemma 2.1 we have $\omega(x_1) \in \left\{\omega(x_0) + \frac{p-1}{2}(k_0 + 1), \omega(x_0) - \frac{p-1}{2}(k_0 + 1)\right\}$, which gives $B(x_1) = \left\{\omega(x_0) + \frac{p-1}{2}(k_0 + 1), \omega(x_0) - \frac{p-1}{2}(k_0 + 1)\right\}$. Hence, in any case, we have $|B(x_1)| = 2$.

Below we shall apply induction to show $|B(x_i)| \geq \min\{i + 1, p\}$ for each $i \in [s + 1]$. The basic case $i = 1$ is proved above. Assume the statement $|B(x_i)| \geq \min\{i + 1, p\}$ holds for any integer at most i. For the case $i + 1$, we shall show that $|B(x_{i+1})| \geq \min\{i + 2, p\}$. Similar as before, if $x_i x_{i+1}$ is an edge of G, then

$$B(x_{i+1}) = \left\{b + \frac{p-1}{2}, b - \frac{p-1}{2} : b \in B(x_i)\right\};$$

if there is a p-cycle between x_i and x_{i+1} consisting of a k_i-thread and a $(p-2-k_i)$-thread, then by Lemma 2.1 we have

$$B(x_{i+1})=\left\{b+\frac{p-1}{2}(k_i+1),\, b-\frac{p-1}{2}(k_i+1):b\in B(x_i)\right\}.$$

Using the notation in Theorem 3.1, we have

$$\text{either}\, B(x_{i+1})=B(x_i)+\left\{\frac{p-1}{2},-\frac{p-1}{2}\right\}\quad\text{or}\quad B(x_{i+1})$$
$$=B(x_i)+\left\{\frac{p-1}{2}(k_i+1),-\frac{p-1}{2}(k_i+1)\right\}.$$

By Theorem 3.1, we obtain that $|B(x_{i+1})|\geq\min\{|B(x_i)|+1,p\}\geq\min\{i+2,p\}$. This proves the claim that $|B(x_i)|\geq\min\{i+1,p\}$ for each $i\in[s+1]$.

(ii) Fix a C_p-coloring ω of $G-(V(N(x_0,x_{s+1}))\setminus\{x_0,x_{s+1}\})$. We show that ω can be extended to a C_p-coloring of G. We still let $B(x_i)$ be the set of available colors of x_i from a coloring of x_{i-1} for each $1\leq i\leq s$, where $B(x_0)=\{\omega(x_0)\}$. By (i), we particularly have that $|B(x_j)|\geq p-1$ for each $p-2\leq j\leq s$. For the s-necklace $N(x_0,x_{s+1})$, $\omega(x_{s+1})$ is a fixed color, and so its restriction requires that $\omega(x_s)\in\left\{\omega(x_{s+1})+\frac{p-1}{2},\omega(x_{s+1})-\frac{p-1}{2}\right\}$ when x_sx_{s+1} is an edge, and $\omega(x_s)\in\left\{\omega(x_{s+1})+\frac{p-1}{2}(k_s+1),\omega(x_{s+1})-\frac{p-1}{2}(k_s+1)\right\}$ when there is a p-cycle between x_s and x_{s+1} consisting of k_s-thread and a $(p-2-k_s)$-thread. Since $|B(x_s)|\geq p-1$, we have both $B(x_s)\cap\left\{\omega(x_{s+1})+\frac{p-1}{2},\omega(x_{s+1})-\frac{p-1}{2}\right\}\neq\varnothing$ and $B(x_s)\cap\left\{\omega(x_{s+1})+\frac{p-1}{2}(k_s+1),\omega(x_{s+1})-\frac{p-1}{2}(k_s+1)\right\}\neq\varnothing$. Therefore, there exists an available color for the choice of x_s in $B(x_s)$, and so ω can be extended to a C_p-coloring of G by appropriately coloring each of $x_1,x_2,\ldots,x_s$ and by Lemma 2.1. □

Lemma 3.3. *For a $(k_1,k_2,\ldots,k_t)$-necklace or a $(k_1,k_2,\ldots,k_t)$-crown-necklace, if it holds that*

$$\max_{1\leq i\leq t}\{k_i\}\leq p-2\quad\text{and}\quad\sum_{i=1}^{t}k_i\geq(p-2)t-p+1,$$

then it is reducible for C_p-coloring.

Proof. Let H be a $(k_1,k_2,\ldots,k_t)$-necklace or a $(k_1,k_2,\ldots,k_t)$-crown-necklace with end vertex set W. For each $i\in[t]$, let x_i,y_i be the end vertices of the k_i-necklace in H, where $x_i\in W$. If H is a $(k_1,k_2,\ldots,k_t)$-necklace, then $y_1=y_2=\cdots=y_t$ is a common vertex. If H is a $(k_1,k_2,\ldots,k_t)$-crown-necklace, then $y_1,y_2,\ldots,y_t$ (may or may not be identical) are lying in a common p-cycle. In the later case, suppose that we color a selected vertex y_0 of the common p-cycle with color b. Then denote the color of y_i by $\varphi(y_i)=b+\frac{p-1}{2}d_i$, where d_i is the distance from y_0 to y_i in the cyclic order for each $1\leq i\leq t$. In the former case, we apply the same notation and set that $y_0=y_1=y_2=\cdots=y_t$, $\omega(y_i)=b$, and $d_i=0$ for each i.

Fix a precoloring ω of W. We show that ω can be extended to a C_p-coloring of H by selecting an appropriate value of b with application of Lemma 3.2(i).

For each $1 \le i \le t$, let B_i be the set of available colors of y_i such that the coloring $\omega(x_i)$ and $\omega(y_i) \in B_i$ can be extended to a C_p-coloring of the k_i-necklace. By Lemma 3.2(i), we have $|B_i| \ge k_i + 2$. Let $D_i = \left\{\beta : \beta = \alpha - \frac{p-1}{2}d_i, \alpha \in B_i\right\}$. Clearly, $|D_i| = |B_i| \ge k_i + 2$. Thus we have

$$\begin{aligned}\left|\bigcap_{i=1}^{t} D_i\right| &\ge \sum_{i=1}^{t}|D_i| - (t-1)\left|\bigcup_{i=1}^{t} D_i\right| \\ &\ge \sum_{i=1}^{t}(k_i + 2) - (t-1)p \\ &= \sum_{i=1}^{t}k_i - (p-2)t + p \ge 1.\end{aligned}$$

Hence $\bigcap_{i=1}^{t} D_i \ne \varnothing$ holds. Then we can select an element $b \in \bigcap_{i=1}^{t} D_i$ and color y_i with $\omega(y_i) = \varphi(y_i) = b + \frac{p-1}{2}d_i$ for each $i \in [t]$. By definition, the coloring $\omega(x_i)$ and $\omega(y_i)$ can be extended to a C_p-coloring of the k_i-necklace for each $i \in [t]$. Therefore, H is reducible for C_p-coloring. □

3.2 | The proof of Theorem 1.3

By Proposition 2.3, $f(k)$ does not exist if $k > 0$ is not an odd prime integer. Proposition 2.4 indicates $p^2 - \frac{5}{2}p + \frac{3}{2} \le f(p)$ for a prime $p \ge 5$. To complete the proof of Theorem 1.3, it suffices to show that every planar graph of girth p without cycles of length from $p+1$ to $2(p-1)(p+2)-1$ is C_p-colorable. In fact, we show the following mild stronger theorem.

Theorem 3.4. *Let G be a plane graph of girth p without cycles of length from $p+1$ to $2(p-1)(p+2)-1$, and let ω be a precoloring of a p-cycle C of G. Then G is $(\omega, V(C))$-colorable.*

Proof. Suppose to the contrary that G is a counterexample with $|E(G)\backslash E(C)|$ minimized. Clearly, we have $E(G)\backslash E(C) \ne \varnothing$ and $|V(G)| > p$. □

Claim 1.

(i) G is 2-connected. In particular, $\delta(G) \ge 2$.
(ii) Every p-cycle in G bounds a face. In particular, C is a facial p-cycle of G.

Proof of Claim 1.

(i) If G is not 2-connected, then there exist proper induced subgraphs G_1 and G_2 of G and a vertex $v \in V(G_2)$ such that $E(G) = E(G_1) \cup E(G_2)$, $V(G_1) \cap V(G_2) \subseteq \{v\}$, and $V(C) \subseteq V(G_1)$. By the minimality of the counterexample, ω can be extended to a C_p-coloring $\tilde{\omega}$ of G_1. Take an edge $uv \in E(G_2)$. If uv is in a p-cycle, then we let C' be a

p-cycle containing uv and let $G_2' = G_2$. Otherwise, we construct a new graph G_2' from G_2 by adding a new $(p-2)$-thread between u and v to form a new p-cycle C'. Note that G_2' contains no cycles of length from $p+1$ to $2(p-1)(p+2)-1$ in any case. Let ω' be a precoloring of C' such that $\omega'(v) = \tilde{\omega}(v)$ (if $v \notin V(G_1)$, then G is not connected and we take $\omega'(v)$ to be an arbitrary color). Since $|E\left(G_2'\right) \backslash E(C')| < |E(G) \backslash E(C)|$ and by the minimality of the counterexample, ω' can be extended to a C_p-coloring $\tilde{\omega}'$ of G_2'. So $\tilde{\omega}'$ and $\tilde{\omega}$ combine to provide a C_p-coloring of G, which is a contradiction.

(ii) Suppose for a contradiction that a p-cycle K of G does not bound a face. Let G_1 be the subgraph of G drawn outside (and including) K, and let G_2 be the subgraph of G drawn inside (and including) K. We may, without loss of generality, assume that $V(C) \subset V(G_1)$. By the minimality of the counterexample, ω can be extended to a C_p-coloring $\tilde{\omega}$ of G_1. Let ω' be the restriction of $\tilde{\omega}$ on $V(K)$. Then ω' can be extended to a C_p-coloring $\tilde{\omega}'$ of G_2 by the minimality of G. The union of $\tilde{\omega}$ and $\tilde{\omega}'$ is a C_p-coloring of G extending ω, which is a contradiction.

