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# Fractional Coloring Planar Graphs under Steinberg-type Conditions

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**Abstract** A Steinberg-type conjecture on circular coloring is recently proposed that for any prime $p \geq 5$, every planar graph of girth $p$ without cycles of length from $p+1$ to $p(p-2)$ is $C_p$-colorable (that is, it admits a homomorphism to the odd cycle $C_p$). The assumption of $p \geq 5$ being prime number is necessary, and this conjecture implies a special case of Jaeger's Conjecture that every planar graph of girth $2p-2$ is $C_p$-colorable for prime $p \geq 5$. In this paper, combining our previous results, we show the fractional coloring version of this conjecture is true. Particularly, the $p = 5$ case of our fractional coloring result shows that every planar graph of girth 5 without cycles of length from 6 to 15 admits a homomorphism to the Petersen graph.

**Keywords** Fractional coloring, circular coloring, planar graphs, girth, homomorphism

**MR(2010) Subject Classification** 05C15, 05C10

## 1 Introduction

Jaeger [12] in 1988 conjectured that every 9-edge-connected graph admits a circular 5/2-flow (or equivalently, admits an orientation such that the indegree is congruent to the outdegree modulo 5 at each vertex). Jaeger observed that his conjecture implies the celebrated 5-Flow Conjecture of Tutte [19]. He also extended Tutte's 3-Flow, 5-Flow Conjectures and proposed a more general circular flow conjecture that every $4k$-edge-connected graph admits a circular $\frac{2k+1}{k}$-flow. Jaeger's Circular Flow Conjecture was confirmed for highly connected graphs by Thomassen [18] and later for $6k$-edge-connected graphs by Lovász et al. [14], but it was disproved for $k \geq 3$ recently in [9]. Tutte's Flow Conjectures remain open as of today. The counterexamples of Jaeger's Circular Flow Conjecture presented in [9] are nonplanar graphs, and so it still remains open for planar graphs, which can be equivalently stated below as homomorphism to odd cycles

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by duality. Here for any graph $H$, a graph is called *$H$-colorable* if it admits a homomorphism to $H$.

**Conjecture 1.1** ([12]) *Every planar graph of girth at least $4k$ is $C_{2k+1}$-colorable.*

Conjecture 1.1 has received considerable attentions, and many progresses have been made in [2, 7, 13, 16, 20]. The current best general result towards Conjecture 1.1 is due to Lovász et al. [14] in 2013, from the dual of their flow results.

**Theorem 1.2** ([14]) *Every planar graph of girth at least $6k$ is $C_{2k+1}$-colorable.*

For the cases of $k \in \{1, 2, 3\}$, better results are known. The case $k = 1$ of Conjecture 1.1 is Grötzsch's 3-Coloring Theorem [8]. Proved in 1959, it stated that every triangle-free planar graph is 3-colorable. The following results on the cases $k = 2, 3$ are obtained by Dvořák and Postle [5], Cranston and Li [4], and Postle and Smith-Roberge [17], respectively.

**Theorem 1.3** (i) ([4, 5]) *Every planar graph of girth at least 10 is $C_5$-colorable.*

(ii) ([4, 17]) *Every planar graph of girth at least 16 is $C_7$-colorable.*

Note that Theorems 1.2 and 1.3 are still valid if we replace girth conditions by odd-girth conditions. Steinberg considered a different approach. Instead of forbidding small (odd) cycles, he asked what if we forbid cycles of certain length. More specifically Steinberg conjectured that every planar graph without cycles of length 4 or 5 is $C_3$-colorable. Motivated by this problem, the authors in [11] studied its generalization on $C_k$-coloring under similar Steinberg-type conditions: *for each odd integer $k \geq 3$, what is the smallest number $f(k)$ such that every planar graph of girth $k$ without cycles of length from $k+1$ to $f(k)$ is $C_k$-colorable?* A known result in [1] and the counterexamples of Steinberg's Conjecture in [3] provide that $f(3) \in \{6, 7\}$. It is proved in [11] that $f(k)$ exists if and only if $k$ is an odd prime, and for any prime $p \geq 5$, $p^2 - \frac{5}{2}p + \frac{3}{2} \leq f(p) \leq 2p^2 + 2p - 5$. Furthermore, it is conjectured that $f(p) \leq p^2 - 2p$, which states the following.

**Conjecture 1.4** ([11]) *For any prime $p \geq 5$, every planar graph of girth $p$ without cycles of length from $p+1$ to $p(p-2)$ is $C_p$-colorable.*

It is observed in [11] that Conjecture 1.4, if true, would imply Conjecture 1.1 for each prime $p = 2k + 1 \geq 5$. The first case $p = 5$ is very special, as it not only implies that planar graphs of girth at least 8 are $C_5$-colorable, but also implies the Five Color Theorem that every planar graph is 5-colorable (see [11]).

The fractional coloring, as introduced in [10], is a well-known generalization of ordinary coloring of graphs. For positive integers $s$ and $t$ with $s \geq t$, a fractional *$(s : t)$-coloring* $\varphi$ of a graph $G$ is a set coloring that assigns a $t$-element subset of $\{1, \ldots, s\}$ to each vertex such that $\varphi(u) \cap \varphi(v) = \emptyset$ for each edge $uv \in E(G)$. Equivalently, a graph is fractional $(s : t)$-colorable if and only if it admits a homomorphism to the Kneser graph $K(s, t)$ (or saying that it is $K(s, t)$-colorable). Since the odd cycle $C_{2k+1}$ is a subgraph of the Kneser graph $K(2k + 1, k)$, we have that every $C_{2k+1}$-colorable graph is fractional $(2k+1 : k)$-colorable, but not vice versa. In particular, fractional $(2k+1 : k)$-coloring can be viewed as a relaxation of $C_{2k+1}$-coloring. When $k = 2$, the Kneser graph $K(5, 2)$ is the well-known Petersen graph, and Dvořák, Škrekovski, and Valla [6] proved the fractional coloring version of Conjecture 1.1 in this case.

**Theorem 1.5** ([6]) *Every planar graph of girth at least* 8 *is fractional* $(5:2)$*-colorable (or equivalently, admits a homomorphism to the Petersen graph).*

In [15], Naserasr proposed a stronger conjecture that every planar graph of odd-girth $2t+3$ is fractional $(2t+1:t)$-colorable (or equivalently, admits a homomorphism to the Kneser graph $K(2t+1,t)$). A fractional coloring result related to Conjecture 1.4 is obtained in [11].

**Theorem 1.6** ([11]) *For any odd integer* $k \geq 5$, *every planar graph of girth* $k$ *without cycles of length from* $k+1$ *to* $\lfloor \frac{22k}{3} \rfloor$ *is fractional* $(k : \frac{k-1}{2})$*-colorable.*

Since $p(p-2) \geq \frac{22p}{3}$ when $p \geq 11$, Theorem 1.6 confirms the fractional coloring version of Conjecture 1.4 for all prime $p \geq 11$. The purpose of this paper is to prove the remaining cases $p = 5, 7$ of the fractional coloring version of Conjecture 1.4.

**Theorem 1.7** *Every planar graph of girth* 5 *without cycles of length from* 6 *to* 15 *is fractional* $(5:2)$*-colorable* (*or equivalently, admits a homomorphism to the Petersen graph*).

**Theorem 1.8** *Every planar graph of girth* 7 *without cycles of length from* 8 *to* 35 *is fractional* $(7:3)$*-colorable.*

**Corollary 1.9** *The fractional coloring version of Conjecture* 1.4 *is true.*

We remark that, other than Theorem 1.5 of Dvořák et al. [6], the fractional coloring version of Conjecture 1.1 is still open for each $k \geq 3$. Although Conjecture 1.4 implies Conjecture 1.1 for each prime $p = 2k+1 \geq 5$ (see [11]), their fractional coloring versions seem not to have this relation. Hence Theorem 1.7 does not imply and in turn is not implied by Theorem 1.5.

At the end of this section, we will give some definitions and notations that will be used throughout this paper. In a graph $G$, a *$k$-vertex* is a vertex of degree $k$. A *$k$-thread of length $k$* (*$k$-thread* for short) of $G$ is a path $uv_1v_2\cdots v_kw$ such that $v_i$ is a 2-vertex for $1 \leq i \leq k$. The end vertices of the path are called the *end vertices* of the thread. A thread with end vertices $x, y$ is also called an $(x, y)$-thread. A *$k^+$-thread* of $G$ is a thread of length at least $k$. A *$(k_1, k_2, \ldots, k_t)$-thread* $T_x$ in $G$ is a subgraph consisting of distinct $k_1$-thread, $k_2$-thread, $\ldots$, $k_t$-thread which share a common end vertex $x$, where $x$ is called the *center* of $T_x$. If $z$ is a 2-vertex of an $(x, y)$-thread, then we say $x$ and $z$ are *weakly adjacent*. For a positive integer $t$, let $[t] = \{1, 2, \ldots, t\}$. Let $\varphi$ be a fractional $(2k+1:k)$-coloring of $G$, and for any $v \in V(G)$, denote $\bar{\varphi}(v) = [2k+1] \setminus \varphi(v)$. Fix a graph $H$ and a vertex subset $S$ of $V(H)$. A *precoloring* $\varphi_k$ of $S$ assigns colors in $\binom{[2k+1]}{k}$ to vertices in $S$ such that $H[S]$ is properly fractional $(2k+1:k)$-colored. We say that $H$ is *$(\varphi_k, S)$-colorable* if the precoloring $\varphi_k$ of $S$ can be extended to all vertices of $H$ to obtain a fractional $(2k+1:k)$-coloring.

The rest of this paper is organized as follows. Section 2 presents the proof of Theorem 1.7, Section 3 is devoted to the proof of Theorem 1.8, and a few concluding remarks are given in Section 4.

## 2 Fractional $(5:2)$-coloring of Planar Graphs

This section is aiming to give the proof of Theorem 1.7. We first study some graphs with precoloring extensions in Subsection 2.1, serving for reducible configurations, and then present the proof of Theorem 1.7 in Subsection 2.2 by a discharging method.

### 2.1 Precoloring for Fractional $(5:2)$-coloring

We first show that certain precoloring of some vertices can be extended.

**Lemma 2.1** *Let $P = v_1v_2\dots v_n$ be a path and $\varphi_2$ be a precoloring of $S = \{v_1, v_n\}$.*

(i) *If $n = 3$, then $P$ is $(\varphi_2, S)$-colorable if and only if $|\varphi_2(v_1) \cup \varphi_2(v_3)| \le 3$.*

(ii) *If $n = 4$, then $P$ is $(\varphi_2, S)$-colorable if and only if $\varphi_2(v_1) \ne \varphi_2(v_4)$.*

(iii) *If $n \ge 5$, then $P$ is $(\varphi_2, S)$-colorable.*

*Proof* (i) This is obvious since we have enough colors in $[5] \setminus (\varphi_2(v_1) \cup \varphi_2(v_3))$ to color $v_2$, and vice versa.

(ii) We select two available colors in $\bar{\varphi}_2(v_1)$ to color $v_2$, say that $v_2$ receives color set $\varphi_2(v_2)$. Now the coloring can be extended to $v_3$ if and only if $|\varphi_2(v_2) \cup \varphi_2(v_4)| \le 3$ by (i). This is possible if and only if $\varphi_2(v_1) \ne \varphi_2(v_4)$.

(iii) When $n = 5$, we have three colors not in $\varphi_2(v_1)$, and it is always possible to color $v_2$ with $\varphi_2(v_2) \subset \bar{\varphi}_2(v_1)$ such that $\varphi_2(v_2) \ne \varphi_2(v_5)$. Thus $P$ is $(\varphi_2, S)$-colorable by (ii). For $n \ge 6$, we can arbitrarily color vertices $v_{n-1}, \dots, v_5$ first, and then extend this coloring to $v_2, v_3, v_4$ as before. □

By Lemma 2.1 (i) (ii), we have the following lemma.

