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THE FLOW INDEX OF REGULAR CLASS I GRAPHS*

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Abstract. For integers k and d with $k \geq 2d > 0$, a circular k/d-flow of a graph G is an orientation together with a mapping from $E(G)$ to $\{\pm d, \pm(d+1), \ldots, \pm(k-d)\}$ such that, for each vertex of G, the sum of images on outgoing edges is equal to the sum of images on incoming edges. Related to the four color problem, a classical result of Tutte shows that a cubic graph admits a circular $4/1$-flow if and only if it is Class I (i.e., 3-edge-colorable). Tutte's 3-flow conjecture implies that every 5-regular Class I graph admits a nowhere-zero 3-flow (equivalently, a circular $6/2$-flow) as a special case. Steffen in 2015 conjectured that every $(2t+1)$-regular Class I graph admits a circular $(2t+2)/t$-flow. He also proposed a more general conjecture that every $(2t+1)$-odd-edge-connected $(2t+1)$-regular graph admits a circular $(2t+2)/t$-flow for any integer $t \geq 2$, which includes the circular flow conjecture of Jaeger (1981) stating that every $2t$-edge-connected graph admits a circular $(2t+2)/t$-flow for any even $t \geq 2$. Jaeger's conjecture was disproved in 2018 for all even $t \geq 6$, and based on these results, Mattiolo and Steffen recently constructed counterexamples to Steffen's conjecture for Class I graphs when $t = 4k+2$ for any integer $k \geq 1$. In this paper, we extend the above results and construct infinitely many $2t$-edge-connected $(2t+1)$-regular Class I graphs without circular $(2t+2)/t$-flows for any integer $t \in \{6, 8, 10\}$ or $t \geq 12$. Our result provides more general counterexamples to Steffen's two conjectures for both even and odd t and simultaneously generalizes the counterexamples of Jaeger's circular flow conjecture to regular Class I graphs.

Key words. nowhere-zero flow, circular flow, modulo orientation, counterexample

MSC codes. 05C21, 05C15, 05C40

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1. Introduction.

We consider loopless graphs with possible multiple edges in this paper. A graph $G = (V(G), E(G))$ is *k-edge-colorable* if there is a mapping from $E(G)$ to a color-set $\{1, 2, \ldots, k\}$ such that any two adjacent edges receive different colors. The *chromatic index* of G, denoted by $\chi'(G)$, is the minimum integer k such that G is k-edge-colorable. The celebrated Vizing's theorem [21] tells us that $\Delta(G) \leq \chi'(G) \leq \Delta(G)+\mu(G)$ for any graph G, where $\Delta(G)$ denotes the maximum degree of G and $\mu(G)$ denotes the multiplicity of G. A graph G is called *Class I* if $\chi'(G) = \Delta(G)$ and *Class II* otherwise.

Following papers [2, 10, 11], for given integers k and d with $k \geq 2d > 0$, a *circular k/d-flow* of a graph G is an orientation D together with a mapping $f : E(G) \mapsto \{\pm d, \pm(d+1), \ldots, \pm(k-d)\}$ such that, for each vertex of G, the sum of images on outgoing edges is equal to the sum of images on incoming edges, that is,

$$\sum_{e \in \partial_D^+(v)} f(e) - \sum_{e \in \partial_D^-(v)} f(e) = 0 \ \forall v \in V(G).$$

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When $d = 1$, it is called a *nowhere-zero k-flow* introduced by Tutte [20] from the dual of planar map coloring problem. The *flow index* $\phi(G)$ of a graph G is defined as the infimum among all rational numbers $\frac{k}{d}$ such that G admits a circular k/d-flow. It is known from [2] that $\phi(G)$ does exist as a rational number for any bridgeless graph G. Clearly, a graph G satisfies $\phi(G) = 2$ if and only if every vertex of G is of even degree (or equivalently, G has no odd edge-cut). Thus odd edge-cuts are critical for many flow problems, as indicated in [8, 9, 12, 22]. A graph is called $(2t+1)$-*odd-edge-connected* if each of its odd edge-cuts has a size at least $2t + 1$.

1.1. Flow conjectures and known results. Tutte's work [20] showed that a plane graph is k-face-colorable if and only if it admits a nowhere-zero k-flow. Hence, the four color theorem is equivalent to saying that $\phi(G) \leq 4$ for every bridgeless planar graph G, and Grötzsch's theorem [3] can be stated as $\phi(G) \leq 3$ for every 5-odd-edge-connected planar graph G. Some generalizations of those classical results in literature [3, 7, 22] suggest the following conjecture.

CONJECTURE 1.1. *For every $(2t+1)$-odd-edge-connected planar graph G, $\phi(G) \leq 2 + \frac{2}{t}$.*

The Petersen graph P_{10} satisfies $\phi(P_{10}) = 5$ (see [18]), which indicates that the case $t = 1$ in Conjecture 1.1 cannot be extended to nonplanar graphs, but Tutte's 5-flow conjecture asserts that $\phi(G) \leq 5$ for every bridgeless graph G regardless of planarity. Tutte also conjectured that the case $t = 2$ in Conjecture 1.1, which is Grötzsch's theorem [3], may be true for all 5-odd-edge-connected graphs, known as Tutte's 3-flow conjecture. Jaeger [7] further proposed a general circular flow conjecture, where he believed that all cases of even t should be true for nonplanar graphs.

Steffen in [19] made a conjecture suggesting that Conjecture 1.1 may be true for nonplanar graphs whenever $t \geq 2$.

CONJECTURE 1.2 ([19]). *Let $t \geq 2$ be an integer. Then for every $(2t + 1)$-odd-edge-connected graph G, $\phi(G) \leq 2 + \frac{2}{t}$.*

Some breakthrough progresses have been made on those problems, and as a result, Jaeger's circular flow conjecture has been disproved for each even $t \geq 6$.

THEOREM 1.3 ([4]). *For each even $t \geq 6$, there exists a $2t$-edge-connected nonplanar graph G satisfying $\phi(G) > 2 + \frac{2}{t}$.*

THEOREM 1.4 ([10, 11, 15]). *Each of the following statements holds.*
(i) [15] *For every bridgeless graph G, $\phi(G) \leq 6$.*
(ii) [11] *For every $(6p - 1)$-odd-edge-connected graph G, $\phi(G) \leq 2 + \frac{2}{2p-1}$.*
(iii) [10] *For every $(6p + 1)$-odd-edge-connected graph G, $\phi(G) \leq 2 + \frac{1}{p}$.*
(iv) [11] *For every $(6p + 3)$-odd-edge-connected graph G, $\phi(G) < 2 + \frac{1}{p}$.*

By a splitting lemma of Zhang [22] for odd-edge-connectivity, many flow problems, such as Conjectures 1.1 and 1.2, can be reduced to regular graphs. In fact, an equivalent version of Conjecture 1.2 on regular graphs was proposed by Steffen in [17, 19]. Tait in 1880 already proved that the four color theorem is equivalent to the statement that every bridgeless cubic planar graph is Class I. A classical result of Tutte shows that a cubic graph G has $\phi(G) \leq 4$ if and only if it is Class I. Steffen [17, 19] also proposed a conjecture on the generalization of Tutte's classical result and suggested that Class I regular graphs may be easier for flow problems.

