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# Realizing Degree Sequences With $\mathcal{S}_3$-Connected Graphs

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## ABSTRACT

A graph $G$ is $\mathcal{S}_3$-connected if, for any mapping $\beta : V(G) \mapsto \mathbb{Z}_3$ with $\sum_{v\in V(G)}\beta(v) \equiv 0 \pmod 3$, there exists a strongly connected orientation $D$ satisfying $d_D^+(v) - d_D^-(v) \equiv \beta(v) \pmod 3$ for any $v \in V(G)$. It is known that $\mathcal{S}_3$-connected graphs are contractible configurations for the property of flow index strictly less than three. In this paper, we provide a complete characterization of graphic sequences that have an $\mathcal{S}_3$-connected realization: A graphic sequence $\pi = (d_1, ..., d_n)$ has an $\mathcal{S}_3$-connected realization if and only if $\min\{d_1, ..., d_n\} \geq 4$ and $\sum_{i=1}^n d_i \geq 6n - 4$. Consequently, every graphic sequence $\pi = (d_1, ..., d_n)$ with $\min\{d_1, ..., d_n\} \geq 6$ has a realization $G$ with flow index strictly less than three. This supports the conjecture of Li, Thomassen, Wu and Zhang [European J. Combin., 70 (2018) 164-177] that every 6-edge-connected graph has a flow index strictly less than three.

## 1 | Introduction

Graphs studied here are finite and may have multiple edges but no loops. A graph is referred to as simple if it contains neither multiple edges nor loops. Let $G = (V(G), E(G))$ be a graph with an orientation $D$. For a vertex $v \in V(G)$, we use $E_D^+(v)$ (or $E_D^-(v)$, respectively) to denote the set of edges with tails (or heads, respectively) at $v$, and use $d_D^+(v)$ (or $d_D^-(v)$, respectively) to denote their sizes. The subscript $D$ may be omitted when $D$ is understood from the context. Terms and notations not defined here are referred to [1].

A function $\beta : V(G) \mapsto \mathbb{Z}_3$ is called a $\mathbb{Z}_3$ boundary function if $\sum_{v\in V(G)}\beta(v) \equiv 0 \pmod 3$. A graph $G$ is $\mathbb{Z}_3$-connected if, for any $\mathbb{Z}_3$ boundary function $\beta$ of $G$, there exists an orientation $D$ of $G$ such that $d_D^+(v) - d_D^-(v) \equiv \beta(v) \pmod 3$ for any $v \in V(G)$. It is well-known that a graph admits a nowhere-zero 3-flow if and only if it admits a modulo 3-orientation [2, 3], i.e., an orientation such that the indegree is congruent to the outdegree modulo 3 for each vertex. As mentioned

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in [4], $\mathbb{Z}_3$-connectivity is a generalization of the concept of nowhere-zero 3-flows and serves as a potent tool in the study of nowhere-zero 3-flows. It is proved by Lovász, Thomassen, Wu and Zhang in [5] that every 6-edge-connected graph is $\mathbb{Z}_3$-connected and therefore admits a nowhere-zero 3-flow.

In [6], the concept of strongly connected modulo 3-orientation is introduced to study the property of flow index strictly less than three. An orientation is strongly connected if, for any two vertices $x, y \in V(G)$, there is a directed path from $x$ to $y$. It is shown in [6] that a graph has flow index strictly less than three if and only if it has a strongly connected modulo 3-orientation. Similar to the definition of $\mathbb{Z}_3$-connectivity, a graph $G$ is defined to be $\mathcal{S}_3$-connected if, for any $\mathbb{Z}_3$ boundary function $\beta$ of $G$, there is a strongly connected orientation $D$ of $G$ such that $d_D^+(v) - d_D^-(v) \equiv \beta(v) \pmod 3$ for any $v \in V(G)$. Note that $\mathcal{S}_3$-connected graphs are contractible configurations for the property of flow index strictly less than three (see [7, 8]). As an expected improvement of the result of [5], Li, Thomassen, Wu and Zhang [6] proposed the following conjecture.