**Lemma 2.2** *Let $C$ be a 5-cycle $u_0u_1u_2u_3u_4u_0$ and $\varphi_2$ be a precoloring of $\{u_0, u_2\}$. Then $C$ is $(\varphi_2, \{u_0, u_2\})$-colorable if and only if $|\varphi_2(u_0) \cup \varphi_2(u_2)| = 3$.*

**Lemma 2.3** *Let $H_1$ be a graph consisting of a path $v_0v_1v_2\dots v_6$ and two edges $v_2w_2, v_4w_4$. Given a precoloring $\varphi_2$ of $V(H_1) \setminus \{v_3\}$, let $\phi_2$ be the restriction of $\varphi_2$ on $S = \{v_0, v_6, w_2, w_4\}$. Then $H_1$ is $(\phi_2, S)$-colorable.*

*Proof* Since $|\bar{\varphi}_2(w_2)| = |\bar{\varphi}_2(w_4)| = 3$, we have $\bar{\varphi}_2(w_2) \cap \bar{\varphi}_2(w_4) \ne \emptyset$. Let $\alpha \in \bar{\varphi}_2(w_2) \cap \bar{\varphi}_2(w_4)$. Note that $\varphi_2(v_0) \ne \varphi_2(w_2)$ and $\varphi_2(v_6) \ne \varphi_2(w_4)$ by Lemma 2.1(ii). We select a color $\beta$ such that $\beta \in \varphi_2(v_0) \cap \bar{\varphi}_2(w_2)$ if $\alpha \notin \varphi_2(v_0)$, and $\beta \in \bar{\varphi}_2(w_2) \setminus \{\alpha\}$ otherwise. Similarly, choose a color $\gamma$ such that $\gamma \in \varphi_2(v_6) \cap \bar{\varphi}_2(w_4)$ if $\alpha \notin \varphi_2(v_6)$, and $\gamma \in \bar{\varphi}_2(w_4) \setminus \{\alpha\}$ otherwise. Define $\phi_2(v_2) = \{\alpha, \beta\}$ and $\phi_2(v_4) = \{\alpha, \gamma\}$. Then $H_1$ is $(\phi_2, S \cup \{v_2, v_4\})$-colorable by Lemma 2.1(i). □

**Lemma 2.4** (i) *Let $H_1$ be a graph consisting of a path $v_0v_1v_2v_3$, a path $v_0u_1u_2$ and an edge $v_0w_1$. Given a precoloring $\varphi_2$ of $V(H_1) \setminus \{v_1, v_2\}$, let $\phi_2$ be the restriction of $\varphi_2$ on $S = \{v_3, u_2, w_1\}$. Then $H_1$ is $(\phi_2, S)$-colorable.*

(ii) *Let $H_2$ be a graph consisting of three paths $v_0v_1v_2$, $v_0u_1u_2$ and $v_0w_1w_2$. Then for any precoloring $\varphi_2$ of $S = \{v_2, u_2, w_2\}$, $H_2$ is $(\varphi_2, S)$-colorable.*

*Proof* (i) Let $\alpha \in \bar{\varphi}_2(w_1) \setminus \varphi_2(v_3)$. Choose $\beta \in \varphi_2(u_2) \cap \bar{\varphi}_2(w_1) \setminus \{\alpha\}$ if possible, and $\beta \in \bar{\varphi}_2(w_1) \setminus \{\alpha\}$ if $\varphi_2(u_2) \cap \bar{\varphi}_2(w_1) \setminus \{\alpha\} = \emptyset$. Define $\phi_2(v_0) = \{\alpha, \beta\}$. Then by Lemma 2.1 (i) (ii) $H_1$ is $(\phi_2, S \cup \{v_0\})$-colorable.

(ii) Since

$$\left|\binom{\bar{\varphi}_2(v_2)}{2}\right| + \left|\binom{\bar{\varphi}_2(u_2)}{2}\right| + \left|\binom{\bar{\varphi}_2(w_2)}{2}\right| = 9 < 10 = \left|\binom{[5]}{2}\right|,$$

there exists $\{\alpha, \beta\} \in \binom{[5]}{2}$ which is not in $\binom{\bar{\varphi}_2(v_2)}{2} \cup \binom{\bar{\varphi}_2(u_2)}{2} \cup \binom{\bar{\varphi}_2(w_2)}{2}$, and so define $\varphi_2(v_0) = \{\alpha, \beta\}$. Then $H_2$ is $(\varphi_2, S \cup \{v_0\})$-colorable by Lemma 2.1 (i). □

**Lemma 2.5** *Let $H$ be a graph consisting of two 5-cycles $v_0v_1v_2v_3v_4$ and $v_0u_1u_2u_3u_4$.*

(i) *For any precoloring $\varphi_2$ of $S = \{v_2, v_3, u_2, u_3\}$, $H$ is $(\varphi_2, S)$-colorable.*

(ii) *Given a precoloring $\varphi_2$ of $V(H) \setminus \{v_1, v_2\}$, let $\phi_2$ be the restriction of $\varphi_2$ on $S = \{v_3, u_1, u_2, u_3\}$. Then $H$ is $(\phi_2, S)$-colorable.*

*Proof* (i) By Lemma 2.1 (i), we only need to color $v_0$ with $\varphi_2(v_0)$ such that $|\varphi_2(v_0) \cap \varphi_2(x)| = 1$ for each $x \in \{v_2, v_3, u_2, u_3\}$. Clearly, we have $(\varphi_2(v_2) \cup \varphi_2(v_3)) \cap \varphi_2(u_2) \neq \emptyset$, w.l.o.g., let $\alpha \in \varphi_2(v_2) \cap \varphi_2(u_2)$. If $\varphi_2(v_3) \cap \varphi_2(u_3) \neq \emptyset$, then we choose $\beta \in \varphi_2(v_3) \cap \varphi_2(u_3)$, and set $\varphi_2(v_0) = \{\alpha, \beta\}$, we are done. Otherwise, $\varphi_2(v_3) \cap \varphi_3(u_3) = \emptyset$ and $\varphi_2(v_3) \cup \varphi_2(u_3) = [5] \setminus \{\alpha\}$. Now we have $\varphi_2(v_2) \cap \varphi_2(u_3) = \varphi_2(v_2) \setminus \{\alpha\}$, say $\varphi_2(v_2) \setminus \{\alpha\} = \{\alpha'\}$. Moreover, $\varphi_2(v_3) \cap \varphi_2(u_2) \neq \emptyset$, and we let $\beta \in \varphi_2(v_3) \cap \varphi_2(u_2)$. Then define $\varphi_2(v_0) = \{\alpha', \beta\}$ as desired.

(ii) We may assume that $\varphi_2(v_3) = \varphi_2(v_0)$; otherwise $H$ is $(\varphi_2, V(H) \setminus \{v_1, v_2\})$-colorable by Lemma 2.1(ii), and thus is $(\phi_2, S)$-colorable as well. By Lemma 2.2, $|\varphi_2(v_0) \cap \varphi_2(u_3)| = 1$. Let $\alpha \in \varphi_2(v_0) \cap \varphi_2(u_3)$ and $\beta \in \bar{\varphi}_2(u_1) \setminus \varphi_2(u_3)$. Define $\phi_2(v_0) = \{\alpha, \beta\}$. Then $|\phi_2(v_0) \cap \phi_2(v_3)| = 1$, and $|\phi_2(v_0) \cap \phi_2(u_3)| = 1$. Thus, by Lemma 2.1(i) and Lemma 2.2, $H$ is $(\phi_2, S)$-colorable. □

**Lemma 2.6** *Let $H$ be a graph consisting of a 5-cycle $v_0v_1v_2v_3v_4v_0$, a 5-cycle $u_0u_1u_2u_3u_4u_0$, and an edge $v_0u_0$. Given a precoloring $\varphi_2$ of $V(H) \setminus \{v_0, v_1, v_4\}$, let $\phi_2$ be the restriction of $\varphi_2$ on $S = \{v_2, v_3, u_2, u_3, u_4\}$. Then $H$ is $(\phi_2, S)$-colorable.*

*Proof* By Lemma 2.2, we have $|\varphi_2(u_2) \cap \varphi_2(u_4)| = 1$. Let $\alpha \in \varphi_2(u_2) \setminus \varphi_2(u_4)$. If $\alpha \notin \varphi_2(v_2) \cup \varphi_2(v_3)$, then we choose $\beta \in \bar{\varphi}_2(u_4) \setminus \varphi_2(u_2)$, and then select a color $\gamma_1 \in \varphi_2(v_2) \setminus \{\beta\}$ and a color $\gamma_2 \in \varphi_2(v_3) \setminus \{\beta\}$. Define $\phi_2(v_0) = \{\gamma_1, \gamma_2\}$ and $\phi_2(u_0) = \{\alpha, \beta\}$. Hence $H$ is $(\phi_2, S \cup \{v_0, u_0\})$-colorable by Lemma 2.1(i). Assume instead that $\alpha \in \varphi_2(v_2) \cup \varphi_2(v_3)$, w.l.o.g., say $\alpha \in \varphi_2(v_2)$. Then we have $\alpha \notin \varphi_2(v_3)$. Let $\gamma_1 \in \varphi_2(v_2) \setminus \{\alpha\}$. Choose $\beta \in \bar{\varphi}_2(u_4) \setminus (\varphi_2(u_2) \cup \{\gamma_1\})$, and let $\gamma_2 \in \varphi_2(v_3) \setminus \{\beta\}$. Define $\phi_2(v_0) = \{\gamma_1, \gamma_2\}$ and $\phi_2(u_0) = \{\alpha, \beta\}$. This shows that $H$ is $(\phi_2, S \cup \{v_0, u_0\})$-colorable by Lemma 2.1 (i) as well. □

**Lemma 2.7** (i) *Let $H_1$ be a graph consisting of a 5-cycle $v_0v_1v_2v_3v_4v_0$ and an edge $v_0u_1$. Given a precoloring $\varphi_2$ of $S = \{v_2, v_3, u_1\}$, if $\varphi_2(u_1) \neq \varphi_2(v_2)$ and $\varphi_2(u_1) \neq \varphi_2(v_3)$, then $H_1$ is $(\varphi_2, S)$-colorable.*

(ii) *Let $H_2$ be a graph consisting of a 5-cycle $v_0v_1v_2v_3v_4v_0$ and a path $v_0u_1u_2$. Given a precoloring $\varphi_2$ of $V(H_2) \setminus \{u_1\}$, let $\phi_2$ be the restriction of $\varphi_2$ on $S = \{v_1, v_2, v_3, u_2\}$. If $\phi_2(u_2) \neq \phi_2(v_1)$, then $H_2$ is $(\phi_2, S)$-colorable.*

*Proof* (i) Since $\varphi_2(u_1) \neq \varphi_2(v_2)$ and $\varphi_2(u_1) \neq \varphi_2(v_3)$, we can choose $\alpha \in \varphi_2(v_2) \setminus \varphi_2(u_1)$ and $\beta \in \varphi_2(v_3) \setminus \varphi_2(u_1)$. Define $\varphi_2(v_0) = \{\alpha, \beta\}$. Then $H_1$ is $(\varphi_2, S \cup \{v_0\})$-colorable by Lemma 2.1 (i).