CONJECTURE 1.5 ([19]). *For every $(2t+1)$-regular Class* I *graph G, $\phi(G) \leq 2+\frac{2}{t}$.*

Note that every $(2t+1)$-regular Class I graph is $(2t+1)$-odd-edge-connected, and thus Conjecture 1.5 is a special case of Conjecture 1.2, but not vice versa. Those problems are related to a conjecture of Seymour [14] that every $(2t+1)$-odd-edge-connected $(2t+1)$-regular planar graph is Class I. One can observe that if both Seymour's conjecture and Conjecture 1.5 are valid, then Conjecture 1.1 follows. It is known from [5, 16] that Seymour's conjecture and Conjectures 1.1, 1.2, and 1.5 are all confirmed for K_4-minor-free graphs.

Conjectures 1.2 and 1.5 are both posted on the Open Problem Garden and rated as two stars [17]. However, Theorem 1.3 has already provided a negative answer to Conjecture 1.2 whenever $t \geq 6$ is even. Recently, modifying the construction methods in [4] with some new coloring ideas, Mattiolo and Steffen [13] disproved Conjecture 1.5 for some even t.

THEOREM 1.6 ([13]). *Let $t = 4k+2$, where $k \geq 1$ is an integer. Then there exists a $(2t+1)$-regular Class* I *graph G such that $\phi(G) > 2 + \frac{2}{t}$.*

1.2. Main result. Our motivation is to further push forward the above problems and give a more extensive construction than Theorems 1.3 and 1.6, especially for odd integers t.

THEOREM 1.7. *For any integer t with $t \geq 12$ or $t \in \{6, 8, 10\}$, there exists a $2t$-edge-connected $(2t+1)$-regular Class* I *graph G such that $\phi(G) > 2 + \frac{2}{t}$.*

It is worth noting that the construction of Theorem 1.6 in [13] contains some small even edge-cuts, while our new methods overcome this barrier. Hence, our Theorem 1.7 generalizes Theorem 1.3 on counterexamples of Jaeger's circular flow conjecture to regular Class I graphs.

Note that the construction in Theorem 1.6 is for even integers t in the form of $t = 4k+2$, but our new construction works for not only all even $t \geq 6$ but also all odd $t \geq 13$. Thus our Theorem 1.7 provides more general counterexamples to Steffen's Conjectures 1.2 and 1.5 not only for even integers with a wider range but also for large odd integers. As far as we know, this is the first response appearing in literature to the above conjectures with negative answer for odd t.

For even integers t, our constructions are mainly based on the methods developed in [4, 13]; especially we modify the construction strategies in [4] and the edge-coloring ideas in [13] to achieve our purpose. For odd integers t, another novelty is that we develop a new method to construct graphs without circular $(2 + \frac{2}{t})$-flows from some newly developed orientation techniques in [11].

We feel that Conjectures 1.2 and 1.5 and Theorems 1.6 and 1.7 are of interest for both even and odd integers t. Recall that, for the case $t = 1$, the counterpart of Theorem 1.7 is Tutte's classical theorem that a cubic graph G is Class I if and only if $\phi(G) \leq 4$. In the case $t = 2$, the truth of Tutte's 3-flow conjecture would imply that every 5-regular Class I graph G satisfies $\phi(G) \leq 3$. For the case $t = 3$, Conjecture 1.2 states that $\phi(G) \leq \frac{8}{3}$ for any 7-odd-edge-connected graph G, whose truth implies a conjecture of Li et al. [10] that every 6-edge-connected graph G satisfies $\phi(G) < 3$. For each integer $t \in \{2, 3, 4, 5, 7, 9, 11\}$, it remains an interesting open problem that whether the statement of Theorem 1.7 or its opposite direction (on Conjectures 1.2 and 1.5) is true.

2. Preliminary. In this section, we will first introduce some more necessary notation and definitions. We use $d_G(v)$ to represent the degree of a vertex v in a graph G. Denote by kG the k-extended graph of G with $V(kG) = V(G)$ and each edge of $E(G)$ replaced by k multiple edges.

Suppose that A and B are two disjoint subsets of $V(G)$. Denote by $\partial_G(A, B)$ the set of edges of G with one end in A and the other end in B. When $A = B^c = V(G)\backslash B$, $\partial_G(A, B)$ is abbreviated as $\partial_G(A)$.

For an orientation D of G, denote by $\partial_D^+(A)$ and $\partial_D^-(A)$ the set of edges with only tails and only heads in A, respectively. Moreover, we use $\partial_D^+(A,B)$ to denote the set of edges in $\partial_D(A,B)$ with heads in B and tails in A and denote the set $\partial_D(A,B) \setminus \partial_D^+(A,B)$ by $\partial_D^-(A,B)$. When the set contains exactly one vertex, say $A = \{x\}$, we omit the brace in the above notation. Especially, $\partial_G(x,y)$ denotes the set of edges between x and y. In addition, denote by $d_D^+(v) = |\partial_D^+(v)|$ the outdegree of v in D and $d_D^-(v) = |\partial_D^-(v)|$ the indegree of v in D. If $d_D^+(v) - d_D^-(v) \equiv 0 \pmod{k}$ for every vertex v of G, we call D a *modulo k-orientation.*

The following two lemmas are vital for relating orientations to circular flows, which will be frequently used through our proofs.

LEMMA 2.1 ([7]). *The flow index $\phi(G)$ of a graph G satisfies $\phi(G) \leq 2 + \frac{1}{p}$ if and only if G admits a modulo $(2p+1)$-orientation.*

LEMMA 2.2 ([11]). *A graph G admits a circular $(2 + \frac{2}{2p-1})$-flow if and only if $2G$ admits an orientation D such that*

$$d_D^+(v) - d_D^-(v) \equiv 4pd_G(v) \pmod{8p} \ \forall v \in V(G). \tag{2.1}$$

LEMMA 2.3. *Let G be a graph with a cycle C. If $G - E(C)$ is bridgeless, then $\phi(G - E(C)) \geq \phi(G)$.*

Proof. Let $G' = G - E(C)$, and let $\phi(G') = \frac{k}{d}$. Suppose that G' has a circular k/d-flow (D', f'). We obtain an orientation D of G from D' by orienting the edges in $E(C)$ clockwise. Let $f(e) = f'(e)$ for each $e \in E(G) - E(C)$, and let $f(e) = d$ for each $e \in E(C)$. Then (D, f) is clearly a circular k/d-flow of G. Thus we have $\phi(G - E(C)) = \frac{k}{d} \geq \phi(G)$. ∎

Motivated from some ideas in [4] (see Definition 3.6 below), we define a new 2-sum operation of two graphs for handling circular $(2 + \frac{2}{2p-1})$-flows.