**Conjecture 1.1** ([6]). *Every 6-edge-connected graph has a strongly connected modulo 3-orientation.*

Note that $K_6$ does not have a strongly connected modulo 3-orientation (see [6]), and therefore the edge-connectivity condition cannot be relaxed in the conjecture. It is proved in [6] that Conjecture 1.1 is true for 8-edge-connected graphs. Additionally, Conjecture 1.1 was also verified for some families of graphs in [7].

An integer-valued sequence $\pi = (d_1, \ldots, d_n)$ is graphic if there is a nontrivial simple graph $G$ with degree sequence $\pi$. In this case, the graph $G$ is referred to as a realization of $\pi$. Denote by $\mathcal{GS}$ the set of all integer-valued non-increasing graphic sequences. Let $\overline{\pi} = (n-1-d_1, \ldots, n-1-d_n)$ be the complementary sequence of $\pi$. Obviously, $\overline{\pi}$ is graphic if and only if $\pi$ is graphic. If an integer-valued non-increasing sequence $\pi$ has a realization $G$ that is simple and $\mathcal{S}_3$-connected (or $\mathbb{Z}_3$-connected, respectively), then we say that $G$ is an $\mathcal{S}_3$-connected (or a $\mathbb{Z}_3$-connected, respectively) realization of $\pi$, denoted by $\pi \in \mathcal{GS}(\mathcal{S}_3)$ (or $\pi \in \mathcal{GS}(\mathbb{Z}_3)$, respectively). Clearly, we can reorder each graphic sequence to obtain a non-increasing one without affecting any graph properties. Therefore, for the remainder of this paper, we always assume that all graphic sequences are non-increasing unless otherwise specified. For simplicity, we use exponents to denote degree multiplicities in a graphic sequence. For example, we write $(6^3, 5^4)$ to represent $(6, 6, 6, 5, 5, 5, 5)$.

The question of characterizing degree sequences with realizations that are $\mathbb{Z}_3$-connected or have nowhere-zero 3-flows has been well-studied. Solving the open problem posed by Archdeacon [9], Luo et al. [10] provided a complete characterization of graphic sequences with realizations having nowhere-zero 3-flows.

**Theorem 1.2** ([10]). *Let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence with $d_1 \geq \cdots \geq d_n \geq 2$. Then the sequence $\pi$ has a realization that admits a nowhere-zero 3-flow if and only if $\pi \neq (3^4, 2), (k, 3^k), (k^2, 3^{k-1})$, where $k$ is an odd integer.*

Furthermore, in [10] and [11], the authors proposed the question of characterizing all graphic sequences with $\mathbb{Z}_3$-connected realizations, and this question was finally solved by Dai and Yin [12]. For $n \geq 5$, let $S_1(n) = \{((n-1)^2, 3^{n-k-2}, 2^k) | 0 \leq k \leq n-4, k \equiv n \pmod 2\}$, and let $S_2(n) = \{(d_1, d_2, d_3, d_4, 2^{n-4}) | n-1 \geq d_1 \geq d_2 \geq d_3 \geq d_4 \geq 3, d_1 + d_2 + d_3 + d_4 = 2n+4\}$. Denote

$$R(n) = \begin{cases} S_1(n) \cup S_2(n), & \text{if } n \text{ is odd;} \\ S_1(n) \cup S_2(n) \cup \{(n-1, 3^{n-1})\}, & \text{if } n \text{ is even.} \end{cases}$$

**Theorem 1.3** ([12]). *Let $n \geq 5$, and let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence with $d_1 \geq \cdots \geq d_n \geq 2$. Then the sequence $\pi$ has a realization that is $\mathbb{Z}_3$-connected if and only if $\sum_{i=1}^n d_i \geq 4n-4$ and $\pi \notin R(n)$, where $R(n)$ is the well-characterized set defined above.*

Similar to the work mentioned above, it is natural to study the degree sequence realization problem concerning the property of flow index strictly less than three or $\mathcal{S}_3$-connectivity. Motivated by Conjecture 1.1 and the aforementioned studies, this paper presents a complete characterization of graphic sequences with $\mathcal{S}_3$-connected realizations.