(ii) As $\varphi_2$ provides a fractional $(5 : 2)$-coloring of 5-cycle $v_0v_1v_2v_3v_4$, we have $|\varphi_2(v_1) \cup \varphi_2(v_3)| = 3$ by Lemma 2.2. Let $\alpha \in \bar{\varphi}_2(v_1) \cap \varphi_2(v_3)$. If $\alpha \in \varphi_2(u_2)$, we choose a color $\beta \in \bar{\varphi}_2(v_1) \setminus \{\alpha\}$ and define $\phi_2(v_0) = \{\alpha, \beta\}$. If $\alpha \notin \varphi_2(u_2)$, we can choose $\beta \in \varphi_2(u_2) \cap \bar{\varphi}_2(v_1)$ as $\varphi_2(u_2) \neq \varphi_2(v_1)$, and then define $\phi_2(v_0) = \{\alpha, \beta\}$. In any case, $H_2$ is $(\phi_2, S \cup \{v_0\})$-colorable by Lemma 2.1 (i). □

### 2.2 Proof of Theorem 1.7

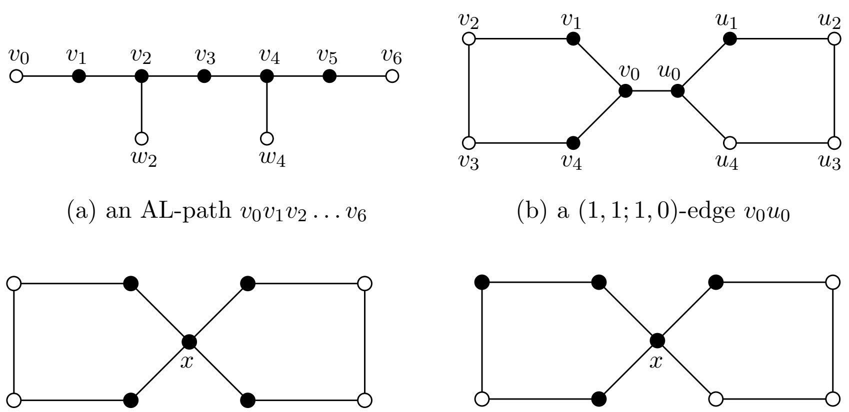
In this subsection, we shall prove Theorem 1.7 by analyzing the structure of the potential minimal counterexample and proceeding with a discharging proof. In the rest of this section, we always let $G$ be a counterexample to Theorem 1.7 such that $|V(G)|+|E(G)|$ is minimized.

#### 2.2.1 Subgraph Structures of a Minimum Counterexample

We start with some basic properties of the minimal counterexample $G$.

**Claim 2.1** $G$ is 2-connected and particularly $\delta(G)\geq 2$.

*Proof* Clearly, $G$ is connected. If $G$ is not 2-connected and contains a cut vertex $v$, then there exist proper induced subgraphs $G_1$ and $G_2$ of $G$ such that $E(G)=E(G_1)\cup E(G_2)$ and $V(G_1)\cap V(G_2)=\{v\}$. By the minimality of the counterexample, $G_1$ has a $(5:2)$-coloring $\varphi$ and $G_2$ has a $(5:2)$-coloring $\psi$. Exchange the colors if necessarily such that $\varphi(v)=\psi(v)$, then $\varphi$ and $\psi$ combine to become a fractional $(5:2)$-coloring of $G$, which is a contradiction. $\square$




(a) an AL-path $v_0v_1v_2\dots v_6$

(b) a $(1,1;1,0)$-edge $v_0u_0$

(c) a $(1,1;1,1)$-vertex $x$

(d) a $(2,1;1,0)$-vertex $x$

Figure 1 An AL-path, a $(1,1;1,0)$-edge, a $(1,1;1,1)$-vertex and a $(2,1;1,0)$-vertex

A subgraph of $G$ consists of a path $v_0v_1v_2\dots v_6$ and two edges $v_2w_2, v_4w_4$ with $d_G(v_1)=d_G(v_3)=d_G(v_5)=2$ and $d_G(v_2)=d_G(v_4)=3$ is called an *alternating-path* (*AL-path* for short). An edge $v_0u_0$ is called a $(1,1;1,0)$-*edge* if $v_0$ and $u_0$ are respectively in vertex-disjoint 5-cycles $v_0v_1v_2v_3v_4v_0$ and $u_0u_1u_2u_3u_4u_0$ with $d_G(v_0)=d_G(u_0)=3$ and $d_G(v_1)=d_G(v_4)=d_G(u_1)=2$, as in Lemma 2.6. A 4-vertex is called a $(1,1;1,1)$-*vertex* if it is a center of a $(1,1,1,1)$-thread and is incident with two edge-disjoint 5-cycles, as in Lemma 2.5 (i). Similarly, a 4-vertex is called a $(2,1;1,0)$-*vertex* if it is a center of a $(2,1,1,0)$-thread and is incident with two edge-disjoint 5-cycles, one consists of a 2-thread and a 1-thread, and the other contains a 1-thread, as in Lemma 2.5 (ii). See Figure 1 for an illustration, where the degrees of the black solid vertices in $G$ equal their degrees in the figure.

By Lemmas 2.1–2.6, we get several reducible structures which do not appear in the minimal counterexample $G$.

**Claim 2.2** $G$ contains none of the following configurations:

(i) a $3^+$-thread,

(ii) an AL-path,
(iii) a $(1,1,1)$-thread or a $(2,1,0)$-thread,
(iv) a $(1,1;1,1)$-vertex or a $(2,1;1,0)$-vertex,
(v) a $(1,1;1,0)$-edge.

*Proof* Suppose for a contradiction that $G$ contains one of the above configurations $H$. Let $S$ be the vertex set defined as in one of Lemmas 2.1–2.6. We obtain a subgraph $G_1$ of $G$ by deleting the vertex set $V(H) \setminus S$. By the minimality of $G$, $G_1$ admits a fractional $(5:2)$-coloring $\varphi$, where each vertex in $S$ receives a color set with certain restrictions. Applying Lemmas 2.1–2.6, $H$ is $(\varphi, S)$-colorable, and so we extend this coloring $\varphi$ to become a fractional $(5:2)$-coloring of $G$, a contradiction. □

2.2.2 Exploring the Subgraph $G'$

Next, unlike some standard methods, we explore further structure of $G$ from its subgraph $G'$.

**From $G$, we obtain a subgraph $G'$ as follows.**

(i) *If there exist two* (*or more*) *adjacent* 2*-vertices in a* 5*-cycle of* $G$*, then we delete all those* 2*-vertices.*

(ii) *If there exist some* 2*-vertices in a* 5*-cycle of* $G$ *but no adjacent* 2*-vertices, then we delete a* 2*-vertex in the* 5*-cycle* (*arbitrarily*).

Clearly, the obtained graph $G'$ is of girth at least 5 and contains no cycles of length from 6 to 15, and moreover each 5-cycle of $G'$ contains no 2-vertices. It is easy to see that $G'$ has minimal degree at least 2 by its construction. Furthermore, in each step of constructing $G'$ we only delete either a single 2-vertex or two adjacent 2-vertices by Claim 2.2 (i).

**Claim 2.3** $G'$ contains no $3^+$-thread.

*Proof* Suppose, for a contradiction, that $xabcy$ is a path of $G'$ with $d_{G'}(a) = d_{G'}(b) = d_{G'}(c) = 2$. Since $G$ contains no adjacent $C_5$, $d_G(b) \leq 4$. We divide our discussion into three cases below.

If $d_G(b) = 4$, then there exist a 5-cycle in $G$ containing $ab$ and another 5-cycle in $G$ containing $bc$. Let $bv_1v_2v_3ab$ and $bu_1u_2u_3cb$ be the corresponding 5-cycles. Since $v_1$ is deleted in $G'$, we have that either $v_1, v_2$ are two adjacent 2-vertices deleted in $G'$, or $v_1$ is a single deleted 2-vertex. In any case, we have $d_G(a) = d_{G'}(a) = 2$. By symmetry, we also obtain that $d_G(c) = d_{G'}(c) = 2$. Hence $b$ is a $(1,1;1,1)$-vertex in $G$, a contradiction to Claim 2.2 (iv).

If $d_G(b) = 3$, w.l.o.g., we may assume that $ab$ is contained in a 5-cycle $bv_1v_2v_3ab$ in $G$. Then $d_G(a) = d_{G'}(a) = 2$, and so $v_1$ is a single deleted 2-vertex by Claim 2.2 (iii). Moreover, we also have $d_G(c) \neq 2$ since $G$ contains no $(1,1,1)$-thread by Claim 2.2 (iii). Thus $d_G(c) = 3$ and $cy$ is contained in a 5-cycle $cyz_1z_2z_3c$, where $z_3$ is a deleted 2-vertex. Now the edge $bc$ is a $(1,1;1,0)$-edge, a contradiction to Claim 2.2 (v).

If $d_G(b) = 2$, since $G$ has no $3^+$-thread by Claim 2.2 (i), one of $a$ and $c$ has degree at least 3, say $d_G(a) \geq 3$. If $d_G(a) = 4$, then $a$ is incident with two 5-cycles, where $a, b, c$ are contained in a 5-cycle $abcz_1z_2a$ of $G$ and $z_2$ is a deleted 2-vertex. By the construction of $G'$, we have $d_G(c) \geq 3$, and thus $y \neq z_1$ and $z_1$ is also a deleted 2-vertex as $d_{G'}(c) = 2$. Thus $a$ is a $(2,1;1,0)$-vertex, a contradiction to Claim 2.2 (iv). So we assume $d_G(a) = 3$ in the following. Since $G$ contains no $(2,1,0)$-thread by Claim 2.2 (iii), we must have $d_G(c) \geq 3$, and so $d_G(c) = 3$ with a similar argument as above. Since $G$ contains no $(2,1,0)$-thread, then $a$, $b$, and $c$ cannot be contained in the same $C_5$. Now let $av_1v_2v_3xa$, $cu_1u_2u_3yc$ be the corresponding 5-cycles containing $ax, cy$, respectively. Hence we have $d_G(v_1) = d_G(u_1) = d_G(b) = 2$ and $d_G(a) = d_G(c) = 3$. This results in an AL-path in $G$, contradicting to Claim 2.2 (ii). This proves Claim 2.3. $\square$

The key of the proof is the following claim to rule out $(2, 2, 2)$-threads in $G'$.

**Claim 2.4** $G'$ contains no $(2, 2, 2)$-thread.

*Proof* Suppose to the contrary that there is a $(2, 2, 2)$-thread consisting of paths $vx_1x_2$, $vy_1y_2$, $vz_1z_2$, where $d_{G'}(v) = 3$ and $d_{G'}(x_i) = d_{G'}(y_i) = d_{G'}(z_i) = 2$ for $1 \le i \le 2$. By the construction of $G'$, $x_2y_2, x_2z_2, y_2z_2 \notin E(G')$. We first show the following fact.

**Subclaim 2.4.1** $d_G(v) = d_{G'}(v) = 3$.

*Proof of Subclaim* 2.4.1 If $d_G(v) \ge 4$, then there exist deleted 2-vertices in $G$, which corresponds to a 5-cycle containing $v$. Since either a single 2-vertex or two adjacent 2-vertices are deleted in constructing $G'$, we may, w.l.o.g., let $vu_1u_2x_2x_1v$ be such a 5-cycle, where $u_1$ is a deleted 2-vertex. If $u_2$ is not a deleted 2-vertex of $G$, then both $x_1$ and $x_2$ are 2-vertices of $G$, which are contained in the 5-cycle $vu_1u_2x_2x_1v$. According to the construction rules of $G'$, we should delete 2-vertices $x_1, x_2$ and keep the vertex $u_1$ in $G'$. This is a contradiction. So we must have that both $u_1$ and $u_2$ are deleted 2-vertices. In this case, $d_G(x_1) = d_{G'}(x_1) = 2$ and $x_2$ is incident with a 1-thread $x_2x_1v$ and a 2-thread $x_2u_2u_1v$. By Claim 2.2 (iii), we have $d_G(x_2) > 3$, and so $d_G(x_2) = 4$, which implies that $x_2$ is contained in another 5-cycle of $G$, say $x_2w_1w_2w_3w_4x_2$, where $w_1$ is a deleted 2-vertex. Hence $x_2$ is a $(2, 1; 1, 0)$-vertex, a contradiction to Claim 2.2 (iv).

Next we obtain further structures around the vertex $v$.

**Subclaim 2.4.2** The vertex $v$ is not contained in any 5-cycle of $G$.

*Proof of Subclaim* 2.4.2 If $v$ is contained in a 5-cycle of $G$, we shall distinguish two cases according to the distribution of the 5-cycle.