DEFINITION 2.4. *Let G_1 and G_2 be two graphs with $x_1, y_1 \in V(G_1)$, $|\partial_{G_1}(x_1,y_1)| \geq 2p-2$, and $x_2y_2 \in E(G_2)$. Define $G = G_1(x_1y_1) \oplus_2^1 G_2(x_2y_2)$, the 2-sum of G_1, G_2 on x_1y_1 and x_2y_2, to be the graph obtained by deleting one edge between x_2, y_2 and $2p-2$ parallel edges between x_1, y_1 and then identifying x_1, x_2 as a new vertex x and y_1, y_2 as a new vertex y.*

When the vertices $x_1, y_1 \in V(G_1)$ and $x_2, y_2 \in V(G_2)$ are clear from context, we usually write $G = G_1 \oplus_2^1 G_2$ below for convenience.

LEMMA 2.5. *Let G_1 and G_2 be two graphs without circular $(2 + \frac{2}{2p-1})$-flows. Assume that x_1, $y_1 \in V(G_1)$, $|\partial_{G_1}(x_1,y_1)| \geq 2p-2$, and $x_2y_2 \in E(G_2)$. Then $G = G_1 \oplus_2^1 G_2$ admits no circular $(2 + \frac{2}{2p-1})$-flow.*

Proof. Suppose to the contrary that $G = G_1(x_1y_1) \oplus_2^1 G_2(x_2y_2)$ admits a circular $(2 + \frac{2}{2p-1})$-flow. By Lemma 2.2, $2G$ has an orientation D such that $d_D^+(v) - d_D^-(v) \equiv 4pd_G(v) \pmod{8p}$ for each $v \in V(G)$. Let F be a set of $2p-2$ edges in $\partial_{G_1}(x_1,y_1)$, and let $2F$ be the set of corresponding $4p-4$ parallel edges in $\partial_{2G_1}(x_1,y_1)$. Denote by D_1 the restriction of D on $E(2(G_1 \setminus F))$, and denote by D_2 the restriction of D on $E(2(G_2 \setminus \{x_2y_2\}))$.

We denote

$$d_{D_1}^+(x_1) - d_{D_1}^-(x_1) \equiv 4pd_{G_1}(x_1) + t_1 \pmod{8p}, \tag{2.2}$$

$$d_{D_2}^+(x_2) - d_{D_2}^-(x_2) \equiv 4pd_{G_2}(x_2) + t_2 \pmod{8p}, \tag{2.3}$$

where integers t_1 and t_2 satisfy that $t_1, t_2 \in \{0, \pm 2, \pm 4, \ldots, \pm(4p-2), 4p\}$.

We first claim that

$$t_1 \in \{\pm(4p-2), 4p\} \text{ and } t_2 \in \{\pm 4, \pm 6, \ldots, \pm(4p-2), 4p\}. \tag{2.4}$$

Note that t_1 and t_2 are even integers, since $d^+_{D_1}(x_i) - d^-_{D_1}(x_i)$ is even for $i \in \{1,2\}$. By contradiction, we suppose that $t_1 \in \{0, \pm 2, \pm 4, \ldots, \pm(4p-4)\}$. Let D'_1 be the orientation obtained by keeping the orientation of D_1 and orienting the edges in $2F$ with $2p-2-\frac{t_1}{2}$ arcs away from x_1 and $2p-2+\frac{t_1}{2}$ arcs into x_1. Then we have

$$d^+_{D'_1}(x_1) - d^-_{D'_1}(x_1) \equiv d^+_{D_1}(x_1) - d^-_{D_1}(x_1) - t_1 \equiv 4pd_{G_1}(x_1) \pmod{8p}.$$

As $d^+_{D'_1}(v) - d^-_{D'_1}(v) \equiv 4pd_{G_1}(v) \pmod{8p}$ for every $v \in V(G_1) \setminus \{y_1\}$, we also have

$$\begin{aligned} d^+_{D'_1}(y_1) - d^-_{D'_1}(y_1) &\equiv \sum_{v \in V(G_1)\setminus\{y_1\}} d^-_{D'_1}(v) - \sum_{v \in V(G_1)\setminus\{y_1\}} d^+_{D'_1}(v) \\ &\equiv -\sum_{v \in V(G_1)\setminus\{y_1\}} 4pd_{G_1}(v) \\ &\equiv -4p(2|E(G_1)| - d_{G_1}(y_1)) \\ &\equiv 4pd_{G_1}(y_1) \pmod{8p}. \end{aligned}$$

Hence G_1 admits a circular $(2+\frac{2}{2p-1})$-flow by Lemma 2.2, a contradiction. So we conclude that $t_1 \in \{\pm(4p-2), 4p\}$.

With a similar argument, if $t_2 \in \{0, \pm 2\}$, then we obtain an orientation D'_2 of $2G_2$ by keeping the orientation of D_2 and orienting the remaining two parallel edges in $\partial_{2G_2}(x_2, y_2)$ with $1 - \frac{t_2}{2}$ arcs from x_2 to y_2 and $1 + \frac{t_2}{2}$ arcs from y_2 to x_2, which implies that D'_2 satisfies $d^+_{D'_2}(v) - d^-_{D'_2}(v) \equiv 4pd_{G_2}(v) \pmod{8p}$ for each $v \in V(G_2)$, resulting in a contradiction to Lemma 2.2. This proves (2.4).

Next, we see from (2.4) that

$$t_1 + t_2 \not\equiv 4p \pmod{8p}.$$

Adding the left and the right of formulas (2.2) and (2.3), respectively, we can get that

$$d^+_{D_1}(x_1) - d^-_{D_1}(x_1) + d^+_{D_2}(x_2) - d^-_{D_2}(x_2) \equiv 4p(d_{G_1}(x_1) + d_{G_2}(x_2)) + t_1 + t_2 \pmod{8p}.$$

Since $d_G(x) = d_{G_1}(x_1) - 2p - 2 + d_{G_2}(x_2) - 1$, we obtain that

$$\begin{aligned} d^+_D(x) - d^-_D(x) &= d^+_{D_1}(x_1) - d^-_{D_1}(x_1) + d^+_{D_2}(x_2) - d^-_{D_2}(x_2) \\ &\equiv 4p(d_{G_1}(x_1) + d_{G_2}(x_2)) + t_1 + t_2 \\ &\equiv 4pd_G(x) + 4p(2p+3) + (t_1 + t_2) \\ &\equiv 4pd_G(x) + 4p + (t_1 + t_2) \\ &\not\equiv 4pd_G(x) \pmod{8p}. \end{aligned}$$

This is a contradiction to Lemma 2.2. Hence G admits no circular $(2+\frac{2}{2p-1})$-flow. ∎

A k-cycle is a cycle on k vertices. Let pC_k be a k-cycle with vertices, $w_1, w_2, \ldots, w_k$, and each edge of which is replaced by p parallel edges. Let $W_{[k]} = K_1 \vee kC_{2k+3}$, where the operation "$\vee$" means connecting each vertex of kC_{2k+3} with a single edge to a K_1 with a single vertex z.