By Claim 1(ii), C must be a facial cycle of G. Re-embedding G on the plane if needed, we can assume that the face bounded by C is the outer face of G, denoted by f_0. Let H be a thread, or a necklace, or a $(k_1, k_2, \ldots, k_t)$-thread, or a $(k_1, k_2, \ldots, k_t)$-necklace, or a $(k_1, k_2, \ldots, k_t)$-crown-necklace of G with end vertex set W. If $V(H) \backslash W \subseteq V(G) \backslash V(C)$, then we say H is *valid* in G. □

Claim 2. Each of the following holds.

(i) G contains no valid $(p-2)^+$-thread.
(ii) G contains no valid $(p-2)^+$-necklace.
(iii) G contains neither a valid $(k_1, k_2, \ldots, k_t)$-necklace nor a valid $(k_1, k_2, \ldots, k_t)$-crown-necklace with $\sum_{i=1}^{t} k_i \geq (p-2)t - p + 1$, where $t \geq 3$.

Proof of Claim 2. Note that an s-thread is a special s-necklace without performing C_p-replacement operation, and so Claim 2(i) follows from Claim 2(ii). Suppose, for a contradiction, that G has a valid $(p-2)^+$-necklace, or a valid $(k_1, k_2, \ldots, k_t)$-necklace, or a valid $(k_1, k_2, \ldots, k_t)$-crown-necklace described above, denoted by H with W being its end vertex set. By the minimality of G, ω can be extended to a C_p-coloring $\tilde{\omega}$ of $G - (V(H) \backslash W)$. By Lemmas 3.2(ii) and 3.3, $\tilde{\omega}$ can be extended to a C_p-coloring of G, that is, ω can be extended to a C_p-coloring of G, a contradiction.

By Claim 1(ii), any p-cycle in G is a facial p-cycle (the boundary of a p-face). Since G contains no cycles of length from $p+1$ to $2(p-1)(p+2)-1$, any two p-cycles have no common edges. A vertex v of a facial p-cycle K with $d_G(v) \geq 3$ is called an *attachment-vertex* of K. Since G is 2-connected by Claim 1(i), every p-cycle contains at least two attachment-vertices.

Next, we construct two graphs G' and G'' modified from G for later proof. Let G' be the graph obtained from G by replacing K with an edge uv for any facial p-cycle K other than C with exactly two attachment-vertices u, v. Note that $d_G(u, v) \geq 2$ by Claim 2(i). By construction, each edge of G' is corresponding to either an edge of G or a p-cycle consisting of a t-thread and a $(p-2-t)$-thread with $p-2 \geq t \geq 2$. Note that the shorter one of t-thread and $(p-2-t)$-thread has length at most $\frac{p-1}{2}$.

Let K be a facial p-cycle of G' other than C with attachment-vertices $v_1, v_2, ..., v_r$. Then $r \geq 3$ by the construction of G'. To *stick* K, we mean to delete all the vertices of $V(K)\backslash\{v_1, v_2, ..., v_r\}$ and add a new vertex v_K^* inside face K to join each vertex of $\{v_1, v_2, ..., v_r\}$. The vertex v_K^* is called a *sticking vertex*, where the degree of v_K^* is at least 3. Let G'' be the graph obtained from G' by sticking all the facial p-cycles of G' except C.

By the construction of G'', we immediately observe the following: G'' is a plane graph with outer face f_0 bounded by C; the minimal degree $\delta(G'') \geq 2$; for any $uv \in E(G'')$, at most one vertex of $\{u, v\}$ is in $V(G'')\backslash V(G)$; each 2-vertex v of G'' is either a 2-vertex of G or an attachment-vertex of a facial p-cycle of G; each vertex $v \in V(G'')\backslash V(G)$ is a sticking vertex with $d_{G''}(v) \geq 3$. These facts will be used implicitly in the rest of the proof.

We further obtain the claim below concerning cycles of G''. □

Claim 3. The new constructed graph G'' is a plane graph of girth p without cycles of length from $p + 1$ to $4(p + 2) - 1$. Furthermore, C is the only one p-cycle of G''.

Proof of Claim 3. Recall that each p-cycle in G is a facial p-cycle by Claim 1(ii). By the construction of G'', C is the only one p-cycle of G''. Let $Q = x_0x_1 \dots x_mx_0$ be a cycle of G'' other than C. If x_i is a sticking vertex, then x_i corresponds to a facial p-cycle K_i of G, and x_{i-1} and x_{i+1} are two attachment-vertices of K_i, thus the two edges $x_{i-1}x_i, x_ix_{i+1}$ together correspond to a segment of K_i whose length is at most $p - 2$ as K_i has at least three attachment-vertices. If both x_j and x_{j+1} are not sticking vertices, then x_jx_{j+1} corresponds to either an edge of G or a p-cycle of G consisting of two threads, where the shorter one has length at most $\frac{p-1}{2}$. It is also clear that any two sticking vertices are not adjacent. Hence, for each sticking vertex in the cycle Q its two incident edges together correspond to a path of length at most $p - 2$ in G, and for each edge in Q not incident to sticking vertex it corresponds to a thread of length at most $\frac{p-1}{2}$. Hence the cycle Q corresponds to a cycle of length at most $\frac{p-1}{2}m$ in G. So we have $\frac{p-1}{2}m \geq 2(p-1)(p+2)$, which gives $m \geq 4(p + 2)$. Therefore, each cycle of G'' except C has length at least $4(p + 2)$.

In the graph G'', a thread or a $(k_1, k_2, ..., k_t)$-thread H with end vertex set W is called *valid* if $V(H)\backslash W \subseteq V(G'')\backslash V(C)$. □

Claim 4. Each valid $(s + 2)$-thread of G'' corresponds to a valid s-necklace of G. In particular, G'' contains no valid p^+-thread by Claim 2(ii).

Proof of Claim 4. Let $P = x_0x_1 \dots x_{s+2}x_{s+3}$ be a valid $(s + 2)$-thread of G''. For any $i \in [s + 2]$, $x_i \in V(G'')\backslash V(C)$ by definition. Noting that x_i is a 2-vertex and by the construction of G'', we have $x_i \in V(G)$. Hence for each $i \in [s + 1]$, the edge x_ix_{i+1} in G'' corresponds to either an edge of $G - V(C)$ or a p-cycle of $G - V(C)$ consisting of two threads by its construction. Therefore, $x_1x_2 \dots x_{s+2}$ corresponds to a valid s-necklace of G. In particular, a valid p^+-thread of G'' corresponds to a valid $(p-2)^+$-necklace of G, and thus G'' contains no valid p^+-thread by Claim 2(ii). □

Claim 5. G'' contains no valid $(k_1, k_2, ..., k_t)$-thread with $\sum_{i=1}^{t} k_i \geq pt - p + 1$ and $t \geq 3$.

Proof of Claim 5. Suppose to the contrary that G'' has a valid $(k_1, k_2, ..., k_t)$-thread T_x such that $\sum_{i=1}^{t} k_i \geq pt - p + 1$ and $t = d_{G''}(x) \geq 3$. For each $i \in [t]$, let x_i be the end vertex (other than x) of the k_i-thread in the $(k_1, k_2, ..., k_t)$-thread, let y_i be the neighbor of x_i on the k_i-thread, and let z_i be the neighbor of x on the k_i-thread. Then $V(T_x)\backslash\{y_1, y_2, ..., y_t\} \subseteq V(G'')\backslash V(C)$. By Claim 4, we have $k_i \leq p - 1$ for any $i \in [t]$. Note that for each $i \in [t]$, y_i is not a sticking vertex since it is a 2-vertex in G''. If x is a vertex in $V(G)$, then the (x, y_i)-thread from x to y_i corresponds to a $(k_i - 1)$-necklace $N(x, y_i)$ in $G - V(C)$ for each $i \in [t]$. Hence G contains a valid $(k_1 - 1, k_2 - 1, ..., k_t - 1)$-necklace N_x with end vertices $y_1, y_2, ..., y_t$. Since $\sum_{i=1}^{t} k_i \geq pt - p + 1$, we have $\sum_{i=1}^{t}(k_i - 1) \geq (p - 1)t - p + 1$, a contradiction to Claim 2(iii).

Assume instead that x is not a vertex in $V(G)$. Then x is a sticking vertex in G'', which corresponds to a p-cycle K_x in $G - V(C)$. Thus, z_i is an attachment-vertex of K_x for each $i \in [t]$. Hence the (z_i, y_i)-thread corresponds to a $(k_i - 2)$-necklace $N(z_i, y_i)$ in $G - V(C)$ for each $i \in [t]$. Thus G contains a valid $(k_1 - 2, k_2 - 2, ..., k_t - 2)$-crown-necklace with end vertices $y_1, y_2, ..., y_t$. Similarly, we have $\sum_{i=1}^{t}(k_i - 2) \geq (p - 2)t - p + 1$ by $\sum_{i=1}^{t} k_i \geq pt - p + 1$, which contradicts Claim 2(iii) again. □

Now we shall complete the proof by a discharging method on G''. Any face other than f_0 is called an *internal face* of G''. The vertices of $V(G'')\backslash V(C)$ are called *internal vertices* of G''. The degree $d_{G''}(f)$ of a face f is the number of edges in its boundary, cut edges being counted twice. Let $F(G'')$ be the set of faces of G''. From Euler Formula, we have

$$\sum_{v\in V(G'')}\left(\frac{p}{2}d_{G''}(v) - (p+2)\right) + \sum_{f\in F(G'')}(d_{G''}(f) - (p+2)) = -2(p+2),$$

which implies

$$\sum_{v\in V(G'')}\left(\frac{p}{2}d_{G''}(v) - (p+2)\right) + (d_{G''}(f_0) + p) + \sum_{f\in F(G'')\backslash\{f_0\}}(d_{G''}(f) - (p+2)) = -2. \quad (2)$$

Assign an initial charge $ch_0(v) = \frac{p}{2}d_{G''}(v) - (p+2)$ for each $v \in V(G'')$, $ch_0(f_0) = 2p$ and $ch_0(f) = d_{G''}(f) - (p+2)$ for each $f \in F(G'')\backslash\{f_0\}$. Hence the total charge is -2 by Equation (2).