**Case 1** Assume that $v$ is contained in a 5-cycle of $G$ which contains one deleted 2-vertex. W.l.o.g., we may assume this 5-cycle to be $vy_1uz_2z_1v$, where $u$ is the deleted 2-vertex. Clearly, there is no 5-cycle of $G$ containing $vx_1$ since $G$ contains no adjacent 5-cycles and $d_G(v) = 3$ by Subclaim 2.4.1, and so we have $d_G(x_1) \le 3$ and $d_G(x_2) \le 3$. If $d_G(x_1) = 3$, then $x_1x_2$ is contained in a 5-cycle $x_1u_1u_2u_3x_2x_1$, where $u_1$ is a deleted 2-vertex. Hence $u_1, x_2, z_1$ are all 2-vertices of $G$ and $d_G(v) = d_G(x_1) = 3$. Thus $vx_1$ is a $(1, 1; 1, 0)$-edge, contradicting to Claim 2.2 (v). Otherwise, we have $d_G(x_1) = 2$. Since $v$ is not in a $(2, 1, 0)$-thread by Claim 2.2 (iii), we have $d_G(x_2) > 2$, and so $d_G(x_2) = 3$. This implies that $x_2$ is contained in a 5-cycle $x_2u_1u_2u_3u_4x_2$, where $u_1$ is a deleted 2-vertex. Hence $z_2z_1vx_1x_2u_1u_2$ is an AL-path, a contradiction to Claim 2.2 (ii).

**Case 2** Assume instead that $v$ is contained in a 5-cycle of $G$ which contains two adjacent 2-vertices deleted. W.l.o.g., we assume this 5-cycle to be $vy_1uwz_1v$, where $u$ and $w$ are deleted 2-vertices. By the minimality of $G$, $G - \{v, y_1, u, w, z_1\}$ has a fractional $(5 : 2)$-coloring $\varphi$. If $d_G(z_1) = 4$, then $z_1z_2$ is in a 5-cycle $z_1a_1a_2a_3z_2z_1$, where $z_2$ and $a_1$ are 2-vertices of $G$. We erase the color of $z_2, a_1$, and let $T_1 = \{\{\alpha, \beta\} : \alpha \in \varphi(a_2), \beta \in \varphi(a_3)\}$. If $d_G(z_1) = 3$, then by Claim 2.2 (iii), we have $z_2$ is in a 5-cycle $z_2a_1a_2a_3a_4z_2$ where $a_1$ is a deleted 2-vertex. We erase the color of $z_2, a_1$, and define $T_1 = \{\{\alpha, \beta\} : \alpha \in \varphi(a_4), \beta \in \bar{\varphi}(a_4) \setminus \varphi(a_2)\}$. Note that we have

$|T_1| \geq 4$ in any case, and $T_1$ contains a subset of type $\{\{\alpha_1, \beta_1\}, \{\alpha_1, \beta_2\}, \{\alpha_2, \beta_1\}, \{\alpha_2, \beta_2\}\}$, where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are distinct. Moreover, any color of $z_1$ in $T_1$ can extend to the erased vertices $z_2$ and $a_1$ by Lemma 2.1. Similarly, we can define a set $T_2$ according to the structure of $y_1$ such that $T_2$ contains a subset of type $\{\{\gamma_1, \theta_1\}, \{\gamma_1, \theta_2\}, \{\gamma_2, \theta_1\}, \{\gamma_2, \theta_2\}\}$, where $\gamma_1, \gamma_2, \theta_1, \theta_2$ are distinct.

If $d_G(x_1) = 2$, then either $d_G(x_2) = 2$ or $d_G(x_2) = 3$ where $x_2$ is contained in a 5-cycle with some 2-vertices deleted. Let $x_3$ be the neighbor of $x_2$ in $G'$ other than $x_1$. In the coloring $\varphi$, we erase the color of $x_1, x_2$ and the deleted 2-vertex in the 5-cycle containing $x_2$ (if exists). Set $T_3 = \binom{[5]}{2} \setminus \{\varphi(x_3)\}$. By Lemma 2.4 (ii) or Claim 2.2 (ii), any color of $v$ in $T_3$ can be extended to $x_1, x_2$ and the deleted 2-vertex. If $d_G(x_1) = 3$, then $x_1$ is contained in a 5-cycle $x_1u_1u_2u_3x_2x_1$, where $u_1$ is a deleted 2-vertex. In the coloring $\varphi$, we erase the color of $u_1, x_1, x_2$ and set $T_3 = \binom{[5]}{2} \setminus \{\varphi(u_2), \varphi(u_3)\}$. Then any color of $v$ in $T_3$ can be extended to $u_1, x_1, x_2$ by Lemma 2.4 (i). Since $\varphi(u_2) \cap \varphi(u_3) = \emptyset$, we have that $T_3$ contains a subset of size 8 with type $\binom{[5]}{2} \setminus \{\{\eta_1, \eta_2\}, \{\eta_3, \eta_4\}\}$, where $\eta_1, \eta_2, \eta_3, \eta_4$ are distinct.

Now it suffices to color $z_1$, $y_1$ and $v$ such that $\varphi(z_1) \in T_1$, $\varphi(y_1) \in T_2$, $\varphi(v) \in T_3$, and $|\varphi(z_1) \cap \varphi(y_1)| = 1$. Then by Lemmas 2.1 and 2.4, $\varphi$ can be extended to $G$. Recall that $\alpha_1, \alpha_2, \beta_1, \beta_2$ are distinct, and $\gamma_1, \gamma_2, \theta_1, \theta_2$ are distinct. In addition, we have either $\{\alpha_1, \beta_1\} \subset \{\gamma_1, \gamma_2, \theta_1, \theta_2\}$ or $\{\alpha_2, \beta_2\} \subset \{\gamma_1, \gamma_2, \theta_1, \theta_2\}$. By symmetry, we may assume that $\{\alpha_1, \beta_1\} \subset \{\gamma_1, \gamma_2, \theta_1, \theta_2\}$ and $\alpha_1 = \gamma_1$. Since $\eta_1, \eta_2, \eta_3, \eta_4$ are distinct, $|\{\beta_1, \beta_2\} \cap \{\eta_1, \eta_2, \eta_3, \eta_4\}| \geq 1$, say $\beta_1 = \eta_1$. If $|\{\alpha_1, \beta_1, \theta_1, \theta_2\} \cap \{\eta_3, \eta_4\}| \geq 1$, let $\theta^* \in \{\theta_1, \theta_2\} \cap \{\eta_3, \eta_4\}$ if $|\{\theta_1, \theta_2\} \cap \{\eta_3, \eta_4\}| \geq 1$, and let $\theta^* \in \{\theta_1, \theta_2\}$ otherwise. Define $\varphi(z_1) = \{\alpha_1, \beta_1\}$, $\varphi(y_1) = \{\alpha_1, \theta^*\}$, and define $\varphi(v) = [5] \setminus (\varphi(y_1) \cup \varphi(z_1))$. Note that $|\varphi(v) \cap \{\eta_1, \eta_2\}| \leq 1$ and $|\varphi(v) \cap \{\eta_3, \eta_4\}| \leq 1$. Then we have $\varphi(v) \in T_3$, $\varphi(y_1) \in T_2$ and $\varphi(z_1) \in T_1$ as desired. Now we assume $|\{\alpha_1, \beta_1, \theta_1, \theta_2\} \cap \{\eta_3, \eta_4\}| = 0$, then $\beta_1 \in \{\theta_1, \theta_2\}$, say $\beta_1 = \theta_1$. Note that $\{\alpha_1, \beta_1, \theta_2, \eta_3, \eta_4\} = [5]$, and $\gamma_2 \notin \{\alpha_1, \beta_1, \theta_2\}$ as $\alpha_1 = \gamma_1, \gamma_2, \beta_1 = \theta_1, \theta_2$ are all distinct. So $\gamma_2 \in \{\eta_3, \eta_4\}$. Define $\varphi(z_1) = \{\alpha_1, \beta_1\}$, $\varphi(y_1) = \{\gamma_2, \beta_1\}$, and define $\varphi(v) = [5] \setminus (\varphi(y_1) \cup \varphi(z_1))$. Then we have $\varphi(v) \in T_3$, $\varphi(y_1) \in T_2$ and $\varphi(z_1) \in T_1$ as desired.

Then we are able to complete the proof of Claim 2.4.

**Subclaim 2.4.3** Such a $(2, 2, 2)$-thread with center vertex $v$ does not exist in $G'$, a contradiction. Hence Claim 2.4 holds.

*Proof of Subclaim* 2.4.3 Let $\varphi$ be a fractional $(5:2)$-coloring of $G - v$. We shall erase the color set of some vertices and then extend the coloring $\varphi$ to $G$. By Subclaim 2.4.2 $v$ is not contained in any 5-cycle of $G$, and so $d_G(x_1) \leq 3$, $d_G(y_1) \leq 3$ and $d_G(z_1) \leq 3$. If $d_G(x_1) = 2$, then either $d_G(x_2) = 2$ or $d_G(x_2) = d_{G'}(x_2) + 1 = 3$ where $x_2$ is contained in a 5-cycle with some 2-vertex deleted. Let $x_3$ be the neighbor of $x_2$ in $G'$ other than $x_1$. In the coloring $\varphi$, we erase the color of $x_1, x_2$ and the deleted 2-vertex in the 5-cycle containing $x_2$ (if exists). Set $L_1 = \{\varphi(x_3)\}$. By Lemma 2.4 (ii) or Claim 2.2 (ii), any color of $v$ not in $L_1$ can be extended to $x_1, x_2$ and the deleted 2-vertex. If $d_G(x_1) = 3$, then $x_1$ is contained in a 5-cycle $x_1u_1u_2u_3x_2x_1$, where $u_1$ is a deleted 2-vertex. In the coloring $\varphi$, we erase the color of $u_1, x_1, x_2$ and set $L_1 = \{\varphi(u_2), \varphi(u_3)\}$. Then any color of $v$ not in $L_1$ can be extended to $u_1, x_1, x_2$ by Lemma 2.4 (i). Similarly, no matter $d_G(y_1) = 2$ or $d_G(y_1) = 3$, we can erase the color of certain vertices and define a list $L_2$ of cardinality 1 or 2 such that any color of $v$ not in $L_2$ can be extended to the uncolored vertices by Lemma 2.4 (i) and (ii). Similarly, there is a corresponding list $L_3$ for $vz_1z_2$. Since $|L_1 \cup L_2 \cup L_3| \leq 6 < 10$, there is an available choice in $\binom{[5]}{2} \setminus (L_1 \cup L_2 \cup L_3)$ to color $v$, which extends the coloring to all the erased vertices. This provides a fractional $(5:2)$-coloring of $G$, a contradiction. $\square$

### 2.2.3 Discharging

Now we are ready to complete the proof by a discharging method on $G'$. Note that $G'$ is clearly planar since it is a subgraph of $G$. In the following, we always assume $G'$ is embedded on the plane. Let $F(G')$ be the set of faces of $G'$. From Euler Formula, we have

$$\sum_{v \in V(G')} \left(\frac{3}{2} d_{G'}(v) - 5\right) + \sum_{f \in F(G')} (d_{G'}(f) - 5) = -10. \tag{2.1}$$

Assign an initial charge $ch_0(v) = \frac{3}{2} d_{G'}(v) - 5$ for each $v \in V(G')$, and $ch_0(f) = d_{G'}(f) - 5$ for each $f \in F(G')$. Hence the total charge is $-10$ by the equation above.

We redistribute the charges according to the following rules.

(R1) *Each* $3^+$*-vertex sends charge* $1$ *to each of its weakly adjacent* $2$*-vertices.*

(R2) *Each* $16^+$*-face sends charge* $\frac{11}{16}$ *to its incident vertices.*

(R3) *After* (R2), *each* $2$*-vertex sends its charge equally to its weakly adjacent* $3^+$*-vertices.*

Let $ch$ denote the charge assignment after performing the charge redistribution using the rules (R1), (R2), and (R3).