LEMMA 2.6 (see [4]). *For any integer $p \geq 1$, $W_{[2p-1]}$ has no circular $(2+\frac{1}{p})$-flow.*

Proof. The proof of this lemma has appeared in [4], and we present the argument here for the reader, which may be helpful in understanding the proof of the next Lemma 2.7.

Suppose to the contrary that $W_{[2p-1]}$ admits a circular $(2+\frac{1}{p})$-flow. According to Lemma 2.1, there is a modulo $(2p+1)$-orientation D of $W_{[2p-1]}$, i.e., $d_D^+(v)-d_D^-(v)\equiv 0 \pmod{2p+1}$ for each vertex $v \in V(W_{[2p-1]})$. Since every vertex v of $V(W_{[2p-1]})$ is of odd degree, we have $d_D^+(v)-d_D^-(v) \in \{\pm(2p+1)\}$. Let $V^+ = \{v \in V(W_{[2p-1]}) \mid d_D^+(v)-d_D^-(v) = 2p+1\}$ and $V^- = \{v \in V(W_{[2p-1]}) \mid d_D^+(v)-d_D^-(v) = -(2p+1)\}$. As there are odd number of vertices in the cycle C_{4p+1}, there must be two adjacent vertices w_j, w_{j+1} such that they are both in V^+ or both in V^-, i.e., $d_D^+(w_j) - d_D^-(w_j) = d_D^+(w_{j+1}) - d_D^-(w_{j+1}) \in \{\pm(2p+1)\}$. Then we have

$$4p = |\partial_G(\{w_j, w_{j+1}\})| \geq ||\partial_D^+(\{w_j, w_{j+1}\})| - |\partial_D^-(\{w_j, w_{j+1}\})|| = 4p+2,$$

a contradiction. Therefore $W_{[2p-1]}$ admits no circular $(2+\frac{1}{p})$-flow for any integer $p \geq 1$. ◻

LEMMA 2.7. *For any integer $p \geq 2$, $W_{[2p-2]}$ has no circular $(2+\frac{2}{2p-1})$-flow.*

Proof. The proof is similar to the approach of proving Lemma 2.6 but uses a different lemma. By contradiction, suppose that $W_{[2p-2]}$ has a circular $(2+\frac{2}{2p-1})$-flow. By Lemma 2.2, there is an orientation D of $2W_{[2p-2]}$ satisfying

$$\begin{aligned} d_D^+(z) - d_D^-(z) &\equiv 4p(4p-1) \pmod{8p} \text{ and} \\ d_D^+(w_i) - d_D^-(w_i) &\equiv 4p(4p-3) \pmod{8p} \text{ for each } 1 \leq i \leq 4p-1. \end{aligned}$$

Thus $d_D^+(v) - d_D^-(v) \in \{\pm 4p\}$ for each $v \in V(W_{[2p-2]})$.

Since $|V(C_{4p-1})|$ is odd, there are two adjacent vertices w_i, w_{i+1} such that $d_D^+(w_i)-d_D^-(w_i) = d_D^+(w_{i+1}) - d_D^-(w_{i+1}) \in \{\pm 4p\}$. Furthermore, we have

$$|\partial_D(\{w_i, w_{i+1}\})| = 8p-4 < 8p = ||\partial_D^+(\{w_i, w_{i+1}\})| - |\partial_D^-(\{w_i, w_{i+1}\})||,$$

which leads to a contradiction. So $W_{[2p-2]}$ has no circular $(2+\frac{2}{2p-1})$-flow. ◻

One of the referees kindly suggested to us that most of the lemmas in this section could also be proved by using the balanced valuations theorem of Bondy [1] and Jaeger [6], in which some arguments might become even shorter. Those arguments are very similar to the proofs above, and readers interested in this approach may take it as an exercise.

3. Proof of Theorem 1.7.

In this section, we prove our main result according to the parity of t. We will first construct some graphs and then verify their desired properties accordingly. The methods and constructions in this part are mainly motivated from [4, 11, 13].

3.1. When t is odd.

Let $t = 2p-1$ with $p \geq 7$. In this subsection, we will construct a $(4p-2)$-edge-connected $(4p+1)$-regular Class I graph J with $\phi(J) > 2+\frac{2}{2p-1}$.

Let $G_0 = K_{4p-2}$ be a complete graph with vertices $v_1, v_2, \ldots, v_{4p-2}$. We will modify the graph G_0 and apply the 2-sum operations to construct a $(4p-1)$-regular graph J first. Later on, we shall present several lemmas to show that J admits no circular $(2+\frac{2}{2p-1})$-flow, J is Class I (or equivalently, $(4p-1)$-edge-colorable), and J is $(4p-2)$-edge-connected, respectively.

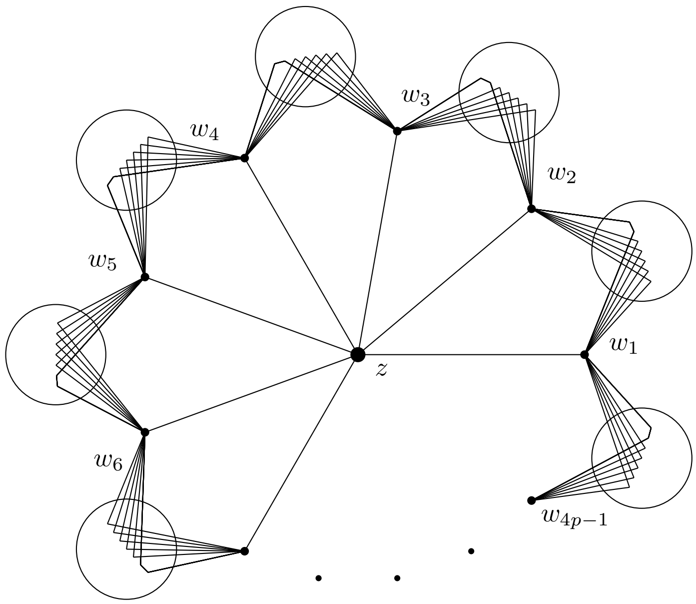

FIG. 1. *The graph J.*

Denote $p = 3r + s$, where r is a nonnegative integer and $s \in \{1, 2, 3\}$. Notice that r and s are unique for fixed p. Define a multiset A of edges with

$$A = \begin{cases} \emptyset & \text{for } s = 1, \\ \{v_{4p-3}v_{4p-2}, v_{4p-3}v_{4p-2}\} & \text{for } s = 2, \\ \{v_{4p-5}v_{4p-4}, v_{4p-4}v_{4p-3}, v_{4p-3}v_{4p-2}, v_{4p-2}v_{4p-5}\} & \text{for } s = 3. \end{cases}$$

Now we are ready to construct the graph J by the following steps.