**Theorem 1.4.** *Let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence with $d_1 \geq \cdots \geq d_n > 0$. Then the sequence $\pi$ has a realization that is $\mathcal{S}_3$-connected if and only if $\sum_{i=1}^n d_i \geq 6n-4$ and $d_n \geq 4$.*

Consequently, Theorem 1.4 implies the following corollary, which provides some supports for Conjecture 1.1.



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**Corollary 1.5.** *Every graphic sequence $\pi = (d_1, ..., d_n)$ with $\min\{d_1, ..., d_n\} \geq 6$ has a realization with a strongly connected modulo 3-orientation.*

The main results of this paper will be served as a tool for characterizing all graphic sequences that have strongly connected modulo 3-orientation realizations in future studies.

The remainder of this paper is organized as follows. In Section 2, we state some results that are used in the following proofs. In Section 3, we handle some special cases and show that $\pi \in \mathcal{GS}(\mathcal{S}_3)$ when there are at least two high degrees. In Section 4, we discuss the situation in which graphic sequences cannot be reduced in order by applying laying off and lifting operations. In Section 5, we prove the main theorem, which fully characterizes all graphic sequences that belong to $\mathcal{GS}(\mathcal{S}_3)$.

## 2 | Preliminaries

In this section, we present some foundational lemmas and special graphs required to prove the main theorem.

### 2.1 | Some $\mathcal{S}_3$-Connected Graphs and the Contraction Methods

Let $\langle \mathcal{S}_3 \rangle$ (or $\langle \mathbb{Z}_3 \rangle$, respectively) be the family of graphs that are $\mathcal{S}_3$-connected (or $\mathbb{Z}_3$-connected, respectively). At first, we list some special graphs that belong to $\langle \mathcal{S}_3 \rangle$, as they are crucial to our proof. Let $K_n$ represent the complete graph on $n$ vertices. Let $mK_2$ denote the graph with two vertices and $m$ parallel edges. The graphs $K_{(1,3,3)}$ and $K_4^*$ are defined to be the graphs respectively depicted in Figure 1.

**Lemma 2.1** ([8]). *Each of the following holds:*

i. $K_n \in \langle \mathcal{S}_3 \rangle$ *if and only if* $n \geq 7$.
ii. $mK_2 \in \langle \mathcal{S}_3 \rangle$ *if and only if* $m \geq 4$.
iii. $K_{(1,3,3)}, K_4^* \in \langle \mathcal{S}_3 \rangle$.

As observed in [7], $\mathcal{S}_3$-connected graphs and $\mathbb{Z}_3$-connected graphs are closely related. Specifically, by adding an arbitrary Hamiltonian cycle to any $\mathbb{Z}_3$-connected graph, we can construct an $\mathcal{S}_3$-connected graph. The presence of the Hamiltonian cycle ensures that the resulting graph is strongly connected while preserving the boundary of each vertex.

**Lemma 2.2** ([7]). *Let $G$ be a graph with a Hamiltonian cycle $C$. If $G - E(C)$ is $\mathbb{Z}_3$-connected, then $G \in \langle \mathcal{S}_3 \rangle$.*

For a graph $G$ with a vertex $u$ of degree 4 or higher, if there are two distinct vertices $v$ and $w$ adjacent to $u$, we define $G_{[u,vw]} = G - u + vw$ as the graph obtained from $G$ by removing the vertex $u$ along with all its incident edges, and adding a new edge $vw$. This operation is called *lifting*. For convenience, the lifting process is denoted as $G \to G_{[u,vw]}$.

**Lemma 2.3** ([7]). *Let $G$ be a graph with a vertex $u$, and let $uv$ and $uw$ be two edges in $E(G)$. If $d_G(u) \geq 4$ and $G_{[u,vw]} \in \langle \mathcal{S}_3 \rangle$, then $G \in \langle \mathcal{S}_3 \rangle$.*

The lifting preserves $\mathcal{S}_3$-connectivity under the assumption of sufficient degree at $u$.