**Claim 2.5** $ch(f) \geq 0$ for each $f \in F(G')$.

*Proof* First we assume $d_{G'}(f) = 5$. Then $ch(f) = ch_0(f) = d_{G'}(f) - 5 = 0$. Now we assume $d_{G'}(f) \geq 16$ as $G'$ contains no cycles of length from 6 to 15. By (R2), $f$ sends charge $\frac{11}{16}$ to each incident vertices, and then

$$ch(f) = ch_0(f) - \frac{11}{16} d_{G'}(f) = \frac{5}{16} d_{G'}(f) - 5 \geq 0. \qquad \square$$

**Claim 2.6** $ch(v) \geq 0$ for each $v \in V(G')$.

*Proof* Recall that $\delta(G') \geq 2$ by its construction. First we assume $d_{G'}(v) = 2$. Then $v$ is weakly adjacent to two $3^+$-vertex by Claim 2.3, and thus $ch(v) = -2 + 2 \times 1 = 0$ by (R1).

Now we assume $d_{G'}(v) \geq 3$. Let $p(v)$ be the number of 2-vertices weakly adjacent to $v$, and let $t(v)$ be the number of 5-cycles incident with $v$.

Notice that there is no 2-vertex in a 5-face of $G'$ by its construction. Since $G'$ has no $3^+$-thread by Claim 2.3, we have

$$p(v) \leq 2(d_{G'}(v) - 2t(v)). \tag{2.2}$$

Let $p(v, f)$ be the number of 2-vertices weakly adjacent to $v$ in $f$. Then

$$\sum_{16^+\text{-face} f \ni v} p(v, f) = 2p(v). \tag{2.3}$$

By (R1), $v$ sends charge $p(v)$ to its weakly adjacent 2-vertices. By (R2), $v$ receives charge $\frac{11}{16}$ from each incident $16^+$-face and receives charge $p(v, f) \times \frac{11}{16} \times \frac{1}{2}$ from its weakly adjacent 2-vertices. Hence for each $3^+$-vertex $v \in V(G')$, it follows from Eqs. (2.2) and (2.3) that

$$ch(v) = \frac{3}{2} d_{G'}(v) - 5 - p(v) + \sum_{16^+\text{-face} f \ni v} \left(\frac{11}{16} + \frac{p(v, f)}{2} \cdot \frac{11}{16}\right)$$

$$
\begin{aligned}
&= \frac{3}{2}d_{G'}(v) - 5 - p(v) + \frac{11}{16}(d_{G'}(v) - t(v)) + \frac{11}{16}p(v) \\
&= \frac{35}{16}d_{G'}(v) - 5 - \frac{11}{16}t(v) - \frac{5}{16}p(v). \qquad (2.4)
\end{aligned}
$$

If $v$ is a $4^+$-vertex, by Eqs. (2.2) and (2.4), we have

$$
\begin{aligned}
ch(v) &= \frac{35}{16}d_{G'}(v) - 5 - \frac{11}{16}t(v) - \frac{5}{16}p(v) \\
&\ge \frac{35}{16}d_{G'}(v) - 5 - \frac{11}{16}t(v) - \frac{5}{16}\cdot 2(d_{G'}(v) - 2t(v)) \\
&= \frac{25}{16}d_{G'}(v) - 5 + \frac{9}{16}t(v) \\
&\ge \frac{25}{16}\cdot 4 - 5 + \frac{9}{16}t(v) \\
&= \frac{5}{4} + \frac{9}{16}t(v) > 0.
\end{aligned}
$$

If $v$ is a 3-vertex, then $t(v) \le 1$ as $G$ contains no $C_4$. When $t(v) = 1$, we have $p(v) \le 2$ by Claim 2.3, and so by Eq. (2.4),

$$
ch(v) = \frac{35}{16}d_{G'}(v) - 5 - \frac{11}{16}t(v) - \frac{5}{16}p(v) \ge \frac{35}{16}\cdot 3 - 5 - \frac{11}{16} - \frac{5}{16}\cdot 2 = \frac{4}{16} > 0.
$$

When $t(v) = 0$, we have $p(v) \le 5$ since $G'$ contains no $(2,2,2)$-thread by Claim 2.4, and hence

$$
ch(v) \ge \frac{35}{16}\cdot 3 - 5 - \frac{5}{16}\cdot 5 = 0.
$$

Therefore, all the vertices of $G'$ receive nonnegative final charge. □

By the fact that the total amount of charge does not change by its redistribution, combining (2.1) and Claims 2.5 and 2.6, we have

$$
-10 = \sum_{v\in V(G')} ch_0(v) + \sum_{f\in F(G')} ch_0(f) = \sum_{v\in V(G')} ch(v) + \sum_{f\in F(G')} ch(f) \ge 0,
$$

a contradiction. This contradiction completes the proof of Theorem 1.7.

## 3 Fractional (7 : 3)-coloring of Planar Graphs

This section is devoted to proving Theorem 1.8. We first present some reducible configurations under precoloring in Subsection 3.1, and then use a discharging method to complete the proof in Subsection 3.2.

### 3.1 Precoloring for Fractional (7 : 3)-coloring

At first, we show that certain precoloring of some vertices can be extended.

**Lemma 3.1** *Let $P = v_1v_2\cdots v_n$ be a path and $\varphi_3$ be a precoloring of $S = \{v_1, v_n\}$. Denote $b = |\varphi_3(v_1) \cap \varphi_3(v_n)|$.*

(i) *If $n = 3$, then $P$ is $(\varphi_3, S)$-colorable if and only if $b \ge 2$.*

(ii) *If $n = 4$, then $P$ is $(\varphi_3, S)$-colorable if and only if $b \le 1$.*

(iii) *If $n = 5$, then $P$ is $(\varphi_3, S)$-colorable if and only if $b \ge 1$.*

(iv) *If $n = 6$, then $P$ is $(\varphi_3, S)$-colorable if and only if $b \le 2$.*

(v) *If $n \ge 7$, then $P$ is $(\varphi_3, S)$-colorable.*

*Proof* (i) This is obvious since there are enough colors in $[7] \setminus (\varphi_3(v_1) \cup \varphi_3(v_3))$ to color $v_2$, and vice versa.

(ii) We choose three available colors in $\bar{\varphi}_3(v_1)$ to color $v_2$, and let $\varphi_3(v_2) \subset \bar{\varphi}_3(v_1)$ be the color set of $v_2$. Now the coloring can be extended to $v_3$ if and only if $|\varphi_3(v_2) \cap \varphi_3(v_4)| \geq 2$ by (i). Meanwhile, this is possible if and only if $|\bar{\varphi}_3(v_1) \cap \varphi_3(v_4)| \geq 2$, which is equivalent to $b = |\varphi_3(v_1) \cap \varphi_3(v_4)| \leq 1$.

The proofs of (iii), (iv), and (v) are similar to (ii). We select a color set $\varphi_3(v_2) \subset \bar{\varphi}_3(v_1)$ to color $v_2$. Then we are trying to color all vertices of path $v_3v_4 \dots v_{n-1}$ by previous facts. For $n = 5$, there exists $\varphi_3(v_2) \subset \bar{\varphi}_3(v_1)$ with $|\varphi_3(v_2) \cap \varphi_3(v_5)| \leq 1$ if and only if $|\varphi_3(v_1) \cap \varphi_3(v_5)| \geq 1$. For $n = 6$, there exists $\varphi_3(v_2) \subset \bar{\varphi}_3(v_1)$ with $|\varphi_3(v_2) \cap \varphi_3(v_6)| \geq 1$ if and only if $|\varphi_3(v_1) \cap \varphi_3(v_6)| \leq 2$. For $n = 7$, there exists $\varphi_3(v_2) \subset \bar{\varphi}_3(v_1)$ with $|\varphi_3(v_2) \cap \varphi_3(v_7)| \leq 2$ for any possible color set $\varphi_3(v_1)$ since $\bar{\varphi}_3(v_1) \setminus \varphi_3(v_7) \neq \emptyset$. Thus (iii), (iv), and (v) are all true. $\square$

By Lemma 3.1 (i)–(iv), we immediately have the following corollary on precoloring of 7-cycle.

**Lemma 3.2** *Let $C = u_0u_1u_2u_3u_4u_5u_6u_0$ be a 7-cycle.*

(i) *For a precoloring $\varphi_3$ of $\{u_0, u_2\}$, $C$ is $(\varphi_3, \{u_0, u_2\})$-colorable if and only if $|\varphi_3(u_0) \cap \varphi_3(u_2)| = 2$.*

(ii) *For a precoloring $\varphi_3$ of $\{u_0, u_3\}$, $C$ is $(\varphi, \{u_0, u_3\})$-colorable if and only if $|\varphi_3(u_0) \cap \varphi_3(u_3)| = 1$.*

In a graph $G$, a *$d$-$C_7$-replacement* operation on a given edge $e = xy \in E(G)$ is to replace the edge $e$ with a 7-cycle $C_7 = v_0v_1 \dots v_6v_0$ by identifying $x$ with $v_0$ and identifying $y$ with $v_d$. When $d$ is not explicitly stated, we just call it a $C_7$-replacement operation on the edge $e \in E(G)$. A necklace in $G$ is a subgraph obtained from a thread by applying $C_7$-replacement operations on some edges. A vertex $z$ is an *end vertex* of the necklace if and only if $z$ is an end vertex of the thread. A necklace with end vertices $x, y$ is also called an $(x, y)$-necklace, denoted by $N(x, y)$.

**Lemma 3.3** *Let $N(x, y)$ be a necklace with a precoloring $\varphi_3$ of $\{x, y\}$. Suppose that the distance between $x$ and $y$ is $d(x, y) = t$.*

(i) *If $t \leq 3$ and $|\varphi_3(x) \cap \varphi_3(y)| = \frac{5}{4} + (-1)^t \cdot \frac{7-2t}{4}$, then $N(x, y)$ is $(\varphi_3, \{x, y\})$-colorable.*

(ii) *If $t = 4$ and $1 \leq |\varphi_3(x) \cap \varphi_3(y)| \leq 2$, then $N(x, y)$ is $(\varphi_3, \{x, y\})$-colorable.*

(iii) *If $t = 5$ and $|\varphi_3(x) \cap \varphi_3(y)| \leq 2$, then $N(x, y)$ is $(\varphi_3, \{x, y\})$-colorable.*

(iv) *If $t \geq 6$, then $N(x, y)$ is $(\varphi_3, \{x, y\})$-colorable.*

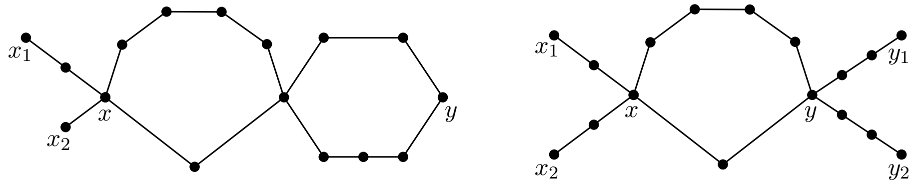
*Proof* (i) The statement is clear when $t = 1, 2$. So we assume $t = 3$ and let $xx_1y_1y$ be a shortest $(x, y)$-path in the necklace $N(x, y)$. If $x$ and $y$ are in the same 7-cycle, then the statement follows by Lemma 3.2 (ii). Otherwise, we may assume, w.l.o.g., that $x_1y_1$ and $y_1y$ are not in a common 7-cycle. So it is enough to color $y_1$ with $\varphi_3(y_1)$ such that $\varphi_3(y_1) \cap \varphi_3(y) = \emptyset$ and $|\varphi_3(y_1) \cap \varphi_3(x)| = 2$. Then by the case $t = 2$ we can extend this coloring $\varphi_3$ to become a fractional $(7 : 3)$-coloring of $N(x, y)$. To construct such a coloring $\varphi_3(y_1)$, we select two colors $\alpha, \beta \in \varphi_3(x) \setminus \varphi_3(y)$ and another color $\gamma \in [7] \setminus (\varphi_3(x) \cup \varphi_3(y))$, and then define $\varphi_3(y_1) = \{\alpha, \beta, \gamma\}$ as desired.