We redistribute the charges according to the following rules.
(RI) *Every $(4p+8)^+$-face of G'' gives charge $\frac{3}{4}$ to each of its incident internal vertices.*
(RII) *Every 3^+-vertex of G'' gives charge $\frac{1}{4}$ to each of its weakly adjacent internal 2-vertices.*
(RIII) *The outer face f_0 gives charge 2 to each of its incident vertices.*
Note that in (RII) each 3^+-vertex of G'' gives no charge to its weakly adjacent 2-vertices in $V(C)$, since each vertex in $V(C)$ is not internal by definition.

Let ch denote the charge assignment after performing the charge redistribution using the rules (RI), (RII), and (RIII).

Claim 6. $ch(f) \geq 0$ for each $f \in F(G'')$.

Proof of Claim 6. By Claim 3, G'' is a plane graph of girth p without cycles of length from $p+1$ to $4(p+2) - 1$. If $d_{G''}(f) = p$, then f must be the outer face f_0, and thus

$ch_0(f_0) = 2p$. By (RIII), f sends charge 2 to each of its incident vertices, and hence $ch(f) = ch_0(f_0) - 2p = 0$. Now assume $d_{G''}(f) \geq 4(p+2)$. Then f sends charge $\frac{3}{4}$ to each incident internal vertices by (RI), and so $ch(f) \geq ch_0(f) - \frac{3}{4}d_{G''}(f) = (d_{G''}(f) - (p+2)) - \frac{3}{4}d_{G''}(f) = \frac{1}{4}(d_{G''}(f) - 4(p+2)) \geq 0$. □

Claim 7. $ch(v) \geq 0$ for each $v \in V(G'')$.

Proof of Claim 7. First we assume $d_{G''}(v) = 2$. Then $ch_0(v) = -2$. If $v \in V(C)$, then v receives charge 2 from f_0 by (RIII). Thus $ch(v) = ch_0(v) + 2 = 0$. For an internal 2-vertex v, by Claims 1 and 4, v is weakly adjacent to two 3^+-vertices, and thus v receives charge $\frac{1}{4} \times 2$ by (RII). By (RI), v receives charge $\frac{3}{4} \times 2$ from its incident faces. Hence $ch(v) = -2 + \frac{1}{2} + \frac{3}{2} = 0$.

Now we assume $d_{G''}(v) \geq 3$. Let $t(v)$ be the number of internal 2-vertices weakly adjacent to v. By (RII), v sends charge $\frac{1}{4}t(v)$ to its weakly adjacent internal 2-vertices. If $v \in V(C)$, then $t(v) \leq (p-1)(d_{G''}(v) - 2)$ as each thread in G'' contains at most $(p-1)$ internal 2-vertices by Claim 4. Note that v receives charge $\frac{3}{4}(d_{G''}(v) - 1)$ from its incident $(4p+8)^+$-faces by (RI), and receives charge 2 from f_0 by (RIII). Then

$$\begin{aligned}
ch(v) &= ch_0(v) - \frac{1}{4}t(v) + \frac{3}{4}(d_{G''}(v) - 1) + 2 \\
&\geq \left(\frac{p}{2}d_{G''}(v) - (p+2)\right) - \frac{1}{4}(p-1)(d_{G''}(v) - 2) + \frac{3}{4}(d_{G''}(v) - 1) + 2 \\
&= \frac{p+4}{4}d_{G''}(v) - \frac{1}{2}p - \frac{5}{4} \\
&\geq \frac{p+4}{4} \cdot 3 - \frac{1}{2}p - \frac{5}{4} \\
&= \frac{p+7}{4} > 0.
\end{aligned}$$

Assume instead that v is an internal vertex. By Claims 4 and 5, $t(v) \leq pd_{G''}(v) - p$. By (RI), v receives charge $\frac{3}{4}d_{G''}(v)$ from its incident faces. Hence

$$\begin{aligned}
ch(v) &= \left(\frac{p}{2}d_{G''}(v) - (p+2)\right) + \frac{3}{4}d_{G''}(v) - \frac{1}{4}t(v) \\
&\geq \frac{p}{2}d_{G''}(v) - p - 2 + \frac{3}{4}d_{G''}(v) - \frac{1}{4}(pd_{G''}(v) - p) \\
&= \frac{p+3}{4}d_{G''}(v) - \frac{3}{4}p - 2 \\
&\geq \frac{p+3}{4} \cdot 3 - \frac{3}{4}p - 2 \\
&= \frac{1}{4} > 0.
\end{aligned}$$

By Equation (2) and Claims 6 and 7, we have

$$-2 = \sum_{v \in V(G'')} ch_0(v) + \sum_{f \in F(G'')} ch_0(f) = \sum_{v \in V(G'')} ch(v) + \sum_{f \in F(G'')} ch(f) \geq 0,$$

a contradiction. This contradiction completes the proof of Theorem 3.4. □

4 | THE FRACTIONAL COLORING

This section is devoted to prove Theorem 1.6. We first study some graphs with precoloring extensions in Section 4.1, serving for reducible configurations, and then present the proof of Theorem 1.6 in Section 4.2 by a discharging method.

4.1 | Precoloring graphs for fractional $\left(k : \frac{k-1}{2}\right)$-coloring

We start with the following property on coloring of paths.

Lemma 4.1. *Let $P = v_1 v_2 \ldots v_t$ be a path with $2 \leq t \leq k$, and let φ be a fractional $\left(k : \frac{k-1}{2}\right)$-coloring of P. Then $|\varphi(v_1) \cap \varphi(v_t)| \geq \frac{k-t}{2}$ if t is odd, and $|\varphi(v_1) \cap \varphi(v_t)| \leq \frac{t-2}{2}$ if t is even.*

Proof. We prove by induction. Since φ is a fractional $\left(k : \frac{k-1}{2}\right)$-coloring of P, we have $\varphi(v_i) \cap \varphi(v_{i+1}) = \varnothing$ for each $i \in [t-1]$. Thus Lemma 4.1 holds for $t = 2$. If $t = 3$, noting that $(\varphi(v_1) \cup \varphi(v_3)) \subseteq [k] \backslash \varphi(v_2)$, then $|\varphi(v_1) \cup \varphi(v_3)| \leq k - |\varphi(v_2)|$, and thus $|\varphi(v_1) \cap \varphi(v_3)| = |\varphi(v_1)| + |\varphi(v_3)| - |\varphi(v_1) \cup \varphi(v_3)| \geq \frac{k-1}{2} + \frac{k-1}{2} - \left(k - \frac{k-1}{2}\right) = \frac{k-3}{2}$. That is, Lemma 4.1 holds for $t = 3$. Assume Lemma 4.1 holds for any value smaller than t. Now we consider $\varphi(v_1) \cap \varphi(v_t)$. First we assume t is even. Then $t-1$ is odd, and $|\varphi(v_1) \cap \varphi(v_{t-1})| \geq \frac{k-t+1}{2}$ by induction hypothesis. Since $\varphi(v_t) \cap \varphi(v_{t-1}) = \varnothing$, we have $\varphi(v_1) \cap \varphi(v_t) \subseteq \varphi(v_1) \backslash \varphi(v_{t-1})$, and thus $|\varphi(v_1) \cap \varphi(v_t)| \leq |\varphi(v_1)| - |\varphi(v_1) \cap \varphi(v_{t-1})| \leq \frac{k-1}{2} - \frac{k-t+1}{2} = \frac{t-2}{2}$. Now we assume t is odd. Then $t-1$ is even, and $|\varphi(v_1) \cap \varphi(v_{t-1})| \leq \frac{t-3}{2}$ by induction hypothesis. As $\varphi(v_t) \cap \varphi(v_{t-1}) = \varnothing$, we have $(\varphi(v_1) \cup \varphi(v_t)) \subseteq [k] \backslash (\varphi(v_{t-1}) \backslash \varphi(v_1))$, which implies $|\varphi(v_1) \cup \varphi(v_t)| \leq k - |\varphi(v_{t-1})| + |\varphi(v_1) \cap \varphi(v_{t-1})|$. Thus $|\varphi(v_1) \cap \varphi(v_t)| = |\varphi(v_1)| + |\varphi(v_t)| - |\varphi(v_1) \cup \varphi(v_t)| \geq \frac{k-1}{2} + \frac{k-1}{2} - k + \frac{k-1}{2} - |\varphi(v_1) \cap \varphi(v_{t-1})| \geq \frac{k-1}{2} + \frac{k-1}{2} - k + \frac{k-1}{2} - \frac{t-3}{2} = \frac{k-t}{2}$. Therefore, Lemma 4.1 holds by induction. □

Recall that, for $S \subset V(H)$, H is φ_S-*colorable* if the precoloring φ of S can be extended to a fractional $\left(k : \frac{k-1}{2}\right)$-coloring of H. Note that the number $\frac{k-2}{4} + (-1)^{d(x,y)} \cdot \frac{k-2d(x,y)}{4}$ is always an integer; in fact it is $\frac{k-1-d(x,y)}{2}$ if $d(x,y)$ is even, and $\frac{d(x,y)-1}{2}$ if $d(x,y)$ is odd.

Lemma 4.2. *Let C be a cycle of length k. Let φ be a precoloring of $\{x, y\} \subseteq V(C)$. Then C is $\varphi_{\{x,y\}}$-colorable if and only if*

$$|\varphi(x) \cap \varphi(y)| = \frac{k-2}{4} + (-1)^{d(x,y)} \cdot \frac{k - 2d(x, y)}{4}.$$

Proof. Denote $C = x_0x_1 \ldots x_{k-1}x_0$, where $x_0 = x, x_t = y$, and $d(x, y) = t \leq \frac{k-1}{2}$.