For an edge set $E' \subseteq E(G)$, the contraction $G/E'$ is the graph obtained from $G$ by identifying the two ends of each edge in $E'$, and then removing the resulting loops. For convenience, $G/E(G')$ is denoted by $G/G'$ if $G'$ is a subgraph of $G$. The following lemma is very powerful for determining whether a graph belongs to $\langle \mathcal{S}_3 \rangle$.

**Lemma 2.4** ([8]). *Let $G$ be a graph with a subgraph $G'$. If $G' \in \langle \mathcal{S}_3 \rangle$ and $G/G' \in \langle \mathcal{S}_3 \rangle$, then $G \in \langle \mathcal{S}_3 \rangle$.*

A subgraph $G'$ of $G$ is called proper, if there exist two distinct vertices $v$ and $w$ in $V(G')$ such that a $(v, w)$-path exists in $G - E(G')$. The following lemma shows that $\mathcal{S}_3$-connectivity can be inherited by contracting a proper subgraph $K_6$.

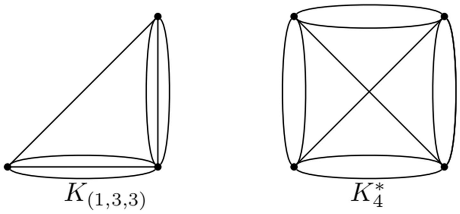


**FIGURE 1** | The graphs $K_{(1,3,3)}$ and $K_4^*$.



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**Lemma 2.5** ([7]). *Let $G$ be a graph. If $K_6$ is a proper subgraph of $G$ and $G/K_6 \in \langle \mathcal{S}_3 \rangle$, then $G \in \langle \mathcal{S}_3 \rangle$.*

## 2.2 | Graphic Sequences and $\mathcal{S}_3$-Preserving Operations

In the inductive arguments of our proofs, a critical part is to determine whether the resulting degree sequences still satisfy the conditions of the induction hypothesis after certain operations. The following useful theorems due to Erdős and Gallai [13] and Hakimi [14, 15] can be applied to verify whether a degree sequence is graphic.

For clarity and formalization, we introduce the function $f(\pi)$ of an integer-valued sequence $\pi = (d_1, ..., d_n)$, defined by:

$$f(\pi) = \max\{i | d_i \geq i, 1 \leq i \leq n\},$$

where $d_i$ denotes the $i$-th term of $\pi$.

**Theorem 2.6** (Erdős-Gallai Theorem [13]). *Let $\pi = (d_1, ..., d_n)$ be an integer-valued sequence, where $n - 1 \geq d_1 \geq \cdots \geq d_n \geq 0$ and $\sum_{i=1}^{n} d_i$ is even. Then the degree sequence $\pi$ is graphic if and only if*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$$

for each integer $k$ with $1 \leq k \leq f(\pi)$.

Let $\pi = (d_1, ..., d_n)$ be an integer-valued sequence with $n - 1 \geq d_1 \geq \cdots \geq d_n \geq 0$. We define the sequence $(d_1 - 1, ..., d_{d_n} - 1, d_{d_n+1}, ..., d_{n-1})$ as the resulting sequence obtained from $\pi$ by laying off $d_n$. Additionally, we introduce the *laying sequence* $\pi' = (d_1', ..., d_{n-1}')$ defined as the non-increasing reordered version of $(d_1 - 1, ..., d_{d_n} - 1, d_{d_n+1}, ..., d_{n-1})$.

**Theorem 2.7** (Hakimi [14, 15]). *Let $\pi = (d_1, ..., d_n)$ be an integer-valued sequence with $n - 1 \geq d_1 \geq \cdots \geq d_n \geq 0$. Then the degree sequence $\pi$ is graphic if and only if the laying sequence $\pi'$ is graphic.*

When the difference between the maximum and minimum degrees is small, the following lemma can be applied to determine whether a degree sequence is graphic.