(ii) Let $xx_1zy_1y$ be a shortest path in the necklace $N(x, y)$. Then there exist two consecutive edges in the path $xx_1zy_1y$ that does not belong to a common 7-cycle. By symmetry, we have two cases as follows. If $x_1z$ and $zy_1$ are not in a common 7-cycle, then it suffices to color $z$ with $\varphi_3(z)$ such that $|\varphi_3(x) \cap \varphi_3(z)| = |\varphi_3(z) \cap \varphi_3(y)| = 2$. With an application of (i), this coloring $\varphi_3$ can be extended to a fractional $(7:3)$-coloring of $N(x,y)$. To this end, we select colors $\alpha \in \varphi_3(x) \cap \varphi_3(y)$, $\beta \in \varphi_3(x) \setminus \varphi_3(y)$, and $\gamma \in \varphi_3(y) \setminus \varphi_3(x)$ to set $\varphi_3(z) = \{\alpha, \beta, \gamma\}$ as required. Assume instead that $zy_1$ and $y_1y$ are not in a common 7-cycle. Hence, by applying (i), it suffices to color $y_1$ with $\varphi_3(y_1)$ such that $|\varphi_3(x) \cap \varphi_3(y_1)| = 1$ and $|\varphi_3(y_1) \cap \varphi_3(y)| = 0$. Now we choose two colors $\alpha, \beta \in [7] \setminus (\varphi_3(x) \cup \varphi_3(y))$ and a color $\gamma \in \varphi_3(x) \setminus \varphi_3(y)$ to define $\varphi_3(y_1) = \{\alpha, \beta, \gamma\}$ as required.

(iii) Let $xx_1z_1z_2y_1y$ be a shortest path of length 5 in the necklace $N(x,y)$. By symmetry, in the path $xx_1z_1z_2y_1y$ there are two cases for the existence of two consecutive edges which do not belong to a common 7-cycle. Assume first that $z_1z_2$ and $z_2y_1$ are not in a common 7-cycle. By applying (i), it is enough to color $z_2$ with $\varphi_3(z_2)$ such that $|\varphi_3(x) \cap \varphi_3(z_2)| = 1$ and $|\varphi_3(z_2) \cap \varphi_3(y)| = 2$. So if $1 \le |\varphi_3(x) \cap \varphi_3(y)| \le 2$ we choose $\alpha \in \varphi_3(x) \cap \varphi_3(y)$, $\beta \in \varphi_3(y) \setminus \varphi_3(x)$, and $\gamma \in [7] \setminus (\varphi_3(x) \cup \varphi_3(y))$; if $|\varphi_3(x) \cap \varphi_3(y)| = 0$ we choose $\alpha, \beta \in \varphi_3(y)$ and $\gamma \in \varphi_3(x)$. Hence we can define $\varphi_3(z_2) = \{\alpha, \beta, \gamma\}$ as desired. Now assume instead that $z_2y_1$ and $y_1y$ are not in a common 7-cycle. By applying (i) and (ii), it suffices to color $y_1$ with $\varphi_3(y_1)$ such that $1 \le |\varphi_3(x) \cap \varphi_3(y_1)| \le 2$ and $|\varphi_3(y_1) \cap \varphi_3(y)| = 0$. If $1 \le |\varphi_3(x) \cap \varphi_3(y)| \le 2$ we choose $\alpha \in \varphi_3(x) \setminus \varphi_3(y)$ and $\beta, \gamma \in [7] \setminus (\varphi_3(x) \cup \varphi_3(y))$; if $|\varphi_3(x) \cap \varphi_3(y)| = 0$ we choose $\alpha, \beta \in \varphi_3(x)$ and $\gamma \in [7] \setminus (\varphi_3(x) \cup \varphi_3(y))$. Thus we can define $\varphi_3(y_1) = \{\alpha, \beta, \gamma\}$ as required. This proves (iii).

(iv) Let $x_1$ be the first cut-vertex of $N(x,y)$ in the shortest $(x,y)$-path. That is, the subpath from $x$ to $x_1$ either lies in a common 7-cycle or is an edge $xx_1$. We divide our discussion according to the distance $d(x,x_1) \in \{1,2,3\}$. If $d(x,x_1) = 1$, then $d(x_1,y) \ge 5$. By induction on $t$ and by applying (iii), it is enough to color $x_1$ such that $\varphi_3(x_1) \subset \bar{\varphi}_3(x)$ and $|\varphi_3(x_1) \cap \varphi_3(y)| \le 2$. This can be done since we can select a color $\alpha \in \bar{\varphi}_3(x) \setminus \varphi_3(y)$ and other two colors $\beta, \gamma \in \bar{\varphi}_3(x) \setminus \{\alpha\}$ to formulate $\varphi_3(x_1) = \{\alpha, \beta, \gamma\}$ as required. If $d(x,x_1) = 2$, then $d(x_1,y) \ge 4$. By induction on $t$ and by applying (ii) and (iii), it suffices to color $x_1$ such that $|\varphi_3(x_1) \cap \varphi_3(x)| = 2$ and $1 \le |\varphi_3(x_1) \cap \varphi_3(y)| \le 2$. When $\varphi_3(x) = \varphi_3(y)$, we choose $\alpha, \beta \in \varphi_3(x) = \varphi_3(y)$ and $\gamma \in \bar{\varphi}_3(x)$; when $\varphi_3(x) \ne \varphi_3(y)$, we select $\alpha \in \varphi_3(x) \setminus \varphi_3(y)$, $\beta \in \varphi_3(y) \setminus \varphi_3(x)$, and $\gamma \in \varphi_3(x) \setminus \{\alpha\}$. Thus we can define $\varphi_3(y_1) = \{\alpha, \beta, \gamma\}$ as required. Finally, assume instead that $d(x,x_1) = 3$, and so $d(x_1,y) \ge 3$. By induction on $t$ and by applying (i)–(iii), it suffices to color $x_1$ such that $|\varphi_3(x_1) \cap \varphi_3(x)| = |\varphi_3(x_1) \cap \varphi_3(y)| = 1$. So if $\varphi_3(x) = \varphi_3(y)$, then we choose $\alpha \in \varphi_3(x) = \varphi_3(y)$ and $\beta, \gamma \in \bar{\varphi}_3(x)$; if $\varphi_3(x) \ne \varphi_3(y)$, then we select $\alpha \in \varphi_3(x) \setminus \varphi_3(y)$, $\beta \in \varphi_3(y) \setminus \varphi_3(x)$, and $\gamma \in [7] \setminus (\varphi_3(x) \cup \varphi_3(y))$. Hence we can set $\varphi_3(y_1) = \{\alpha, \beta, \gamma\}$ as desired. This completes the proof. □

Let $H_t(a,b;c,d)$ be the graph obtained from a necklace $N(x,y)$ with $d(x,y) = t$ by adding an $(x_1,x)$-thread and an $(x_2,x)$-thread at $x$ with $d(x,x_1) = a$ and $d(x,x_2) = b$, and adding a $(y_1,y)$-thread and a $(y_2,y)$-thread at $y$ with $d(y,y_1) = c$ and $d(y,y_2) = d$ (possibly $c = d = 0$, and in this case $y = y_1 = y_2$). Define $W = \{x_1, x_2, y_1, y_2\}$ to be the *end vertices* of $H_t(a,b;c,d)$. See Figure 2 for an illustration.




$H_5(2,1;0,0)$ $H_2(2,2;3,3)$

Figure 2 The graphs $H_5(2,1;0,0)$ and $H_2(2,2;3,3)$.

**Lemma 3.4** *Let $W = \{x_1, x_2, y_1, y_2\}$ be the end vertices of $H_t(a,b;c,d)$, and let $\varphi_3$ be a precoloring of $W$.*

(i) *If $|\varphi_3(x_1) \cap \varphi_3(x_2)| = 1$, then $H_5(2,1;0,0)$ and $H_4(2,2;0,0)$ are $(\varphi_3, W)$-colorable.*

(ii) *If $|\varphi_3(x_1) \cap \varphi_3(x_2)| = 0$, then $H_3(3,3;0,0)$ is $(\varphi_3, W)$-colorable.*

(iii) *If $|\varphi_3(x_1) \cap \varphi_3(x_2)| = 1$ and $|\varphi_3(y_1) \cap \varphi_3(y_2)| = 1$, then $H_3(2,2;2,2)$ is $(\varphi_3, W)$-colorable.*

(iv) *If $|\varphi_3(x_1) \cap \varphi_3(x_2)| = 1$ and $|\varphi_3(y_1) \cap \varphi_3(y_2)| = 0$, then $H_2(2,2;3,3)$ is $(\varphi_3, W)$-colorable.*

*Proof* (i) To show that $H_5(2,1;0,0)$ is $(\varphi_3, W)$-colorable, it is enough to color $x$ with $\varphi_3(x)$ such that $|\varphi_3(x) \cap \varphi_3(x_1)| = 2$, $|\varphi_3(x) \cap \varphi_3(x_2)| = 0$ and $|\varphi_3(x) \cap \varphi_3(y)| \le 2$, and then the rest follows from Lemma 3.1 and Lemma 3.3 (iii). To this end, we choose a color $\alpha \in \bar{\varphi}_3(x_2) \setminus \varphi_3(y)$, and then choose other two colors $\beta, \gamma \in \bar{\varphi}_3(x_2) \setminus \{\alpha\}$ appropriately such that $|\{\alpha, \beta, \gamma\} \cap \varphi_3(x_1)| = 2$. This is possible since $|\varphi_3(x_1) \cap \varphi_3(x_2)| = 1$ and $|\bar{\varphi}_3(x_1) \cap \varphi_3(x_2)| = 2$. Now we define $\varphi_3(x) = \{\alpha, \beta, \gamma\}$, which provides a desired coloring with $|\varphi_3(x) \cap \varphi_3(y)| \le 2$ since $\alpha \notin \varphi_3(y)$.

To verify that $H_4(2,2;0,0)$ is $(\varphi_3, W)$-colorable, by Lemma 3.1 and Lemma 3.3 (ii), it suffices to color $x$ with $\varphi_3(x)$ such that $|\varphi_3(x) \cap \varphi_3(x_1)| = |\varphi_3(x) \cap \varphi_3(x_2)| = 2$ and $1 \le |\varphi_3(x) \cap \varphi_3(y)| \le 2$. Denote $\varphi_3(x_1) \cap \varphi_3(x_2) = \{\alpha\}$. If $\alpha \in \varphi_3(y)$, then we select a color $\beta \in (\varphi_3(x_1) \cup \varphi_3(x_2)) \setminus \varphi_3(y)$, w.l.o.g., say $\beta \in \varphi_3(x_1) \setminus \varphi_3(y)$, and then we choose a color $\gamma \in \varphi_3(x_2) \setminus \{\alpha\}$. If $\alpha \notin \varphi_3(y)$, then we choose a color $\beta \in (\varphi_3(x_1) \cup \varphi_3(x_2)) \cap \varphi_3(y)$, w.l.o.g., say $\beta \in \varphi_3(x_1) \cap \varphi_3(y)$, and then we select a color $\gamma \in \varphi_3(x_2) \setminus \{\alpha\}$. Thus we can define $\varphi_3(x) = \{\alpha, \beta, \gamma\}$, which satisfies $1 \le |\varphi_3(x) \cap \varphi_3(y)| \le 2$ as desired.