Assume that C is $\varphi_{\{x,y\}}$-colorable, and let $\tilde{\varphi}$ be a fractional $\left(k : \frac{k-1}{2}\right)$-coloring of C extended by φ. Denote $P_1 = x_0x_1 \ldots x_t$ and $P_2 = x_0x_{k-1}x_{k-2} \ldots x_t$. Then P_1 is a path of order $t + 1$ and P_2 is a path of order $k - t + 1$. Note that $\tilde{\varphi}$ also provides a fractional $\left(k : \frac{k-1}{2}\right)$-coloring of P_1 and of P_2. If t is even, then by Lemma 4.1, we have $|\tilde{\varphi}(x_0) \cap \tilde{\varphi}(x_t)| \geq \frac{k-t-1}{2}$ as $|V(P_1)| = t + 1$ is odd and $|\tilde{\varphi}(x_0) \cap \tilde{\varphi}(x_t)| \leq \frac{k-t-1}{2}$ as $|V(P_2)| = k - t + 1$ is even. Thus $|\varphi(x_0) \cap \varphi(x_t)| = |\tilde{\varphi}(x_0) \cap \tilde{\varphi}(x_t)| = \frac{k-t-1}{2}$. If t is odd, then by Lemma 4.1, we have $|\tilde{\varphi}(x_0) \cap \tilde{\varphi}(x_t)| \leq \frac{t-1}{2}$ as $|V(P_1)| = t + 1$ is even and $|\tilde{\varphi}(x_0) \cap \tilde{\varphi}(x_t)| \geq \frac{t-1}{2}$ as $|V(P_2)| = k - t + 1$ is odd. Hence $|\varphi(x_0) \cap \varphi(x_t)| = |\tilde{\varphi}(x_0) \cap \tilde{\varphi}(x_t)| = \frac{t-1}{2}$.

Conversely, assume that $a = |\varphi(x) \cap \varphi(y)| = \frac{k-2}{4} + (-1)^t \cdot \frac{k-2t}{4}$. Without loss of generality, we may assume $\varphi(x_0) = \left\{1, 2, \ldots, \frac{k-1}{2}\right\}$. If t is even, we assume $\varphi(x_0) \cap \varphi(x_t) = \{1, 2, \ldots, a\}$, and $\varphi(x_t) \backslash \varphi(x_0) = \left\{\frac{k+1}{2} + a + 1, \frac{k+1}{2} + a + 2, \ldots, k\right\}$. If t is odd, we assume $\varphi(x_0) \cap \varphi(x_t) = \left\{\frac{k-1}{2} - a + 1, \frac{k-1}{2} - a + 2, \frac{k-1}{2}\right\}$, and $\varphi(x_t) \backslash \varphi(x_0) = \left\{\frac{k-1}{2} + 1, \frac{k-1}{2} + 2, \ldots, k - a - 1\right\}$. We define a coloring by setting $\varphi(x_{2i}) = \left\{1, 2, \ldots, \frac{k-1}{2} - i\right\} \cup \{k - i + 1, k - i + 2, \ldots, k\}$ and $\varphi(x_{2i+1}) = \left\{\frac{k+1}{2} - i, \frac{k+1}{2} - i + 1, \ldots, k - i - 1\right\}$ for $0 \leq i \leq \frac{k-1}{2}$. It is routine to check that φ is a fractional $\left(k : \frac{k-1}{2}\right)$-coloring of C. □

Lemma 4.3. *Let $N(x, y)$ be a necklace with a precoloring φ of $\{x, y\}$. Suppose that the distance between x and y is $d(x, y) = t \leq \frac{k+1}{2}$. If*

$$|\varphi(x) \cap \varphi(y)| = \frac{k-2}{4} + (-1)^t \cdot \frac{k-2t}{4},$$

then $N(x, y)$ is $\varphi_{\{x,y\}}$-colorable.

Proof. We prove by induction. The statement holds for $t = 0, 1$. Assume that it holds for any value smaller than t. If x and y are in the same k-cycle, then the statement holds from Lemma 4.2. Otherwise, we can always find a vertex u in the shortest (x, y)-path $xz_1 \ldots z_{t-1}y$ which divides the necklace into two separated necklaces that one is from x to u and the other is from u to y. More precisely, if xz_1 is not contained in a k-cycle, then we choose $u = z_1$; otherwise, we choose $u = z_j$ where j is the largest index such that $z_{j-1}z_j$ is in the k-cycle containing xz_1. Note that u is a cut vertex of H that divides the necklace H into two separated necklaces. Now we shall try to provide a coloring $\varphi(u)$ of u and then apply induction on the (x, u)-necklace and on the (u, y)-necklace. This can be achieved if we can find a colors from $\varphi(x)\backslash\varphi(y)$, b colors from $\varphi(x) \cap \varphi(y)$, c colors from $\varphi(y)\backslash\varphi(x)$, and the rest colors from $[k]\backslash(\varphi(x) \cup \varphi(y))$ to formulate $\varphi(u)$ satisfying the induction hypothesis.

Let $d(x, u) = s$. Then $d(u, y) = t - s$. Formally, we need to find a nonnegative integer solution (a, b, c) of the following system of inequalities:

$$\begin{cases} 0 \leq a \leq |\varphi(x)\backslash\varphi(y)| = \dfrac{k}{4} - (-1)^t \cdot \dfrac{k-2t}{4}, \\ 0 \leq b \leq |\varphi(x) \cap \varphi(y)| = \dfrac{k-2}{4} + (-1)^t \cdot \dfrac{k-2t}{4}, \\ 0 \leq c \leq |\varphi(y)\backslash\varphi(x)| = \dfrac{k}{4} - (-1)^t \cdot \dfrac{k-2t}{4}, \\ 0 \leq \dfrac{k-1}{2} - a - b - c \leq |[k]\backslash(\varphi(x) \cup \varphi(y))| = 1 + \dfrac{k-2}{4} + (-1)^t \cdot \dfrac{k-2t}{4}, \\ a + b = \dfrac{k-2}{4} + (-1)^s \cdot \dfrac{k-2s}{4}, \\ b + c = \dfrac{k-2}{4} + (-1)^{t-s} \cdot \dfrac{k-2(t-s)}{4}. \end{cases}$$

Let

$$\alpha = (-1)^t \cdot \frac{k-2t}{4}, \quad \beta = (-1)^s \cdot \frac{k-2s}{4}, \quad \text{and} \quad \gamma = (-1)^{t-s} \cdot \frac{k-2(t-s)}{4}.$$

Plugging $a = -b + \frac{k-2}{4} + \beta$ and $c = -b + \frac{k-2}{4} + \gamma$ into the above system of inequalities, we have

$$\begin{cases} \alpha + \beta - \frac{1}{2} \leq b \leq \frac{k-2}{4} + \beta, \\ \qquad\quad 0 \leq b \leq \frac{k-2}{4} + \alpha, \\ \alpha + \gamma - \frac{1}{2} \leq b \leq \frac{k-2}{4} + \gamma, \\ \beta + \gamma - \frac{1}{2} \leq b \leq \frac{k}{4} + \alpha + \beta + \gamma. \end{cases}$$

Let

$$M = \max\left\{\alpha + \beta - \frac{1}{2}, 0, \alpha + \gamma - \frac{1}{2}, \beta + \gamma - \frac{1}{2}\right\} \quad \text{and}$$

$$N = \min\left\{\frac{k-2}{4} + \beta, \frac{k-2}{4} + \alpha, \frac{k-2}{4} + \gamma, \frac{k}{4} + \alpha + \beta + \gamma\right\}.$$

We can actually show that $0 \leq M \leq N$ by a one-by-one compression, and then setting $b = M$ provides a valid solution of the above system of inequalities. This method will be applied in a similar but more complicated Lemma 4.5 below.

Here an alternative way to do so is to check case by case on the parity as follows.

- If t is odd and s is odd, then set $b = M = N = \dfrac{s-1}{2}$.
- If t is odd and s is even, then set $b = M = N = \dfrac{t-s-1}{2}$.
- If t is even and s is odd, then set $b = M = N = 0$.
- If t is even and s is even, then set $b = M = N = \dfrac{k-1-t}{2}$.

Then this solution (a, b, c) provides a coloring $\varphi(u)$ as desired. □

We present this version of the proof of Lemma 4.3 to provide an overview of the more complicated Lemma 4.5 below when $d(x, y)$ is relatively large. We also need the following technical inequality.

Proposition 4.4. *Let s, t be integers with $1 \le s \le \frac{k-1}{2}$ and $\frac{k+1}{2} \le t \le s + \frac{k+1}{2}$. Denote*

$$\beta = (-1)^s \cdot \frac{k-2s}{4} \quad \text{and} \quad \gamma = (-1)^{t-s} \cdot \frac{k-2(t-s)}{4}.$$

Let ℓ be a fixed integer with $\frac{k-t}{2} - \frac{(-1)^t+1}{4} \le \ell \le \frac{t-1}{2} - \frac{(-1)^t+1}{4}$. Define

$$\mathbf{M} = \max\left\{\beta + \ell - \frac{k}{4}, 0, \gamma + \ell - \frac{k}{4}, \beta + \gamma - \frac{1}{2}\right\} \quad \text{and}$$
$$\mathbf{N} = \min\left\{\frac{k-2}{4} + \beta, \ell, \frac{k-2}{4} + \gamma, \beta + \gamma + \ell + \frac{1}{2}\right\}.$$

Then $\mathbf{M}$ and $\mathbf{N}$ are integers satisfying

$$0 \le \mathbf{M} \le \mathbf{N}.$$

Proof. It is routine to check that each term in $\mathbf{M}$ and in $\mathbf{N}$ is an integer by discussing the parity of t and s. To show that $\mathbf{M} \le \mathbf{N}$, is suffices to check 16 inequalities one by one.

- $\beta + \ell - \frac{k}{4} \le \mathbf{N} = \min\left\{\frac{k-2}{4} + \beta, \ell, \frac{k-2}{4} + \gamma, \beta + \gamma + \ell + \frac{1}{2}\right\}$;
- $0 \le \mathbf{N} = \min\left\{\frac{k-2}{4} + \beta, \ell, \frac{k-2}{4} + \gamma, \beta + \gamma + \ell + \frac{1}{2}\right\}$;
- $\gamma + \ell - \frac{k}{4} \le \mathbf{N} = \min\left\{\frac{k-2}{4} + \beta, \ell, \frac{k-2}{4} + \gamma, \beta + \gamma + \ell + \frac{1}{2}\right\}$;
- $\beta + \gamma - \frac{1}{2} \le \mathbf{N} = \min\left\{\frac{k-2}{4} + \beta, \ell, \frac{k-2}{4} + \gamma, \beta + \gamma + \ell + \frac{1}{2}\right\}$.