(1) The graph G_1 is derived from G_0 by adding two new vertices x_1, x_2, two parallel edges x_1x_2, and edge-set $\{x_1v_1, x_2v_2, v_1v_2\} \bigcup \{x_iv_j | i \in \{1, 2\}, j \in \{3, 4, \ldots, 2p\}\}$.

(2) Let G_2 be the graph obtained from G_1 by adding edges of $2r$ vertex-disjoint triangles $v_{2p+3i-2}v_{2p+3i-1}v_{2p+3i}$, where $1 \leq i \leq 2r$.

(3) Denote by Q the graph derived from G_2 by adding all edges of the multiset A.

(4) Let $Q_i (1 \leq i \leq 4p-1)$ be copies of Q, where the corresponding vertex of v is written as v^i. For each $i \in \{1, 2, \ldots, 4p-1\}$, apply the 2-sum operation defined in Definition 2.4 on w_iw_{i+1} of $W_{[2p-2]}$ and $x_1^i x_2^i$ of Q_i, where $w_{4p} = w_1$, and then delete the edges of cycle $w_1w_2 \cdots w_{4p-1}w_1$. The final graph is J (see Figure 1).

THEOREM 3.1. *The graph J is a $(4p-1)$-regular, $(4p-2)$-edge-connected, Class* I *graph without circular $(2 + \frac{2}{2p-1})$-flows for any integer $p \geq 7$.*

It is straightforward to check that J is $(4p-1)$-regular. So the proof of Theorem 3.1 follows from the lemmas below. We shall prove the facts that J admits no circular $(2 + \frac{2}{2p-1})$-flow and J is Class I, respectively.

LEMMA 3.2. *The graph Q has no circular $(2 + \frac{2}{2p-1})$-flows for any integer $p \geq 7$.*

Proof. Suppose to the contrary that Q admits a circular $(2+\frac{2}{2p-1})$-flow. By Lemma 2.2, there is an orientation D of $2Q$ such that

$$d_D^+(v_i) - d_D^-(v_i) \equiv 4p(4p-1) \pmod{8p} \ \forall 1 \le i \le 4p-2, \text{ and}$$
$$d_D^+(x_j) - d_D^-(x_j) \equiv 4p(2p+1) \pmod{8p} \ \text{ for any } j \in \{1,2\}.$$

Thus $d_D^+(v) - d_D^-(v) \in \{\pm 4p\}$ for any $v \in V(Q)$.

Let

$$V^+ = \{v \in V(Q) \mid d_D^+(v) - d_D^-(v) = 4p\} \text{ and}$$
$$V^- = \{v \in V(Q) \mid d_D^+(v) - d_D^-(v) = -4p\}.$$

Clearly, $|V^+| = |V^-| = 2p$. Moreover, x_1, x_2 are not in the same part of V^+, V^-. Otherwise, we must have

$$8p-4 = |\partial_D(\{x_1,x_2\})| \ge |(d_D^+(x_1) - d_D^-(x_1)) + (d_D^+(x_2) - d_D^-(x_2))| = 8p,$$

a contradiction.

For each $i \in \{3,4,\ldots,2p\}$, no matter which part v_i is in, there is exactly one edge of $\{v_ix_1, v_ix_2\}$ in $\partial_Q(V^+,V^-)$. The path $x_1v_1v_2x_2$ provides at most 3 edges to $\partial_Q(V^+,V^-)$. For any triangle added inside the complete graph, there are at most 2 edges in $\partial_Q(V^+,V^-)$. Recall that $r = \frac{p-s}{3}$, where $s \in \{1,2,3\}$. Hence, considering all the $2r$ triangles, there are at most $4r$ edges between V^+ and V^- in Q. In addition, a 4-cycle contributes at most 4 edges to $\partial_Q(V^+,V^-)$. As D is an orientation of $2Q$, we obtain the following inequalities.

When $s=1$, we have

$$|\partial_D(V^+,V^-)| \le 2[(2p-1)^2 + 2 + (2p-2) + 3 + 4r] = 8p^2 - \frac{4p}{3} + \frac{16}{3}.$$

When $s=2$, we have

$$|\partial_D(V^+,V^-)| \le 2[(2p-1)^2 + 2 + (2p-2) + 3 + 4r + 2] = 8p^2 - \frac{4p}{3} + \frac{20}{3}.$$

When $s=3$, we have

$$|\partial_D(V^+,V^-)| \le 2[(2p-1)^2 + 2 + (2p-2) + 3 + 4r + 4] = 8p^2 - \frac{4p}{3} + 8.$$

But, in each case, the above inequalities provide $|\partial_D(V^+,V^-)| < 8p^2 = 4p|V^+| = |\partial_D^+(V^+,V^-)|$ when $p \ge 7$. This is a contradiction. So Q admits no circular $(2+\frac{2}{2p-1})$-flow. □

LEMMA 3.3. *The graph J admits no circular $(2+\frac{2}{2p-1})$-flow for any integer $p \ge 7$.*

Proof. Let $J_0' = W_{[2p-2]}$ and $J_i' = Q_i(x_1^i, x_2^i) \oplus_2^1 J_{i-1}'(w_iw_{i+1})$ for $i \in \{1,\ldots,4p-1\}$, where $w_{4p} = w_1$. By applying Lemma 2.5 repeatedly, J_i' has no circular $(2+\frac{2}{2p-1})$-flow for any i. It follows from Lemma 2.3 that the graph J, derived from J_{4p-1}' by deleting the edges of cycle $w_1w_2\cdots w_{4p-1}$, admits no circular $(2+\frac{2}{2p-1})$-flow. □

Given an edge-coloring of a graph, we say that a vertex v sees a color α if v is incident with at least one edge of color α, and a vertex v sees a color-set S if S is composed of all colors that v sees.