(ii) For proving that $H_3(3,3;0,0)$ is $(\varphi_3, W)$-colorable, we shall color $x$ with $\varphi_3(x)$ such that $|\varphi_3(x) \cap \varphi_3(x_1)| = |\varphi_3(x) \cap \varphi_3(x_2)| = |\varphi_3(x) \cap \varphi_3(y)| = 1$, and so the statement holds by Lemma 3.1 and Lemma 3.3 (i). Assume that $|\varphi_3(x_1) \cap \varphi_3(y)| \ge |\varphi_3(x_2) \cap \varphi_3(y)|$. As $|\varphi_3(x_1) \cap \varphi_3(x_2)| = 0$, we have $|\varphi_3(x_1) \cap \varphi_3(y)| \ge 1$. When $|\varphi_3(x_1) \cap \varphi_3(y)| = 3$, we select $\alpha \in \varphi_3(x_1)$, $\beta \in \beta(x_2)$, and $\gamma \in [7] \setminus (\varphi_3(x_1) \cup \varphi_3(x_2))$. When $|\varphi_3(x_1) \cap \varphi_3(y)| = 2$, we choose $\alpha \in \varphi_3(x_1) \setminus \varphi_3(y)$ and $\beta \in \varphi_3(y) \setminus \varphi_3(x_1)$. Let $\gamma \in \varphi_3(x_2)$ if $\beta \notin \varphi_3(x_2)$, and let $\gamma \in [7] \setminus (\varphi_3(x_1) \cup \varphi_3(x_2))$ if $\beta \in \varphi_3(x_2)$. When $|\varphi_3(x_1) \cap \varphi_3(y)| = 1$, let $\alpha \in \varphi_3(x_1) \setminus \varphi_3(y)$, $\beta \in \varphi_3(x_2) \setminus \varphi_3(y)$, and $\gamma \in \varphi_3(y) \setminus (\varphi_3(x_1) \cup \varphi_3(x_2))$. In any case, we always define $\varphi_3(x) = \{\alpha, \beta, \gamma\}$ satisfying $|\varphi_3(x) \cap \varphi_3(x_1)| = |\varphi_3(x) \cap \varphi_3(x_2)| = |\varphi_3(x) \cap \varphi_3(y)| = 1$ as required. This proves (ii).

(iii) For convenience, we may assume that $\varphi_3(x_1) = \{1,2,3\}$, $\varphi_3(x_2) = \{1,4,5\}$, $\varphi_3(y_1) = \{\alpha, \beta_1, \gamma_1\}$, and $\varphi_3(y_2) = \{\alpha, \beta_2, \gamma_2\}$, where $\alpha, \beta_1, \gamma_1, \beta_2, \gamma_2$ are distinct colors.

Now we show that $H_3(2,2;2,2)$ is $(\varphi_3, W)$-colorable. The arguments are divided into three cases as follows. First, assume that $\alpha = 1$. As $\{2,3,4,5\} \cap \{\beta_1, \beta_2, \gamma_1, \gamma_2\} \neq \emptyset$, we may, w.l.o.g., assume that $\beta_1 = 2$. Then we choose $\theta_1 \in \{4,5\} \setminus \{\gamma_1\}$ and define $\varphi_3(x) = \{1, 2, \theta_1\}$. Similarly, we choose $\theta_2 \in \{\beta_2, \gamma_2\} \setminus \{\theta_1\}$ and define $\varphi_3(y) = \{1, \gamma_1, \theta_2\}$. Hence we have $|\varphi_3(x) \cap \varphi_3(y)| = 1$, and we can extend $\varphi_3$ to become a fractional $(7:3)$-coloring of $H_3(2,2;2,2)$ by Lemma 3.1 and Lemma 3.3 (i). Second, we assume that $\alpha \in \{2,3,4,5\}$, and w.l.o.g., say $\alpha = 2$. Recall that $\beta_1, \gamma_1, \beta_2, \gamma_2$ are distinct colors, and we have $1 \notin \{\beta_1, \gamma_1\}$ or $1 \notin \{\beta_2, \gamma_2\}$. W.l.o.g., we assume that $1 \notin \{\beta_2, \gamma_2\}$. Note that $1 \neq \beta_1$ or $1 \neq \gamma_1$, say $1 \neq \beta_1$. Choose $\theta_1 \in \{4,5\} \setminus \{\beta_1\}$ and $\theta_2 \in \{\beta_2, \gamma_2\} \setminus \{\theta_1\}$. Define $\varphi_3(x) = \{1, 2, \theta_1\}$ and $\varphi_3(y) = \{2, \beta_1, \theta_2\}$. Hence we have $|\varphi_3(x) \cap \varphi_3(y)| = 1$, and thus we are done by Lemma 3.1 and Lemma 3.3 (i). At last, we assume $\alpha \in \{6,7\}$, say $\alpha = 6$. Note that $|\{2,3,4,5\} \cap \{\beta_1, \gamma_1, \beta_2, \gamma_2\}| \geq 1$, w.l.o.g., say $\beta_1 = 2$. Choose $\theta_2 \in \{\beta_2, \gamma_2\} \setminus \{1\}$ and $\theta_1 \in \{4,5\} \setminus \{\theta_2\}$. Define $\varphi_3(x) = \{1, 2, \theta_1\}$ and $\varphi_3(y) = \{2, 6, \theta_2\}$. Hence always get $|\varphi_3(x) \cap \varphi_3(y)| = 1$, and we are done by Lemma 3.1 and Lemma 3.3 (i). This finishes the proof of (iii).

(iv) For convenience, we may denote $\varphi_3(y_1) = \{1,2,3\}$, $\varphi_3(y_2) = \{4,5,6\}$, $\varphi_3(x_1) = \{\alpha, \beta_1, \gamma_1\}$, and $\varphi_3(x_2) = \{\alpha, \beta_2, \gamma_2\}$, where $\alpha, \beta_1, \gamma_1, \beta_2, \gamma_2$ are all distinct colors. When $\alpha = 7$, as $\beta_1 \in \{1,2,3,4,5,6\}$, we may, w.l.o.g., assume $\beta_1 = 1$. Now we define $\varphi_3(x) = \{1, 7, \beta_2\}$ and choose $\theta \in \{4,5,6\} \setminus \{\beta_2\}$ to define $\varphi_3(y) = \{1, 7, \theta\}$. When $\alpha \neq 7$, we may, w.l.o.g., assume $\alpha = 1$. As $\{\alpha, \beta_1, \gamma_1, \beta_2, \gamma_2\} \cap \{4,5,6\} \neq \emptyset$, we may, w.l.o.g., assume $\beta_1 = 4$. Then we define $\varphi_3(y) = \{1,4,7\}$ and choose $\theta \in \{\beta_2, \gamma_2\} \setminus \{7\}$ to define $\varphi_3(x) = \{1, 4, \theta\}$. Thus we always have $|\varphi_3(x) \cap \varphi_3(y)| = 2$ in any case, and then $H_2(2,2;3,3)$ is $(\varphi_3, W)$-colorable by Lemma 3.1 and Lemma 3.3 (i). This completes the proof. □

## 3.2 Completing the Proof of Theorem 1.8

Now we prove Theorem 1.8 restated below for convenience.

**Theorem 1.8.** *Every plane graph of girth at least* 7 *without cycles of length from* 8 *to* 35 *is fractional* $(7:3)$*-colorable.*

*Proof* By contradiction, suppose that Theorem 1.8 is false. Let $G$ be a counterexample with $|V(G)| + |E(G)|$ minimized. Then we have the following claim, whose proof is the same as that of Claim 2.1 and thus omitted.

**Claim 3.1** $G$ is 2-connected. In particular, $\delta(G) \geq 2$.

For $3 \geq a \geq b \geq 1$, define $B_t(a,b;0,0)$ as the graph obtained from an $H_t(a,b;0,0)$ by joining a new $(x_1,x_2)$-path of length $7-a-b$, where the vertices in the new $(x_1,x_2)$-path (including $x_1, x_2$) may have arbitrary degrees in $G$. Let

$$\mathcal{B}_1 = \{B_5(2,1;0,0), B_4(2,2;0,0), B_3(3,3;0,0)\}.$$

For $3 \geq a \geq b \geq 1$ and $3 \geq c \geq d \geq 1$, define $B_t(a,b;c,d)$ to be the graph obtained from an $H_t(a,b;c,d)$ by joining a new $(x_1,x_2)$-path of length $7-a-b$ and a new $(y_1,y_2)$-path of length $7-c-d$, where the vertices in each new $(x_1,x_2)$-path and new $(y_1,y_2)$-path (including $x_1, x_2, y_1, y_2$) may have arbitrary degrees in $G$. Denote

$$\mathcal{B}_2 = \{B_3(2,2;2,2), B_2(2,2;3,3)\} \quad \text{and} \quad \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2.$$

That is, the graphs in $\mathcal{B}_1$ consist of a necklace $N(x,y)$ and a 7-cycle $C_x$ with a common vertex $x$, where in $C_x$ there exist an $(a-1)$-thread and a $(b-1)$-thread starting at $x$; the graphs in $\mathcal{B}_2$ consist of a necklace $N(x,y)$ and two 7-cycles $C_x$ and $C_y$ with common vertices $x$ and $y$, respectively, where in $C_x$ there exist an $(a-1)$-thread and a $(b-1)$-thread starting at $x$ and in $C_y$ there exist a $(c-1)$-thread and a $(d-1)$-thread starting at $y$.

**Claim 3.2** (i) $G$ contains no necklace $N(x,y)$ with $d_G(x,y) \geq 6$.

(ii) $G$ contains none of the graphs in $\mathcal{B}$.

*Proof of Claim* 3.2 (i) Suppose to the contrary that $G$ contains a necklace $N(x,y)$ with $d_G(x,y) \geq 6$. By the minimality of $G$, $G - (V(N(x,y)) \setminus \{x,y\})$ has a fractional $(7:3)$-coloring $\varphi$. By Lemma 3.3 (iv), $\varphi$ can be extended to a fractional $(7:3)$-coloring of $N(x,y)$, and thus it results in a fractional $(7:3)$-coloring of $G$, a contradiction.

(ii) Let $B$ be a graph in $\mathcal{B}$ with end vertices $x_1, x_2, y_1, y_2$. (In some situation, we may have $y_1 = y_2 = y$.) Then $G - (V(B) \setminus \{x_1, x_2, y_1, y_2\})$ admits a fractional $(7:3)$-coloring $\varphi$ by the minimality of $G$. Applying Lemma 3.4 (i)–(iv), the precoloring $\{\varphi(x_1), \varphi(x_2), \varphi(y_1), \varphi(y_2)\}$ of $\{x_1, x_2, y_1, y_2\}$ can be extended to a fractional $(7:3)$-coloring of $B$. Combining the coloring $\varphi$ of $G - V(B)$, we obtain a fractional $(7:3)$-coloring of $G$, a contradiction.

From $G$, we obtain a subgraph $G'$ as follows: *for each facial* 7*-cycle $C$ of $G$, if there exists a* 2*-vertex in $C$, then we delete all the* 2*-vertices of a longest thread of $C$*. Clearly, the obtained graph $G'$ is a plane graph of girth at least 7, and it contains no cycles of length from 8 to 35; moreover, each facial 7-cycle of $G'$ contains no 2-vertices. It is also easy to check that $G'$ has minimal degree at least 2 by its construction.

Let $T(v_0, v_{t+1})$ be a $(v_0, v_{t+1})$-thread of $G'$. If $v_0v_1$ is in a facial 7-cycle $C_0$ of $G$ whose 2-vertices of a longest thread is deleted in $G'$, then we say that $v_0$ is a *bad end vertex* of $T(v_0, v_{t+1})$; otherwise, $v_0$ is called a *good end vertex* of $T(v_0, v_{t+1})$.

**Claim 3.3** Let $T(v_0, v_{t+1}) = v_0v_1v_2 \dots v_tv_{t+1}$ be a $t$-thread of $G'$. If $v_0$ is a good end vertex of $T(v_0, v_{t+1})$, then $t \leq 4$ (i.e., $d(v_0, v_{t+1}) \leq 5$).