It turns out to become the following:

- $\ell \le \frac{k-1}{2}, \beta \le \frac{k}{4}, \beta - \gamma \le \frac{k-1}{2} - \ell, -\gamma \le \frac{k+2}{4}$;

- $-\beta \leq \frac{k-2}{4}$, $0 \leq \ell$, $-\gamma \leq \frac{k-2}{4}$, $-\gamma-\beta \leq \ell+\frac{1}{2}$;
- $\gamma-\beta \leq \frac{k-1}{2}-\ell$, $\gamma \leq \frac{k}{4}$, $\ell \leq \frac{k-1}{2}$, $-\beta \leq \frac{k+2}{4}$;
- $\gamma \leq \frac{k}{4}$, $\beta+\gamma \leq \ell+\frac{1}{2}$, $\beta \leq \frac{k}{4}$, $0 \leq \ell+1$.

Except some trivial ones that $|\beta| \leq \frac{k}{4}$, $|\gamma| \leq \frac{k}{4}$, $0 \leq \ell \leq \frac{k-1}{2}$, *this reduces to the following:*

- $-\gamma-\beta \leq \ell+\frac{1}{2}$, $\beta+\gamma \leq \ell+\frac{1}{2}$, $\beta-\gamma \leq \frac{k-1}{2}-\ell$, *and* $\gamma-\beta \leq \frac{k-1}{2}-\ell$.

Those inequalities above are all true since

- $|\gamma+\beta| \leq |\gamma|+|\beta| \leq \frac{k-2s}{4}+\frac{k-2(t-s)}{4}=\frac{k-t}{2} \leq \ell+\frac{1}{2}$ *and*
- $|\beta-\gamma|+\ell \leq |\beta|+|\gamma|+\ell \leq \frac{k-2s}{4}+\frac{k-2(t-s)}{4}+\frac{t-1}{2}=\frac{k-1}{2}$.

This proves that $0 \leq \mathbf{M} \leq \mathbf{N}$. □

Lemma 4.5. *Let* $N(x, y)$ *be a necklace with a precoloring* φ *of* $\{x, y\}$. *Suppose that the distance between* x *and* y *satisfies* $d(x, y)=t \geq \frac{k+1}{2}$. *If*

$$\frac{k-t}{2}-\frac{(-1)^t+1}{4} \leq |\varphi(x) \cap \varphi(y)| \leq \frac{t-1}{2}-\frac{(-1)^t+1}{4},$$

then H *is* $\varphi_{\{x,y\}}$*-colorable.*

Proof. The basic case $t=\frac{k+1}{2}$ has already been handled in Lemma 4.3. We shall prove Lemma 4.5 by induction. Similarly, there exists a cut vertex u of H in the shortest (x, y)-path that divides the necklace into two parts (two separated necklaces), one is from x to u and the other is from u to y. We choose such cut vertex u with the smallest distance from x. So either xu is an edge or x and u are in the same k-cycle, and hence we have $d(x, u)=s \leq \frac{k-1}{2}$. We shall divide the discussion into two cases depending on the value of $d(u, y)=t-s$.

Case 1. $d(u, y)=t-s \leq \frac{k+1}{2}$.

Note that in this case $t \leq s+\frac{k+1}{2} \leq k$. Now we shall try to find a colors from $\varphi(x) \backslash \varphi(y)$, b colors from $\varphi(x) \cap \varphi(y)$, c colors from $\varphi(y) \backslash \varphi(x)$, and the rest colors from $[k] \backslash(\varphi(x) \cup \varphi(y))$ to formulate $\varphi(u)$ satisfying the induction hypothesis. Formally, similar to the proof of Lemma 4.3, we need to find a nonnegative integer solution (a, b, c) of the following system of inequalities:

$$\begin{cases} 0 \le a \le |\varphi(x) \backslash \varphi(y)|, \\ 0 \le b \le |\varphi(x) \cap \varphi(y)|, \\ 0 \le c \le |\varphi(y) \backslash \varphi(x)|, \\ 0 \le \dfrac{k-1}{2} - a - b - c \le |[k] \backslash (\varphi(x) \cup \varphi(y))|, \\ a + b = \dfrac{k-2}{4} + (-1)^s \cdot \dfrac{k-2s}{4}, \\ b + c = \dfrac{k-2}{4} + (-1)^{t-s} \cdot \dfrac{k-2(t-s)}{4}. \end{cases}$$

Let $\ell = |\varphi(x) \cap \varphi(y)|$ be a fixed number with $\frac{k-t}{2} - \frac{(-1)^t+1}{4} \le \ell \le \frac{t-1}{2} - \frac{(-1)^t+1}{4}$.

Denote

$$\beta = (-1)^s \cdot \frac{k-2s}{4} \quad \text{and} \quad \gamma = (-1)^{t-s} \cdot \frac{k-2(t-s)}{4}.$$

Then by plugging a and c into the above system of inequalities, it reduces to the following:

$$\begin{cases} \beta + \ell - \frac{k}{4} \le b \le \frac{k-2}{4} + \beta, \\ \qquad 0 \le b \le \ell, \\ \gamma + \ell - \frac{k}{4} \le b \le \frac{k-2}{4} + \gamma, \\ \beta + \gamma - \frac{1}{2} \le b \le \beta + \gamma + \ell + \frac{1}{2}. \end{cases}$$

Let

$$\mathbf{M} = \max\left\{\beta + \ell - \frac{k}{4}, 0, \gamma + \ell - \frac{k}{4}, \beta + \gamma - \frac{1}{2}\right\} \quad \text{and}$$

$$\mathbf{N} = \min\left\{\frac{k-2}{4} + \beta, \ell, \frac{k-2}{4} + \gamma, \beta + \gamma + \ell + \frac{1}{2}\right\}.$$

By Proposition 4.4, $\mathbf{M}$ and $\mathbf{N}$ are integers satisfying $0 \le \mathbf{M} \le \mathbf{N}$. Therefore, we choose

$$b = \mathbf{M}, a = \frac{k-2}{4} + (-1)^s \cdot \frac{k-2s}{4} - \mathbf{M}, \text{ and}$$

$$c = \frac{k-2}{4} + (-1)^{t-s} \cdot \frac{k-2(t-s)}{4} - \mathbf{M},$$

providing a desired nonnegative integer solution (a, b, c).

Case 2. $d(u, y) = t - s \ge \frac{k+3}{2}$.

We are still trying to find a colors from $\varphi(x)\backslash\varphi(y)$, b colors from $\varphi(x)\cap\varphi(y)$, c colors from $\varphi(y)\backslash\varphi(x)$, and the rest colors from $[k]\backslash(\varphi(x)\cup\varphi(y))$ to form $\varphi(u)$ satisfying the induction hypothesis. This formulates similar system of inequalities as follows:

$$\begin{cases} 0 \le a \le |\varphi(x)\backslash\varphi(y)|, \\ 0 \le b \le |\varphi(x)\cap\varphi(y)|, \\ 0 \le c \le |\varphi(y)\backslash\varphi(x)|, \\ 0 \le \dfrac{k-1}{2} - a - b - c \le |[k]\backslash(\varphi(x)\cup\varphi(y))|, \\ a + b = \dfrac{k-2}{4} + (-1)^s \cdot \dfrac{k-2s}{4}, \\ \dfrac{k-t+s}{2} - \dfrac{(-1)^{t-s}+1}{4} \le b + c \le \dfrac{t-s-1}{2} - \dfrac{(-1)^{t-s}+1}{4}. \end{cases}$$

Notice that $\dfrac{k-2-(-1)^{\frac{k+1}{2}}}{4}$ is an integer (this value comes from the case $d(x,y)=\frac{k+1}{2}$), and we have

$$\frac{k-t+s}{2} - \frac{(-1)^{t-s}+1}{4} \le \frac{k-2-(-1)^{\frac{k+1}{2}}}{4} \le \frac{t-s-1}{2} - \frac{(-1)^{t-s}+1}{4}.$$

So it is enough to find a solution (a,b,c) with the last inequality replaced by

$$b + c = \frac{k-2-(-1)^{\frac{k+1}{2}}}{4}.$$

Let $\ell = |\varphi(x)\cap\varphi(y)|$ be a fixed number with $\frac{k-t}{2} - \frac{(-1)^t+1}{4} \le \ell \le \frac{t-1}{2} - \frac{(-1)^t+1}{4}$. Note that $0 \le \ell \le \frac{k-1}{2}$.

Denote

$$\beta = (-1)^s \cdot \frac{k-2s}{4} \quad \text{and} \quad \gamma = -(-1)^{\frac{k+1}{2}} \cdot \frac{1}{4}.$$

Then with similar calculation, by plugging a and c into the above system of inequalities, it becomes the following:

$$\begin{cases} \beta + \ell - \frac{k}{4} \le b \le \frac{k-2}{4} + \beta, \\ 0 \le b \le \ell, \\ \gamma + \ell - \frac{k}{4} \le b \le \frac{k-2}{4} + \gamma, \\ \beta + \gamma - \frac{1}{2} \le b \le \beta + \gamma + \ell + \frac{1}{2}. \end{cases}$$

We still let

$$\mathbf{M} = \max\left\{\beta + \ell - \frac{k}{4}, 0, \gamma + \ell - \frac{k}{4}, \beta + \gamma - \frac{1}{2}\right\} \quad \text{and}$$

$$\mathbf{N} = \min\left\{\frac{k-2}{4} + \beta, \ell, \frac{k-2}{4} + \gamma, \beta + \gamma + \ell + \frac{1}{2}\right\}.$$

By Proposition 4.4 with $t - s = \frac{k+1}{2}$, $\mathbf{M}$ and $\mathbf{N}$ are integers satisfying $0 \le \mathbf{M} \le \mathbf{N}$. Therefore, we can choose $b = \mathbf{M}$ and corresponding a and c to form a desired solution (a, b, c). This completes the proof. □

By Lemma 4.5, we have the following corollary.