Denote by H the graph obtained by deleting the two parallel edges x_1x_2 from Q. We may first give a suitable edge-coloring of H and then apply this fact to verify that the graph J is Class I. More specifically, we would precolor some edges of H and then color the rest edges of H with certain restricted properties. Here, our edge-coloring methods are mainly motivated from the ideas of Mattiolo and Steffen [13] with certain modifications.

LEMMA 3.4. *The graph H is $(4p-1)$-edge-colorable, and there is $(4p-1)$-edge-coloring of H such that x_1 and x_2 see the same color-set.*

Proof. We follow the notation from the construction of Q except the labels of vertices in $V(Q) \setminus \{x_1, x_2\}$. For convenience, reformulate the labels of these vertices as $v_0, v_1, v_2, \dots, v_{4p-4}, v_\infty$ such that each of the following statements holds:

- The labels are taken modulo $4p-3$, that is, we set $v_i = v_j$ if $i \equiv j \pmod{4p-3}$. Besides, the vertex v_∞ has a unique distinguished label.
- The set of vertices adjacent to x_1 or x_2 is $\{v_j : j \in \{0, \pm1, \pm2, \dots, \pm(p-1), \infty\}$, and a parallel edge is added between v_0 and v_1.
- The $2r$ triangles are added as $v_{p+j}v_{p+1+j}v_{p+2+j}$ and $v_{-(p+j)}v_{-(p+1+j)}v_{-(p+2+j)}$ for each $j \in \{0, 3, 6, \dots, 3(r-1)\}$.
- When $s = 2$, $A = \{v_{2p-2}v_{-(2p-2)}, v_{2p-2}v_{-(2p-2)}\}$, and when $s = 3$, $A = \{v_{2p-2}v_{2p-3}, v_{2p-3}v_{-(2p-2)}, v_{-(2p-2)}v_{-(2p-3)}, v_{-(2p-3)}v_{2p-2}\}$.

Denote by T the set of edges of the $2r$ added triangles, and denote

$$X = \{v_0x_1, v_1x_2\} \cup \{v_jx_i \mid i \in \{1, 2\}, j \in \{-1, \infty, \pm2, \pm3, \dots, \pm(p-1)\}\}.$$

Let $\{0, 1, \dots, 4p-2\}$ be the colors that we need. When $s \in \{2, 3\}$ (i.e., $A \neq \emptyset$), we first properly color the edges of A by colors $4p-3$ and $4p-2$. Then we give the way of coloring for the other edges as follows.

Step 1. Color the edges of $M_j = \{v_jv_\infty\} \cup \{v_{-i+j}v_{i+j} \mid i \in \mathbb{Z}_{4p-3} \setminus \{0\}\}$ by j, where $0 \leq j \leq 4p-4$. Denote $\varphi(e)$ the color of e for each $e \in \cup_{j=0}^{j=4p-4} M_j$.

Step 2. Consider the even cycle C': $v_0v_1v_2 \cdots v_{p-1}v_\infty v_{-(p-1)} \cdots v_{-2}v_{-1}v_0$. Let $c_0 = v_0, c_1 = v_1, \dots, c_{p-1} = v_{p-1}, c_p = v_\infty, c_{p+1} = v_{-(p-1)}, \dots$, and $c_{2p-1} = v_{-1}$. Notice that each edge of C' has been assigned with a different color in Step 1, including $2p$ colors of a color-set K with $K = \{2p-1, 2p, \dots, 3p-3, p-1, 3p-2, p, \dots, 2p-4, 2p-3, 2p-2\}$. Now recolor $E(C')$ by colors $4p-3, 4p-2$ alternately. Then we use the colors in K to color edge-set X by $\theta : X \mapsto K$ as follows:

$$\theta(x_1c_i) = \varphi(c_{i-1}c_i) \text{ for } i \in \mathbb{Z}_{2p} \setminus \{1\} \text{ and} \tag{3.1}$$

$$\theta(x_2c_j) = \varphi(c_jc_{j+1}) \text{ for } j \in \mathbb{Z}_{2p} \setminus \{0\}. \tag{3.2}$$

Notice that the color $\varphi(c_0c_1)$ is not used for any edge of X. So we can use it to properly color the added parallel edge v_0v_1.

Step 3. For edges in T, consider the 6-cycle

$$C_j'' : v_{p+j}v_{-(p+1+j)}v_{p+2+j}v_{-(p+2+j)}v_{p+1+j}v_{-(p+j)}v_{p+j},$$

where $j \in \{0, 3, 6, \dots, 3(r-1)\}$. The edges of C_j'' were colored with $0, 2p-1, 2p-2$ in Step 1. Now we recolor $E(C_j'')$ with colors $4p-3, 4p-2$ alternately. Then the original colors $0, 2p-1, 2p-2$ can be assigned to the corresponding triangles $v_{p+j}v_{p+1+j}v_{p+2+j}v_{p+j}$ and $v_{-(p+j)}v_{-(p+1+j)}v_{-(p+2+j)}v_{-(p+j)}$ for each $j \in \{0, 3, 6, \dots, 3(r-1)\}$. The process is shown in Figure 2.

Note that the edges in A have been properly colored by colors $4p-3$ and $4p-2$ before Step 1. Thus throughout the steps of the construction, we use $4p-1$ colors

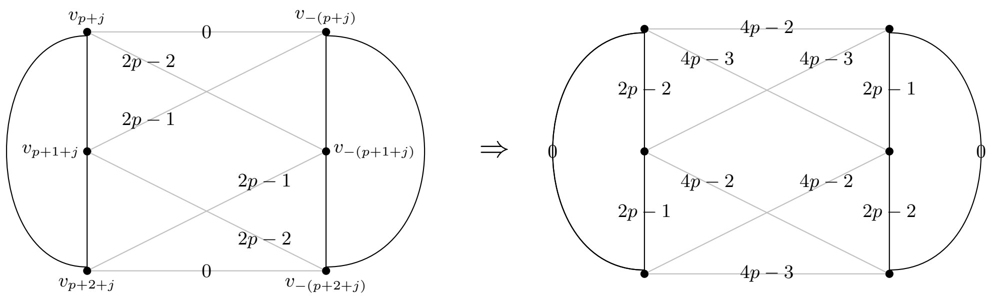

FIG. 2. *The coloring of Step 3 in Lemma 3.4.*

in total, and the edge-coloring always remains proper. Moreover, in Step 2, we have guaranteed that x_1 and x_2 see the identical color-set. In conclusion, the lemma holds. ∎

Notice that there are lots of permutations for the color classes in edge-colorings. So by Lemma 3.4 and by permutating colors, we can assign the colors of the edges incident with x_1^i and x_2^i such that w_i in J receives different colors for distinct i. In the process, let us color the edges incident to x_j^i with color-set $i+\{2,4,\dots,4p-2\}$, where $j \in \{1,2\}$ and $i \in \mathbb{Z}_{4p-1}$. Then we can check that w_i receives $4p-2$ different colors, i.e., colors $i+\{2,4,\dots,4p-2\}$ from x_1^i and colors $i+\{1,3,5,\dots,4p-3\}$ from x_2^{i-1}, where colors are taken modulo $4p-1$. Finally, we color zw_i by i for $1 \le i \le 4p-1$. Consequently, this provides a proper edge-coloring of J using $4p-1$ colors, and we obtain the following conclusion.