**Lemma 2.8** ([16]). *Let $\pi = (d_1, ..., d_n)$ be an integer-valued sequence, where $n - 1 \geq d_1 \geq \cdots \geq d_n > 0$ and $\sum_{i=1}^{n} d_i$ is even. If $n \geq \frac{1}{d_n}\left\lfloor \frac{(d_1 + d_n + 1)^2}{4} \right\rfloor$, then $\pi$ is graphic.*

Similar to the laying sequence, we define the *lifting sequence* $\pi_l = (d_1^l, ..., d_{n-1}^l)$ as the non-increasing reordered sequence of $(d_1 - 1, ..., d_{d_n-2} - 1, d_{d_n-1}, ..., d_{n-1})$. Now we show that the lifting operation can be used to obtain degree sequences that have $\mathcal{S}_3$-connected realizations.

**Lemma 2.9.** *Let $\pi = (d_1, ..., d_n) \in \mathcal{GS}$ with $d_n \geq 4$. If the lifting sequence $\pi_l \in \mathcal{GS}(\mathcal{S}_3)$ and*

$$d_1 + \cdots + d_{d_n-2} - (d_n - 2) < d_{d_n-1} + \cdots + d_{n-1},$$

then $\pi \in \mathcal{GS}(\mathcal{S}_3)$.

*Proof.* Let $G_l$ be an $\mathcal{S}_3$-connected realization of the lifting sequence $\pi_l$. For $1 \leq i \leq d_n - 2$, we denote $v_i$ as the vertex of $G_l$ with $d_{G_l}(v_i) = d_i - 1$. Now we have $d_1 + \cdots + d_{d_n-2} - (d_n - 2) < d_{d_n-1} + \cdots + d_{n-1}$, which ensures the existence of a suitable edge $uw$ for a valid lifting operation. That is, there exists an edge $uw \in E(G_l)$ such that $u, w \notin \{v_1, ..., v_{d_n-2}\}$. Then we introduce a new vertex $v_n$ and construct a new graph $G$ from $G_l$ by setting $V(G) = V(G_l) \cup \{v_n\}$ and $E(G) = E(G_l) \cup \{v_1v_n, ..., v_{d_n-2}v_n, uv_n, wv_n\} - \{uw\}$. Clearly, $G$ is simple and its degree sequence is $\pi$. Since $G_{[v_n,uw]} = G_l \in \langle \mathcal{S}_3 \rangle$, by Lemma 2.3 we know that $G \in \langle \mathcal{S}_3 \rangle$, i.e., $\pi \in \mathcal{GS}(\mathcal{S}_3)$. □

Now we show that if the laying sequence $\pi'$ has an $\mathcal{S}_3$-connected realization, then the original degree sequence $\pi$ also has one.



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**Lemma 2.10.** *Let $\pi = (d_1, \ldots, d_n) \in \mathcal{GS}$ with $d_n \geq 4$. If the laying sequence $\pi' \in \mathcal{GS}(\mathcal{S}_3)$, then $\pi \in \mathcal{GS}(\mathcal{S}_3)$.*

*Proof.* Let $G'$ be an $\mathcal{S}_3$-connected realization of $\pi'$. For each $1 \leq i \leq d_n$, we denote $v_i$ to be the vertex of $G'$ with degree $d_i - 1$. Let $u$ be a new vertex. Now we construct a new graph $G$ such that $V(G) = V(G') \cup \{u\}$ and $E(G) = E(G') \cup \{v_1u, v_2u, \ldots, v_{d_n}u\}$. Obviously, $G$ is a realization of $\pi$. Since $G/G' = d_nK_2$ with $d_n \geq 4$, we have $G/G' \in \langle \mathcal{S}_3 \rangle$ by Lemma 2.1 (ii). Therefore, by Lemma 2.4, $G \in \langle \mathcal{S}_3 \rangle$, i.e., $\pi \in \mathcal{GS}(\mathcal{S}_3)$. □

## 2.3 | Some Special Sequences With $\mathcal{S}_3$-Connected Realizations

In this subsection, we characterize some special degree sequences that have $\mathcal{S}_3$-connected realizations. In [11, 17], the authors provided several $\mathbb{Z}_3$-connected graphs that can be used to construct $\mathcal{S}_3$-connected graphs by Lemma 2.2. The graph $W_4$ is constructed by adding a new vertex to a 4-cycle and connecting it to each vertex of the cycle (see Figure 2).