Corollary 4.6. *Let $N(x, y)$ be a necklace. If $d(x, y) \ge k$, then $N(x, y)$ is $\varphi_{\{x,y\}}$-colorable for any precoloring φ of $\{x, y\}$.*

Recall Definition 2.7 that a $(k_1, k_2; k_3)$-bull-necklace is a subgraph obtained from a (k_1, k_2, k_3)-thread by applying C_k-replacement operation on some edges of the k_3-thread. For $1 \le t \le \frac{k-1}{2}$, let $B_v(t, s)$ be a $(t-1, t-1; r)$-bull-necklace N_v with end vertices x, y, z and $d(v, z) = s$.

Lemma 4.7. *For a bull-necklace $B_v(t, s)$ with end vertices x, y, z, if $1 \le t \le \frac{k-1}{2}$ and $t + s \ge k$, then $B_v(t, s)$ is $\varphi_{\{x,y,z\}}$-colorable for any precoloring φ of $\{x, y, z\}$ satisfying $|\varphi(x) \cap \varphi(y)| = \frac{k-1-2t}{2}$.*

Proof. Let φ be a precoloring of $\{x, y, z\}$ such that $|\varphi(x) \cap \varphi(y)| = \frac{k-1-2t}{2}$. Denote $A = \varphi(x)\backslash\varphi(y)$, $B = \varphi(x) \cap \varphi(y)$, $C = \varphi(y)\backslash\varphi(x)$, and $D = [k]\backslash(\varphi(x) \cup \varphi(y))$. Then $|A| = |C| = t$, $|B| = \frac{k-1-2t}{2}$, and $|D| = \frac{k+1-2t}{2}$. Let S be a subset of $[k]$ such that $S = B$ if t is even and $S = D$ if t is odd. Then $|S| = \frac{k+1-2t}{2} - \frac{(-1)^t+1}{2}$. Denote $S_1 = S\backslash\varphi(z)$ and $S_2 = S \cap \varphi(z)$. □

We first make the following claim.

Claim 1. Each of the following holds:

(i) either $|A \cap \varphi(z)| \ge \frac{t+1}{2} - \frac{(-1)^t+1}{4} - |S_2|$ or $|C \cap \varphi(z)| \ge \frac{t+1}{2} - \frac{(-1)^t+1}{4} - |S_2|$;

(ii) either $|A\backslash\varphi(z)| \ge \frac{t+1}{2} - \frac{(-1)^t+1}{4} - |S_1|$ or $|C\backslash\varphi(z)| \ge \frac{t+1}{2} - \frac{(-1)^t+1}{4} - |S_1|$.

Proof of Claim 1.

(i) Notice that

$$\begin{aligned}|A \cap \varphi(z)| + |C \cap \varphi(z)| &= |(A \cup C \cup S) \cap \varphi(z)| - |S_2| \\ &= |A \cup C \cup S| + |\varphi(z)| - |(A \cup C \cup S) \cup \varphi(z)| \\ &\quad - |S_2| \\ &\geq \left[t + t + \left(\frac{k+1-2t}{2} - \frac{(-1)^t + 1}{2}\right)\right] + \frac{k-1}{2} \\ &\quad - k - |S_2| \\ &= t - \frac{(-1)^t + 1}{2} - |S_2| \\ &\geq 2\left(\frac{t+1}{2} - \frac{(-1)^t + 1}{4} - |S_2|\right) - 1.\end{aligned}$$

Hence (i) holds.

(ii) Similarly, notice that

$$\begin{aligned}|A\backslash\varphi(z)| + |C\backslash\varphi(z)| &= |(A \cup C \cup S)\backslash\varphi(z)| - |S_1| \\ &\geq \left(2t + \frac{k+1-2t}{2} - \frac{(-1)^t + 1}{2}\right) - \frac{k-1}{2} - |S_1| \\ &= t + 1 - \frac{(-1)^t + 1}{2} - |S_1| \\ &\geq 2\left(\frac{t+1}{2} - \frac{(-1)^t + 1}{4} - |S_1|\right).\end{aligned}$$

Thus (ii) holds. □

Next, we show that there are certain subsets of A and C of large size for candidates of $\varphi(v)$.

Claim 2. There exist $A_1 \subseteq A\backslash\varphi(z)$, $A_2 \subseteq A \cap \varphi(z)$, $C_1 \subseteq C\backslash\varphi(z)$, and $C_2 \subseteq C \cap \varphi(z)$ such that $|A_1| + |A_2| = \frac{t-1}{2} + \frac{(-1)^t+1}{4}$, $|C_1| + |C_2| = \frac{t-1}{2} + \frac{(-1)^t+1}{4}$, $|A_1| + |S_1| + |C_1| \geq \frac{t+1}{2} - \frac{(-1)^t+1}{4}$, and $|A_2| + |S_2| + |C_2| \geq \frac{t+1}{2} - \frac{(-1)^t+1}{4}$.

Proof of Claim 2. By Claim 1(i), we may assume without loss of generality that $|A \cap \varphi(z)| \geq \frac{t+1}{2} - \frac{(-1)^t+1}{4} - |S_2|$.

If $|C\backslash\varphi(z)| \geq \frac{t+1}{2} - \frac{(-1)^t+1}{4} - |S_1|$, then we can choose $C_1 = C\backslash\varphi(z)$ and $C_2 \subseteq C \cap \varphi(z)$ such that $|C_1| + |C_2| = \frac{t-1}{2} + \frac{(-1)^t+1}{4}$ and $|C_1| \geq \frac{t+1}{2} - \frac{(-1)^t+1}{4} - |S_1|$. This is feasible since $|C| = t \geq \frac{t-1}{2} + \frac{(-1)^t+1}{4}$. By $|A \cap \varphi(z)| \geq \frac{t+1}{2} - \frac{(-1)^t+1}{4} - |S_2|$, we can also choose $A_1 = A\backslash\varphi(z)$ and $A_2 \subseteq A \cap \varphi(z)$ such that $|A_1| + |A_2| = \frac{t-1}{2} + \frac{(-1)^t+1}{4}$ and $|A_2| \geq \frac{t+1}{2} - \frac{(-1)^t+1}{4} - |S_2|$. Hence we have $|A_1| + |S_1| + |C_1| \geq |S_1| + |C_1| \geq \frac{t+1}{2} - \frac{(-1)^t+1}{4}$ and $|A_2| + |S_2| + |C_2| \geq |A_2| + |S_2| \geq \frac{t+1}{2} - \frac{(-1)^t+1}{4}$.

Assume instead that $|C\backslash\varphi(z)| < \frac{t+1}{2} - \frac{(-1)^t+1}{4} - |S_1|$. Notice that

$$
\begin{aligned}
|C \cap \varphi(z)| &= |C| - |C \backslash \varphi(z)| \\
&\geq t - \left(\frac{t+1}{2} - \frac{(-1)^t + 1}{4} - |S_1|\right) \\
&= t - \frac{t+1}{2} + \frac{(-1)^t + 1}{4} + (|S| - |S_2|) \\
&= \frac{t-1}{2} + \frac{(-1)^t + 1}{4} + \left(\frac{k+1-2t}{2} - \frac{(-1)^t + 1}{2}\right) - |S_2| \\
&= \frac{k-t}{2} - \frac{(-1)^t + 1}{4} - |S_2| \\
&\geq \frac{t+1}{2} - \frac{(-1)^t + 1}{4} - |S_2|.
\end{aligned}
$$

Hence we can choose $C_1 = C \backslash \varphi(z)$ and $C_2 \subseteq C \cap \varphi(z)$ such that $|C_1| + |C_2| = \frac{t-1}{2} + \frac{(-1)^t+1}{4}$ and $|C_2| \geq \frac{t+1}{2} - \frac{(-1)^t+1}{4} - |S_2|$. By Claim 1(ii) and as $|C \backslash \varphi(z)| < \frac{t+1}{2} - \frac{(-1)^t+1}{4} - |S_1|$, we have $|A \backslash \varphi(z)| \geq \frac{t+1}{2} - \frac{(-1)^t+1}{4} - |S_1|$. Thus we can select $A_1 = A \backslash \varphi(z)$ and $A_2 \subseteq A \cap \varphi(z)$ such that $|A_1| + |A_2| = \frac{t-1}{2} + \frac{(-1)^t+1}{4}$ and $|A_1| \geq \frac{t+1}{2} - \frac{(-1)^t+1}{4} - |S_1|$. Therefore, we have $|A_1| + |S_1| + |C_1| \geq |A_1| + |S_1| \geq \frac{t+1}{2} - \frac{(-1)^t+1}{4}$ and $|A_2| + |S_2| + |C_2| \geq |S_2| + |C_2| \geq \frac{t+1}{2} - \frac{(-1)^t+1}{4}$ as desired.

Now we choose such $A_1 \subseteq A \backslash \varphi(z)$, $A_2 \subseteq A \cap \varphi(z)$, $C_1 \subseteq C \backslash \varphi(z)$, and $C_2 \subseteq C \cap \varphi(z)$ as in Claim 2. Let $\varphi(v) = A_1 \cup A_2 \cup S_1 \cup S_2 \cup C_1 \cup C_2$. Then

$$
\begin{aligned}
|\varphi(v)| &= \left(\frac{t-1}{2} + \frac{(-1)^t + 1}{4}\right) + \left(\frac{k+1-2t}{2} - \frac{(-1)^t + 1}{2}\right) + \left(\frac{t-1}{2} + \frac{(-1)^t + 1}{4}\right) \\
&= \frac{k-1}{2}.
\end{aligned}
$$

Moreover, $|\varphi(v) \cap \varphi(x)| = |A_1| + |A_2| + |S_1| + |S_2| = \frac{k-1-t}{2}$ if t is even and $|\varphi(v) \cap \varphi(x)| = |A_1| + |A_2| = \frac{t-1}{2}$ if t is odd; $|\varphi(v) \cap \varphi(y)| = |C_1| + |C_2| + |S_1| + |S_2| = \frac{k-1-t}{2}$ if t is even and $|\varphi(v) \cap \varphi(x)| = |C_1| + |C_2| = \frac{t-1}{2}$ if t is odd.