LEMMA 3.5. *The graph J is $(4p-1)$-edge-colorable.*

Combining Lemmas 3.3 and 3.5, we need only to prove that J is $(4p-2)$-edge-connected for Theorem 3.1. Assume $S \subseteq V(J)$ and $\partial(S)$ is an edge-cut with size less than $4p-2$ in J. Since J is $(4p-1)$-regular, $\partial(S)$ is a nontrivial edge-cut of J. For any vertex-set $A_i \subset V(K_{4p}^i) \subset V(J)$, we have $\partial_J(A_i) \ge \min\{4p-1, 8p-8\}$. Thus $\partial(S)$ cannot separate the vertices of K_{4p}^i for any $1 \le i \le 4p-1$. It is similar for the vertex-subset of $\{z, w_1, w_2, \dots, w_{4p-1}\}$. Therefore, the small edge-cut $\partial(S)$ only exists between $W_{[2p-2]}$ and $Q_i \setminus \{x_1^i, x_2^i\}$, where there are exactly $4p-2$ edges. Hence, J is $(4p-2)$-edge-connected, and Theorem 3.1 follows.

3.2. When t is even. For the even case, the main ideas follow from [4, 13], and the proof is similar to the case that t is odd. So our proof will be brief.

DEFINITION 3.6 ([4]). *Let H_1 and H_2 be two graphs with $u_1, v_1 \in V(H_1)$, $|\partial_{H_1}(u_1,v_1)| \ge 2p-1$, and $u_2, v_2 \in V(H_2)$. Define $H = H_1 \oplus_2^2 H_2$, the 2-sum of H_1 and H_2, to be the graph obtained from H_1 and H_2 by deleting $2p-1$ parallel edges between u_1 and v_1 in H_1 and then identifying u_1 and u_2 as a new vertex u and identifying v_1 and v_2 as a new vertex v.*

LEMMA 3.7 ([4]). *Let $H = H_1 \oplus_2^2 H_2$ be a 2-sum of H_1 and H_2 defined in Definition 3.6. If neither H_1 nor H_2 admits a modulo $(2p+1)$-orientation, then $H = H_1 \oplus_2^2 H_2$ admits no modulo $(2p+1)$-orientation.*

Denote $p = 3r+s$, where r is a nonnegative integer and $s \in \{0,1,2\}$. Define B as a mulitiset of edges with

$$B = \begin{cases} \emptyset & \text{for } s=0, \\ \{v_{4p-1}v_{4p}, v_{4p-1}v_{4p}\} & \text{for } s=1, \\ \{v_{4p}v_{4p-3}, v_{4p-3}v_{4p-2}, v_{4p-2}v_{4p-1}, v_{4p-1}v_{4p}\} & \text{for } s=2. \end{cases}$$

Recalling the definition of $W_{[2p-1]}$ stated in the above Lemma 2.6, we use the same notation here. Let G be a complete graph with $4p$ vertices: $v_1, v_2, \ldots, v_{4p-1}, v_{4p}$.

(1) The graph G_1 is constructed from G by adding two new vertices x_1, x_2 and edges of $\{x_1x_2\} \bigcup \{x_iv_j \mid i \in \{1,2\}, j \in \{1,2,\ldots,2p\}\}$.

(2) Let G_2 be the graph derived from G_1 by adding edges of $2r$ disjoint triangles $v_{2p+3i-2}v_{2p+3i-1}v_{2p+3i}$, $1 \le i \le 2r$.

(3) The graph obtained from G_2 by adding all edges from B is denoted by M'.

(4) Let $M'_i (1 \le i \le 4p+1)$ be the $4p+1$ copies of M'. Denote the vertex v in the ith copy of M' by v^i. For each $i \in \mathbb{Z}_{4p+1}$, apply the 2-sum operation defined in Definition 3.6 on w_iw_{i+1} of $W_{[2p-1]}$ and $x_1^ix_2^i$ of M'_i. Then delete the edges of cycle $w_1w_2\cdots w_{4p+1}w_1$, and the obtained graph is denoted by M.

Our target in this subsection is to prove the even case of Theorem 1.7 as follows, which generalizes Theorem 1.3 on the counterexamples of Jaeger's circular flow conjecture [4] to regular Class I graphs and also extends Theorem 1.6 of Mattiolo and Steffen [13] to all even integers $t = 2p \ge 6$.

THEOREM 3.8. *For any integer $p \ge 3$, the graph M is a $(4p+1)$-regular, $4p$-edge-connected, Class* I *graph without circular $(2+\frac{1}{p})$-flows.*

It is easy to check that M is $(4p+1)$-regular. The proof that M is $4p$-edge-connected is similar to the proof of J aforementioned and thus omitted. So we will only prove that M admits no circular $(2+\frac{1}{p})$-flow and show how to color $E(M)$ with $4p+1$ colors briefly.

LEMMA 3.9. *The graph M admits no circular $(2+\frac{1}{p})$-flow for any integer $p \ge 3$.*

By Lemmas 3.7 and 2.3, Lemma 3.9 follows from the fact that M' has no circular $(2+\frac{1}{p})$-flow. To this end, by Lemma 2.1 we just need to prove that M' has no modulo $(2p+1)$-orientation as follows.

LEMMA 3.10. *The graph M' has no modulo $(2p+1)$-orientation for any integer $p \ge 3$.*

Proof. Notice that $d_{M'}(x_1) = d_{M'}(x_2) = 2p+1$ and $d_{M'}(v_i) = 4p+1$ for each $i \in \{1,2,\ldots,4p\}$. Suppose to the contrary that there is a modulo $(2p+1)$-orientation D of M'. For each vertex $v \in V(M')$, since the degree of v is odd, we have $d_D^+(v) - d_D^-(v) \in \{\pm(2p+1)\}$.

Let $V^+ = \{v \in M' \mid d_D^+(v) - d_D^-(v) = 2p+1\}$ and $V^- = \{v \in M' \mid d_D^+(v) - d_D^-(v) = -(2p+1)\}$. Clearly, $|V^+| = |V^-| = 2p+1$. Furthermore, x_1, x_2 are not in the same part of V^+, V^-. Otherwise, there is no appropriate orientation for the edge x_1x_2.