*Proof of Claim* 3.3. Suppose to the contrary that $t \geq 5$. If $v_{t+1}$ is a good end vertex of $T(v_0, v_{t+1})$, then the thread $T(v_0, v_{t+1})$ in $G'$ transfers to a necklace $N(v_0, v_{t+1})$ in $G$ with $d_G(v_0, v_{t+1}) = t + 1 \geq 6$, which is a contradiction to Claim 3.2 (i). So we assume $v_tv_{t+1}$ is in a facial 7-cycle $C_t$ of $G$ whose 2-vertices of a longest thread is deleted in $G'$. We denote $j$ be the index such that $v_{j-1}v_j \notin E(C_t)$ and $v_jv_{j+1} \in E(C_t)$, and we define $y = v_j$. By the construction of $G'$, we have $j \geq t-2$. If $j = t-2$, then $G$ contains a $B_3(3,3;0,0)$; if $j = t-1$, then $G$ contains a $B_4(2,2;0,0)$; and if $j = t$, then $G$ contains a $B_5(2,1;0,0)$. Thus $G$ contains a graph in $\mathcal{B}_1$, a contradiction to Claim 3.2 (ii).

**Claim 3.4** $G'$ contains no $8^+$-thread.

*Proof of Claim* 3.4 Suppose to the contrary that $G'$ has an $8^+$-thread $T(v_0, v_{t+1}) = v_0v_1v_2 \cdots v_tv_{t+1}$ with $t \geq 8$. By Claim 3.3, we have that $v_0$ and $v_{t+1}$ are bad end vertices of $T(v_0, v_{t+1})$. Let $C_0$ be the 7-cycle of $G$ containing $v_0v_1$ whose 2-vertices of a longest thread is deleted in $G'$. We denote by $i$ the index such that $v_{i-1}v_i \in E(C_0)$ and $v_iv_{i+1} \notin E(C_0)$. By the construction of $G'$, we have $i \leq 3$. Then $T(v_i, v_{t+1}) = v_iv_{i+1}v_{i+2} \dots v_tv_{t+1}$ is a $(t-i)$-thread (where $t - i \geq 5$) of $G'$ with $v_i$ being a good end vertex, which is a contradiction to Claim 3.3.

**Claim 3.5** $G'$ contains no $(k_1, k_2, k_3)$-thread such that $k_1 + k_2 + k_3 \geq 16$.

*Proof of Claim* 3.5 Suppose to the contrary that $G'$ has a $(k_1, k_2, k_3)$-thread with center vertex $x$ and end vertices $u, v, w$ such that $d_{G'}(x,u) = k_1 + 1$, $d_{G'}(x,v) = k_2 + 1$, $d_{G'}(x,w) = k_3 + 1$

and $k_1+k_2+k_3 \geq 16$. Let $xu_1$ ($xv_1, xw_1$, resp.) be the edge incident with $x$ in the $(x,u)$-thread ($(x,v)$-thread, $(x,w)$-thread, resp.) of $G'$. Assume that any two of $xu_1, xv_1, xw_1$ are not in a common facial 7-cycle of $G$. Note that

$$\max\{d_{G'}(x,u), d_{G'}(x,v), d_{G'}(x,w)\} \geq \left\lceil \frac{k_1+k_2+k_3}{3}+1 \right\rceil \geq 7,$$

w.l.o.g., say $d_{G'}(x,u) \geq 7$. Hence $G'$ contains an $(x,u)$-thread with $x$ being a good end vertex, a contradiction to Claim 3.3.

Assume instead that two of $xu_1, xv_1, xw_1$ are in a common facial 7-cycle $C_x$ of $G$ whose 2-vertices of a longest thread is deleted in $G'$, we may, w.l.o.g., assume $xv_1, xw_1 \in E(C_x)$ and $xu_1 \notin E(C_x)$. Let $v'$ be the common neighbor of $C_x$ and the $(x,v)$-thread $T(x,v)$ such that $d_G(x,v')$ is as large as possible, and let $w'$ be the common vertex of $C_x$ and the $(x,w)$-thread $T(x,w)$ such that $d_G(x,w')$ is as large as possible. By Claim 3.3, we have that $d_{G'}(x,u) \leq 5$, $d_{G'}(v',v) \leq 5$ and $d_{G'}(w',w) \leq 5$. Then $d_{G'}(x,v') + d_{G'}(x,w') \geq 19-5-5-5 = 4$. By the construction of $G'$, we have $d_{G'}(x,v') + d_{G'}(x,w') \leq \frac{2\times 7}{3}$. Hence $d_{G'}(x,v') + d_{G'}(x,w') = 4$, $d_{G'}(v,w') = 3$, $d_{G'}(x,u) = 5$, $d_{G'}(v',v) = 5$ and $d_{G'}(w',w) = 5$. Note that $d_{G'}(x,v') \geq 2$ or $d_{G'}(x,w') \geq 2$, say $d_{G'}(x,v') \geq 2$. If $v$ is a good end vertex of $T(v,v')$, then the thread $T(v,v')$ in $G'$ transfers to a necklace $N(v,v')$ in $G$, and hence $G$ contains a $B_5(2,1;0,0)$, a contradiction to Claim 3.2 (ii). Assume that $C_v$ is the 7-cycle of $G$ containing $vy$ whose 2-vertices of a longest thread is deleted in $G'$, where $y$ is the neighbor of $v$ in the $(v,x)$-thread. Let $y'$ be the common neighbor of $C_v$ and the $(v,x)$-thread $T(v,x)$ such that $d_G(v,y')$ is as large as possible. By the construction of $G'$, we have $d(v,y') \leq 3$. Notice that the thread $T(y',v')$ in $G'$ transfers to a necklace $N(y',v')$ in $G$. If $d(v,y')=3$, then $G$ contains a $B_2(3,3;2,2)$; and if $d(v,y')=2$, then $G$ contains a $B_3(2,2;2,2)$; and if $d(v,y')=1$, then $G$ contains a $B_4(2,2;2,1)$, hence $G$ contains a $B_4(2,2;0,0)$. In any case, $G$ contains a graph in $\mathcal{B}$, a contradiction to Claim 3.2 (ii).

Now we are ready to complete the proof by a discharging method on $G'$.

Let $F(G')$ be the set of faces of $G'$. From Euler Formula, we have

$$\sum_{v\in V(G')} \left(\frac{5}{2}d_{G'}(v)-7\right) + \sum_{f\in F(G')} (d_{G'}(f)-7) = -14. \tag{3.1}$$

Assign an initial charge $ch_0(v) = \frac{5}{2}d_{G'}(v)-7$ for each $v \in V(G')$, and $ch_0(f) = d_{G'}(f)-7$ for each $f \in F(G')$. Hence the total charge is $-14$ by Eq. (3.1).

We redistribute the charges according to the following rules.

(RI) *Every* $36^+$*-face of* $G'$ *gives charge* $\frac{29}{36}$ *to each of its incident vertices.*

(RII) *Every* $3^+$*-vertex of* $G'$ *gives charge* $\frac{7}{36}$ *to each of its weakly adjacent* 2*-vertices.*

Let $ch$ denote the charge assignment after performing the charge redistribution using rules (RI) and (RII).

**Claim 3.6** We have $ch(f) \geq 0$ for each $f \in F(G')$ and $ch(v) \geq 0$ for each $v \in V(G')$.

*Proof of Claim* 3.6 Clearly, each 7-face $f$ has charge $ch(f) = ch_0(f) = 0$. Each $36^+$-face $f$ sends charge $\frac{29}{36}$ to each incident vertices by (RI). So

$$ch(f) = ch_0(f) - \frac{29}{36}d_{G'}(f) = d_{G'}(f) - 7 - \frac{29}{36}d_{G'}(f) = \frac{7}{36}d_{G'}(f) - 7 \geq 0.$$

Hence $ch(f) \geq 0$ for each $f \in F(G')$, and it remains to show that $ch(v) \geq 0$ for each $v \in V(G')$.

First we assume $d_{G'}(v) = 2$. Then $ch_0(v) = -2$. By Claims 3.1 and 3.4, $v$ is weakly adjacent to two $3^+$-vertices, and thus $v$ receives charge $\frac{7}{36} \times 2$ from them by (RII). By (RI), $v$ receives charge $\frac{29}{36} \times 2$ from its two incident faces. Thus $ch(v) = -2 + \frac{7}{36} \times 2 + \frac{29}{36} \times 2 = 0$.

Next we assume $d_{G'}(v) \geq 3$. Let $t(v)$ be the number of 2-vertices weakly adjacent to $v$. Suppose $v$ is adjacent to $r(v)$ facial 7-cycles. Since $G'$ contains no cycles of length from 8 to 35, any two 7-cycles of $G'$ have no common edge, and thus $r(v) \leq \frac{d_{G'}(v)}{2}$. By Claim 3.4 and by the construction of $G'$, each thread incident with $v$ contains at most seven 2-vertices and each 7-cycle contains no 2-vertices, and so we have $t(v) \leq 7(d_{G'}(v) - 2r(v))$. By (RI), $v$ receives charge $\frac{29}{36}(d_{G'}(v) - r(v))$ from its incident faces. By (RII), $v$ sends 7/36 to each of its weakly adjacent 2-vertices. Therefore, we have

$$ch(v) = \left(\frac{5}{2}d_{G'}(v) - 7\right) + \frac{29}{36}\left(d_{G'}(v) - r(v)\right) - \frac{7}{36}t(v). \tag{3.2}$$

Assume that $d_{G'}(v) \geq 4$. By Eq. (3.2), it follows from $t(v) \leq 7(d_{G'}(v) - 2r(v))$ that

$$\begin{aligned} ch(v) &\geq \frac{5}{2}d_{G'}(v) - 7 + \frac{29}{36}(d_{G'}(v) - r(v)) - \frac{7}{36} \cdot 7(d_{G'}(v) - 2r(v)) \\ &= \frac{35}{18}d_{G'}(v) - 7 + \frac{69}{36}r(v) \\ &\geq \frac{35}{18} \cdot 4 - 7 = \frac{7}{9} > 0. \end{aligned}$$

Assume instead that $d_{G'}(v) = 3$. Then $ch_0(v) = \frac{1}{2}$ and $r(v) \leq 1$. If $r(v) = 1$, then $t(v) \leq 7$ by Claim 3.4. Thus by Eq. (3.2) we have

$$ch(v) \geq \frac{1}{2} + \frac{29}{36} \cdot 2 - \frac{7}{36} \cdot 7 = \frac{27}{36} > 0.$$

If $r(v) = 0$, then $t(v) \leq 15$ by Claim 3.5. Thus it follows from Eq. (3.2) that

$$ch(v) \geq \frac{1}{2} + \frac{29}{36} \cdot 3 - \frac{7}{36} \cdot 15 = 0.$$

This proves Claim 3.6.

By Eq. (3.1) and Claim 3.6, we have

$$-14 = \sum_{v \in V(G')} ch_0(v) + \sum_{f \in F(G')} ch_0(f) = \sum_{v \in V(G')} ch(v) + \sum_{f \in F(G')} ch(f) \geq 0,$$

a contradiction. This contradiction finishes the proof of Theorem 1.8. □

## 4 Concluding Remarks

In this paper, we complete the proof of the fractional version of Conjecture 1.4, namely, every planar graph of girth $p$ without cycles of length from $p+1$ to $p(p-2)$ is fractional $(p : \frac{p-1}{2})$-colorable for any prime $p \geq 5$. However, the related fractional version of Conjecture 1.1 is still open for $k \geq 3$. In view of Theorem 1.3 (ii), it would be interesting to attempt to make an improvement on its fractional version, that is, to show that every planar graph of girth at least 14 is fractional $(7 : 3)$-colorable. There are also some versions of those circular and fractional coloring problems concerning forbidden odd cycles, and we refer the readers to [11] for more details.

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