Notice that $(A_1 \cup S_1 \cup C_1) \subset [k] \backslash \varphi(z)$ and $(A_2 \cup S_2 \cup C_2) \subset \varphi(z)$. Hence by Claim 2 we have

$$
\frac{t+1}{2} - \frac{(-1)^t + 1}{4} \leq |\varphi(v) \cap \varphi(z)| \leq \frac{k-1}{2} - \left(\frac{t+1}{2} - \frac{(-1)^t + 1}{4}\right).
$$

Since $s + t \geq k$, we have

$$
\begin{aligned}
\frac{k-s}{2} - \frac{(-1)^s + 1}{4} \leq \frac{t+1}{2} - \frac{(-1)^t + 1}{4} \quad \text{and} \quad & \frac{k-1}{2} - \left(\frac{t+1}{2} - \frac{(-1)^t + 1}{4}\right) \\
\leq \frac{s-1}{2} - \frac{(-1)^s + 1}{4},
\end{aligned}
$$

which implies

$$\frac{k-s}{2}-\frac{(-1)^s+1}{4}\le|\varphi(v)\cap\varphi(z)|\le\frac{s-1}{2}-\frac{(-1)^s+1}{4}.$$

Thus $B_v(t,s)$ is $\varphi_{\{x,y,z,v\}}$-colorable by Lemmas 4.3 and 4.5. □

4.2 | The proof of Theorem 1.6

Now we are ready to prove Theorem 1.6 restated below in terms of plane graph.

Theorem 1.6. *For any odd integer $k \ge 5$, every plane graph of girth at least k without cycles of length from $k+1$ to $\left\lfloor \frac{22}{3}k \right\rfloor$ is fractional $\left(k : \frac{k-1}{2}\right)$-colorable.*

Proof. Suppose, for a contradiction, that G is a counterexample with $|V(G)| + |E(G)|$ minimized. □

Claim 1. G is 2-connected. In particular, $\delta(G) \ge 2$.

Proof of Claim 1. Clearly, G is connected. If G is not 2-connected, then there exist proper induced subgraphs G_1 and G_2 of G and a vertex $v \in V(G_2)$ such that $E(G) = E(G_1) \cup E(G_2)$ and $V(G_1) \cap V(G_2) = \{v\}$. By the minimality of G, G_1 has a fractional $\left(k : \frac{k-1}{2}\right)$-coloring φ_1 and G_2 has a fractional $\left(k : \frac{k-1}{2}\right)$-coloring φ_2. Exchange the colors if needed such that $\varphi_1(v) = \varphi_2(v)$, then φ_1 and φ_2 combine to become a $\left(k : \frac{k-1}{2}\right)$-coloring of G, which is a contradiction. □

For $1 \le t \le \frac{k-1}{2}$, let $F_v(t,s)$ be a graph obtained from a bull-necklace $B_v(t,s)$ with end vertices x, y, z by joining a new (x,y)-path of length $k-2t$ connecting x and y, where the vertices in the new (x,y)-path may have arbitrary degrees in G. That is, $F_v(t,s)$ consists of a k-cycle C_v and a necklace $N(v,z)$ with a common vertex v, where in the k-cycle C_v there exist two $(t-1)$-threads, one is from x to v and the other is from y to v.

Claim 2. G contains no $F_v(t,s)$ with $1 \le t \le \frac{k-1}{2}$ and $t+s \ge k$, where $s = d(v,z)$.

Proof of Claim 2. Suppose to the contrary that G contains an $F_v(t,s)$ with $1 \le t \le \frac{k-1}{2}$ and $t+s \ge k$. By the minimality of G, $G - (V(N(v,z))\setminus\{v,z\})$ has a $\left(k : \frac{k-1}{2}\right)$-coloring φ. If $2t \le \frac{k-1}{2}$, then $d_G(x,y) = 2t$, and if $2t \ge \frac{k+1}{2}$, then $d_G(x,y) = k-2t$. By Lemma 4.2, we always have $|\varphi(x) \cap \varphi(y)| = \frac{k-1-2t}{2}$. Let φ' be the restriction of φ to $G - (V(B_v(t,s))\setminus\{x,y,z\})$. As $|\varphi'(x) \cap \varphi'(y)| = \frac{k-1-2t}{2}$, $B_v(t,s)$ is $\varphi'_{\{x,y,z\}}$-colorable by Lemma 4.7. That is, φ' can be extended to a $\left(k : \frac{k-1}{2}\right)$-coloring of G, which is a contradiction. □

From G, we obtain a subgraph G' as follows: *f*or each facial k-cycle C of G, if there exists a 2-vertex in C, then we delete all the 2-vertices of a longest thread of C. Clearly, the obtained graph G' is a plane graph of girth at least k, and contains no cycles of length from $k+1$ to $\left\lfloor \frac{22k}{3} \right\rfloor$; furthermore, each facial k-cycle of G' contains no 2-vertices. It is easy to see that G' has minimal degree at least 2 by its construction.

Let $T(v,x)$ be a (v,x)-thread of G' and let $u = N_{G'}(v) \cap V(T(v,x))$. If there exists $w \in N_{G'}(v) \backslash \{u\}$ such that vu and vw are in a common k-cycle of G, then we say v is a *bad end vertex* of $T(v,x)$; otherwise, v is called a *good end vertex* of $T(v,x)$.

Claim 3. Let $T(v,x)$ be a (v,x)-thread of G' with a good end vertex v. Then $d_{G'}(v,x) \le k-1$.

Proof of Claim 3. Suppose to the contrary that $d_{G'}(v,x) \ge k$. If x is also a good end vertex of $T(v,x)$, then the thread $T(v,x)$ in G' corresponds to a necklace H with end vertices v, x in G. By the minimality of G, $G - (V(H) \backslash \{v,x\})$ has a $\left(k : \frac{k-1}{2}\right)$-coloring φ. Since $d_G(v,x) = d_{G'}(v,x) \ge k$ by construction, H is $\varphi_{\{v,x\}}$-colorable by Corollary 4.6. That is, φ can be extended to a $\left(k : \frac{k-1}{2}\right)$-coloring of G, which is a contradiction.

Therefore we assume that x is a bad end vertex of $T(v,x)$. By definition, let $y = N_{G'}(x) \cap V(T(v,x))$ such that there exists a k-cycle C_x of G containing both xy and xz for some $z \in N_{G'}(x) \backslash \{y\}$. Let $w \in V(C_x) \cap V(T(v,x))$ such that $d_G(x,w)$ as large as possible. By the construction of G', we obtain that the (x,w)-thread in G satisfies $d(x,w) \le \frac{k-1}{2}$, and that there is a deleted thread from w to some vertex, say (w,u)-thread, in the k-cycle C_x such that $d(u,w) \ge d(x,w)$. Thus G contains a bull-necklace $B_w(d(w,x), d(w,v))$, which provides an $F_w(d(w,x), d(w,v))$ in G, contradicting to Claim 2. □

Claim 4. G' contains no $\left(\frac{3k-3}{2}\right)^+$-thread.

Proof of Claim 4. Suppose to the contrary that G' has a $\left(\frac{3k-3}{2}\right)^+$-thread $T(v,x)$. Then $d_G(v,x) = d_{G'}(v,x) \ge \frac{3k-1}{2}$. By Claim 3, v and x are both bad end vertices of $T(v,x)$. Let u be the neighbor of v in $T(v,x)$. Then there exists a k-cycle C_v of G containing both vu and vw for some $w \in N_{G'}(v) \backslash \{u\}$. Let $y \in V(C_v) \cap V(T(v,x))$ such that $d_G(v,y)$ is as large as possible. By the construction of G', we have $d(v,y) \le \frac{k-1}{2}$, and so $d(x,y) = d(v,x) - d(v,y) \ge k$. Now $T(v,x) - (V(T(v,x)) \cap V(C_v) \backslash \{y\})$ is an (x,y)-thread from x to y in G' with y being a good end vertex, which is a contradiction to Claim 3. □

Claim 5. G' contains no (k_1, k_2, k_3)-thread such that $k_1 + k_2 + k_3 \ge \frac{11k-17}{3}$.

Proof of Claim 5. Suppose to the contrary that G' has a (k_1, k_2, k_3)-vertex v such that $k_1 + k_2 + k_3 \ge \frac{11k-17}{3}$ with end vertices x, y, z. Then $d_{G'}(v,x) + d_{G'}(v,y) + d_{G'}(v,z) \ge \frac{11k-8}{3}$.

If there are no two edges incident to v in G' lying in a common k-cycle of G, then we may assume, without loss of generality, that $d_{G'}(v,x) \geq \frac{1}{3}(d_{G'}(v,x) + d_{G'}(v,y) + d_{G'}(v,z)) > k$. Now the (x,v)-thread from x to v has length at least k with v as a good end vertex, which is a contradiction to Claim 3.

If there exist two edges incident to v in G' containing in a k-cycle C_v of G, then we may suppose that C_v has no common vertex other than v with the (v,z)-thread $T(v,z)$. Thus v is a good end vertex of the (v,z)-thread $T(v,z)$, and so $d_{G'}(v,z) \leq k-1$ by Claim 3. Let u be the common vertex of C_v and the (v,x)-thread $T(v,x)$ such that $d_G(v,u)$ as large as possible, and let w be the common vertex of C_v and the (v,y)-thread $T(v,y)$ such that $d_G(v,w)$ as large as possible. By the construction of G', we have $d_{G'}(v,u) + d_{G'}(v,w) \leq \frac{2k}{3}$, since the deleted (u,w)-thread is a longest thread in C_v. Now we have

$$
\begin{aligned}
d_{G'}(x,u) + d_{G'}(y,w) &= d_{G'}(v,x) + d_{G'}(v,y) + d_{G'}(v,z) - (d_{G'}(v,u) + d_{G'}(v,w)) \\
&\quad - d_{G'}(v,z) \\
&\geq \frac{11k-8}{3} - \frac{2k}{3} - (k-1) = 2k - \frac{5}{3}.
\end{aligned}
$$

Thus $\max\{d_{G'}(x,u), d_{G'}(y,w)\} \geq k$, say $d_{G'}(x,u) \geq k$. Hence the (x,u)-thread $T(x,u)$ is of length at least k with u as a good end vertex, a contradiction to Claim 3. □

Now we complete the proof by a discharging method on G'.