Recall that $p = 3r+s$, where $s \in \{0,1,2\}$. For any $i \in \{1,2,\ldots,2p\}$, no matter which part v_i is in, there is exactly one edge incident to v_i in $\partial_D(V^+,V^-)$. For each triangle added inside the complete graph, there are at most 2 edges in $\partial_D(V^+,V^-)$. Therefore we obtain the following inequalities.

When $s=0$, we have

$$|\partial_D(V^+,V^-)| \le (2p)^2 + 2p + 1 + 4r = 4p^2 + \frac{10p}{3} + 1.$$

When $s=1$, we have

$$|\partial_D(V^+,V^-)| \le (2p)^2 + 2p + 1 + 4r + 2 = 4p^2 + \frac{10p}{3} + \frac{5}{3}.$$

When $s=2$, we have

$$|\partial_D(V^+,V^-)| \le (2p)^2 + 2p + 1 + 4r + 4 = 4p^2 + \frac{10p}{3} + \frac{7}{3}.$$

In any case, this derives a contradiction from $|\partial_D(V^+, V^-)| < (2p+1)^2 \leq |\partial_D^+(V^+, V^-)|$ when $p \geq 3$. Therefore M' admits no modulo $(2p+1)$-orientations as desired. ∎

Similar to the last subsection, we give a proper edge-coloring with $4p+1$ colors of $H = M' \setminus \{x_1x_2\}$ first.

LEMMA 3.11. *There is a proper edge-coloring of H which uses $4p+1$ colors such that x_1 and x_2 see the same color-set.*

Proof. Let $\{0, 1, 2, \ldots, 4p\}$ be the colors that we need. Except x_1, x_2, we label the vertices of H from $\{v_0, v_1, \ldots, v_{4p-2}, v_\infty\}$ as follows:

- Except for the unique distinguished vertex v_∞, the indices are taken modulo $4p-1$, that is, we define $v_i = v_j$ if $i \equiv j \pmod{4p-1}$.
- The set of vertices adjacent to x_1 or x_2 is $\{v_j \mid j \in \{0, \pm1, \pm2, \ldots, \pm(p-1), \infty\}\}$, and we denote X as the edge-set $\{v_jx_i | i \in \{1,2\}, j \in \{0, \infty, \pm1, \pm2, \ldots, \pm(p-1)\}\}$.
- The $2r$ triangles are added as $v_{p+j}v_{p+1+j}v_{p+2+j}$ and $v_{-(p+j)}v_{-(p+1+j)}v_{-(p+2+j)}$ for each $j \in \{0, 3, 6, \ldots, 3(r-1)\}$, and the set of these edges is denoted by T.
- When $s = 1$, $B = \{v_{2p-1}v_{-(2p-1)}, v_{2p-1}v_{-(2p-1)}\}$. When $s = 2$, $B = \{v_{2p-2}v_{-(2p-1)}, v_{-(2p-1)}v_{2p-1}, v_{2p-1}v_{-(2p-1)}, v_{-(2p-1)}v_{2p-2}\}$.

When $B \neq \emptyset$, we color the edges of B by colors $4p-1$ and $4p$, alternatively. Then we color the other edges of $E(H)$ as follows:

Step 1. Color the edges of $M_j = \{v_jv_\infty\} \cup \{v_{-i+j}v_{i+j} \mid i \in \mathbb{Z}_{4p-1} \setminus \{0\}\}$ by j for $0 \leq j \leq 4p$.

Step 2. Consider the even cycle $v_0v_1 \cdots v_{p-1}v_\infty v_{-(p-1)} \cdots v_{-1}v_0$. Notice that each edge of the cycle has been assigned with a different color in Step 1. The set of colors used for the cycle is $K = \{2p, 2p+1, \ldots, 3p-2, p-1, 3p, p+1, p+2, \ldots, 2p-1\}$. Now recolor the edges of the cycle with colors $4p-1, 4p$ alternately. Then use the colors of K to color the edges of X. Suppose that the cycle is $u_1u_2 \cdots u_{2p}u_1$, and u_iu_{i+1} is colored by α_i. Then x_2u_i and x_1u_{i+1} are colored by α_i for $i \in \mathbb{Z}_{2p}$.

Step 3. For edges in T, consider the 6-cycle

$$C : v_{p+j}v_{-(p+j)}v_{p+1+j}v_{-(p+2+j)}v_{p+2+j}v_{-(p+1+j)}v_{p+j},$$

where $j \in \{0, 3, 6, \ldots, 3(r-1)\}$. Note that C has been colored with colors $0, 2p, 2p+1$ in Step 1. Now we recolor $E(C)$ with colors $4p-1, 4p$ alternately, and then the colors in $\{0, 2p, 2p+1\}$ can be assigned to the corresponding triangles: $v_{p+j}v_{p+1+j}v_{p+2+j}$ and $v_{-(p+j)}v_{-(p+1+j)}v_{-(p+2+j)}$ for each $j \in \{0, 3, 6, \ldots, 3(r-1)\}$. The process is similar to Step 3 of Lemma 3.4.

The desired edge-coloring of H has been given. ∎

Now we are constructing the following coloring of M modulo $4p+1$. Using suitable labels of the colors in Lemma 3.11, we assign the colors of edges incident to x_1^i and x_2^i such that w_i receives colors $i + \{2, 4, \ldots, 4p-2, 4p\}$ from x_1^i and $i + \{1, 3, 5, \ldots, 4p-3, 4p-1\}$ from x_2^{i-1}. Then for each $i \in \mathbb{Z}_{4p+1}$, w_i receives exactly $4p$ different colors. Finally, we color zw_i by i. Thus M is $(4p+1)$-edge-colorable, i.e., Class I.

By Lemmas 3.9 and 3.11, for any $p \geq 3$, there is a $(4p+1)$-regular $4p$-edge-connected and Class I graph M without circular $(2+\frac{1}{p})$-flows. Combining Theorem 3.1 for the odd case, Theorem 1.7 follows.

Remark. Note that the graphs constructed in Theorems 3.1 and 3.8 contain many parallel edges. But we can easily modify them to obtain simple graphs by replacing each vertex with a certain graph H. Here, for the graph J, H can be a $(4p-1)$-regular $(4p-1)$-edge-connected Class I simple graph with one vertex deleted; for the graph M, H can be a $(4p+1)$-regular $(4p+1)$-edge-connected Class I simple graph with one vertex deleted.

Although Conjecture 1.5 is false for $t \geq 12$ and $t \in \{6, 8, 10\}$, it might be still possible that Conjecture 1.5 is true for some small value t. The truth of the case $t = 2$ in Conjecture 1.5 is implied by Tutte's 3-flow conjecture, and as learned from Yezhou Wu in 2017 (personal communication with the first author), the following weaker problem is still open: *Is it true that* $\phi(G) < 4$ *for every* 5*-regular Class* I *graph* G*?*

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