Let $F(G')$ be the set of faces of G'. From Euler Formula, we have

$$
\sum_{v \in V(G')} \left(\frac{k-2}{2} d_{G'}(v) - k \right) + \sum_{f \in F(G')} (d_{G'}(f) - k) = -2k. \tag{3}
$$

Assign an initial charge $ch_0(v) = \frac{k-2}{2} d_{G'}(v) - k$ for each $v \in V(G')$, and $ch_0(f) = d_{G'}(f) - k$ for each $f \in F(G')$. Hence the total charge is $-2k$ by Equation (3).

We redistribute the charges according to the following rules.

(R1) Every $\left\lceil \frac{22k}{3} \right\rceil^{+}$-face of G' gives charge $\frac{19}{22}$ to each of its incident vertices.

(R2) Every 3^{+}-vertex of G' gives charge $\frac{3}{22}$ to each of its weakly adjacent 2-vertices.

Let ch denote the charge assignment after performing the charge redistribution using rules (R1) and (R2).

Claim 6. $ch(f) \geq 0$ for $f \in F(G')$.

Proof of Claim 6. Clearly, each k-face f has charge $ch(f) = ch_0(f) = 0$. Each $\left\lceil \frac{22k}{3} \right\rceil^{+}$-face f sends charge $\frac{19}{22}$ to each incident vertices by (R1). So $ch(f) = ch_0(f) - \frac{19}{22} d_{G'}(f) = (d_{G'}(f) - k) - \frac{19}{22} d_{G'}(f) = \frac{3}{22} d_{G'}(f) - k \geq 0$ as $d_{G'}(f) \geq \left\lceil \frac{22k}{3} \right\rceil$. □

Claim 7. $ch(v) \geq 0$ for $v \in V(G')$.

Proof of Claim 7. Let v be a vertex of G'. Then $d_{G'}(v) \geq 2$ by Claim 1 and the construction of G'.

First we assume $d_{G'}(v) = 2$. Then $ch_0(v) = -2$. By Claims 1 and 4, v is weakly adjacent to two 3^+-vertex, and thus v receives charge $\frac{3}{22} \times 2$ by (R2). By (R1), v receives charge $\frac{19}{22} \times 2$ from its two incident faces. Hence $ch(v) = -2 + \frac{3}{22} \times 2 + \frac{19}{22} \times 2 = 0$.

Now we assume $d_{G'}(v) \geq 3$. Let $t(v)$ be the number of 2-vertices weakly adjacent to v. Suppose v is adjacent to $r(v)$ facial k-cycles. Since G' contains no cycles of length from $k+1$ to $\left\lfloor \frac{22k}{3} \right\rfloor$, any two k-cycles of G' have no edges in common, and thus $r(v) \leq \frac{d_{G'}(v)}{2}$. By Claim 4 and by the construction of G', each thread incident to v contains at most $\left(\frac{3k-3}{2} - 1\right)$ 2-vertices and each k-cycle contains no 2-vertices, and so we have $t(v) \leq \frac{3k-5}{2}(d_{G'}(v) - 2r(v))$. By (R1), v receives charge $\frac{19}{22}(d_{G'}(v) - r(v))$ from its incident faces. By (R2), v sends 3/22 to each of its weakly adjacent 2-vertices. Therefore, we have

$$ch(v) = \left(\frac{k-2}{2} d_{G'}(v) - k\right) + \frac{19}{22}(d_{G'}(v) - r(v)) - \frac{3}{22} t(v). \tag{4}$$

Assume that $d_{G'}(v) \geq 4$. By Equation (4), it follows from $t(v) \leq \frac{3k-5}{2}(d_{G'}(v) - 2r(v))$ that

$$\begin{aligned} ch(v) &\geq \frac{k-2}{2} d_{G'}(v) - k + \frac{19}{22}(d_{G'}(v) - r(v)) - \frac{3}{22} \times \frac{3k-5}{2}(d_{G'}(v) - 2r(v)) \\ &= \frac{13k+9}{44} d_{G'}(v) - k + \frac{9k-34}{22} r(v) \\ &\geq \frac{13k+9}{44} d_{G'}(v) - k \\ &\geq \frac{13k+9}{44} \cdot 4 - k \\ &= \frac{2k+9}{11} > 0. \end{aligned}$$

Assume instead that $d_{G'}(v) = 3$. Then $ch_0(v) = \frac{k-6}{2}$ and $r(v) \leq 1$. If $r(v) = 1$, then $t(v) \leq \frac{3k-5}{2}$ by Claim 4. Thus by Equation (4) we have $ch(v) = \frac{k-6}{2} + \frac{19}{22} \times 2 - \frac{3}{22} \times \frac{3k-5}{2} = \frac{13k-41}{44} \geq \frac{6}{11}$. If $r(v) = 0$, then $t(v) \leq \frac{1}{3}(11k - 17)$ by Claim 5. Thus by Equation (4) we have $ch(v) = \frac{k-6}{2} + \frac{19}{22} \times 3 - \frac{3}{22} \times \frac{11k-17}{3} = \frac{4}{11} > 0$. This proves Claim 7.

Combining Equation (3), Claims 6 and 7, we have

$$-2k = \sum_{v \in V(G')} ch_0(v) + \sum_{f \in F(G')} ch_0(f) = \sum_{v \in V(G')} ch(v) + \sum_{f \in F(G')} ch(f) \geq 0,$$

a contradiction. This contradiction finishes the proof of Theorem 1.6. □

5 | CONCLUDING REMARKS

In this paper, we obtain two Steinberg-type results on circular coloring and fractional coloring as Theorems 1.3 and 1.6. Improving the bound to $f(p) \le p(p-2)$ would provide solutions to Conjecture 1.1 for $t = p - 1$ when $p \ge 5$ is a prime, and completely determining the value $f(p)$ seems to be more challenging. Theorem 1.6 confirms the fractional coloring version of Conjecture 1.4 for $p \ge 11$, since $\frac{22p}{3} \le p(p-2)$ when $p \ge 11$. In a follow-up paper [18], we also verify the remaining cases ($p = 5, 7$) of the fractional coloring version of Conjecture 1.4 with refined arguments and additional configurations. Those results provide evidence to Conjectures 1.1 and 1.4.

A nature question is to consider variations of Question 1.2 concerning odd cycles. However, naive odd cycle versions of Theorems 1.3 and 1.6 are false, that is, for any $t > \frac{k-1}{2}$, there exist planar graphs G of odd girth k without odd cycles of length from $k+2$ to $2t+1$ satisfying $\chi_c(G) \ge \chi_f(G) > \frac{2k}{k-1}$. To see this, we construct a graph G by taking $2t$ disjoint copies of k-cycle, where each k-cycle contains two distinguished edges $x_i y_i, y_i z_i$ for each $i \in [2t]$, adding edges $x_i y_{i+1}, z_i y_{i+1}$ for each $i \in [2t-1]$, and adding a new vertex v to connect edges vx_{2t}, vz_{2t}, vy_1. See Figure 3 for the construction of G. We claim that $\chi_f(G) > \frac{2k}{k-1}$. In fact, if φ is a fractional $\left(ka, \frac{k-1}{2}a\right)$-coloring of G, then it is easy to show, by an argument similar to Lemmas 4.1 and 4.2, that $|\varphi(x_i) \cap \varphi(z_i)| = \frac{k-1}{2}a - a$. This implies $\varphi(y_i) = \varphi(y_{i+1})$ for each $i \in [2t-1]$ and $\varphi(y_{2t}) = \varphi(v)$, which indicates $\varphi(y_1) = \varphi(v)$. But there is an edge $y_1 v$ between y_1 and v, a contradiction. Hence $\chi_c(G) \ge \chi_f(G) > \frac{2k}{k-1}$.

Two k-cycles are called *adjacent* if they share at least one common edge. Notice that the above-constructed graph G contains adjacent k-cycles. It would be possible to consider the following modified odd cycle versions without adjacent k-cycles.

Question 5.1. Does there exist a smallest number $g(p)$ for each prime $p \ge 3$ such that every planar graph of odd girth p without adjacent p-cycles and without odd cycles of length from $p+2$ to $g(p)$ is C_p-colorable?

The results from [7,10,27] imply that $g(3) = 7$. It would be interesting to show the existence of $g(p)$ for every prime $p \ge 5$. Furthermore, is it true that $g(p) \le f(p) + 1$?

A similar question arises for fractional coloring.

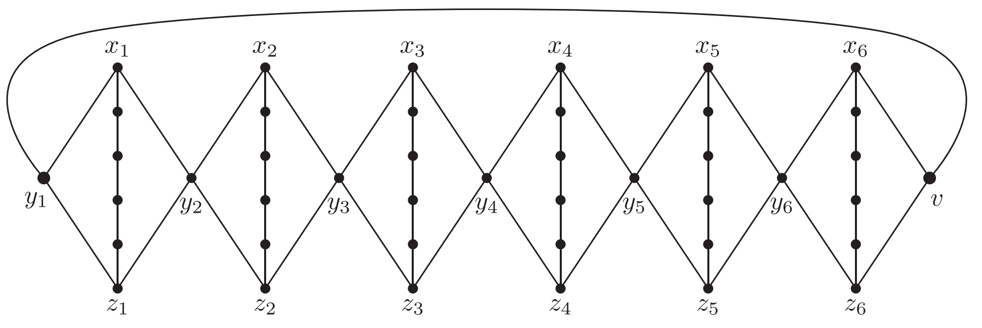


FIGURE 3 Construction of G when $t = 3$ and $k = 7$

Question 5.2. Does there exist a smallest number $h(k)$ for each odd integer $k \geq 3$ such that every planar graph of odd girth k without adjacent k-cycles and without odd cycles of length from $k+2$ to $h(k)$ is fractional $\left(k : \frac{k-1}{2}\right)$-colorable?

From Theorem 1.6, it is plausible that $h(k)$ exists as a linear function of k.

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