**Lemma 2.11.** *Each of the following holds:*

i. (*Proposition* 3.6 *of* [17]) *The graph $W_4$ is $\mathbb{Z}_3$-connected.*
ii. (*Lemma* 2.2 *of* [11]) *Let $G$ be a graph with a subgraph $W_4$. If $G/W_4$ is $\mathbb{Z}_3$-connected, then $G$ is $\mathbb{Z}_3$-connected.*
iii. (*Lemmas* 2.8 *and* 2.9 *of* [11]) *Each of the graphs in Figure 3 is $\mathbb{Z}_3$-connected.*

**Lemma 2.12.** *If $\pi = (4^5, 3^4)$, then $\pi$ has a $\mathbb{Z}_3$-connected realization as depicted in Figure 2.*

*Proof.* Let $G$ denote a realization of $(4^5, 3^4)$ in Figure 2. Let $G'$ be the subgraph of $G$ induced by the vertices $v_5$, $v_6$, $v_7$, $v_8$ and $v_9$. Obviously, both $G'$ and $G/G'$ are isomorphic to $W_4$. Therefore, by Lemma 2.11 (i) (ii), $G$ is a $\mathbb{Z}_3$-connected graph, which means that $(4^5, 3^4) \in \mathcal{GS}(\mathbb{Z}_3)$. □

Next, we provide some $\mathcal{S}_3$-connected realizations for certain specific graphic sequences.

**Lemma 2.13.** *If*

$$\pi \in \{(7^4, 4^4), (7^3, 6, 5, 4^3), (7^3, 5^3, 4^2), (7^2, 6^3, 4^3), (8^3, 5^2, 4^4), (8^3, 6, 4^5), (9^3, 5, 4^6), (10^3, 4^8)\},$$

then $\pi$ has an $\mathcal{S}_3$-connected realization.

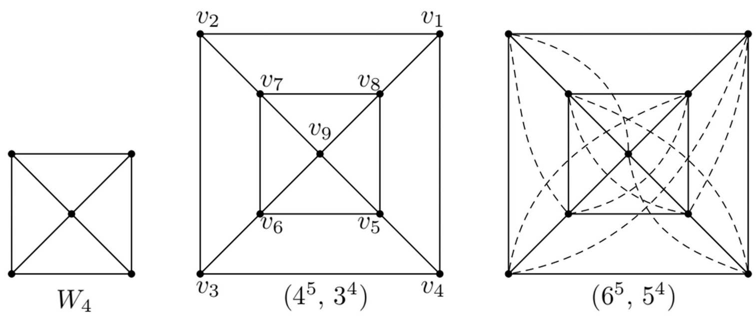


**FIGURE 2** | The graphs $W_4$, $(4^5, 3^4)$-realization, and $(6^5, 5^4)$-realization.

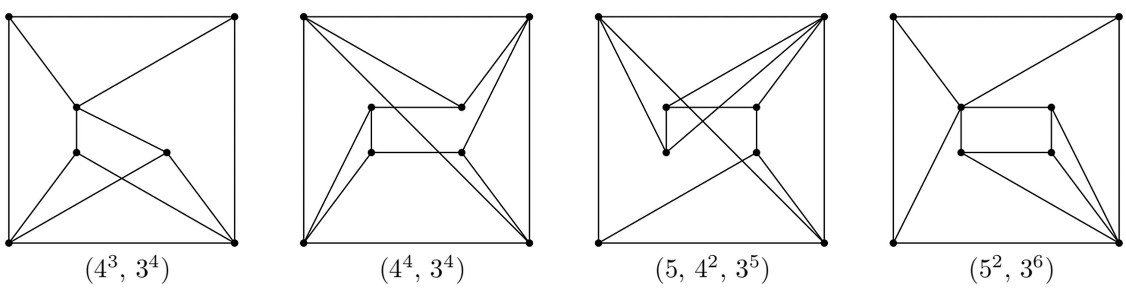


**FIGURE 3** | The $\mathbb{Z}_3$-connected realizations of $(4^3, 3^4)$, $(4^4, 3^4)$, $(5, 4^2, 3^5)$, and $(5^2, 3^6)$.



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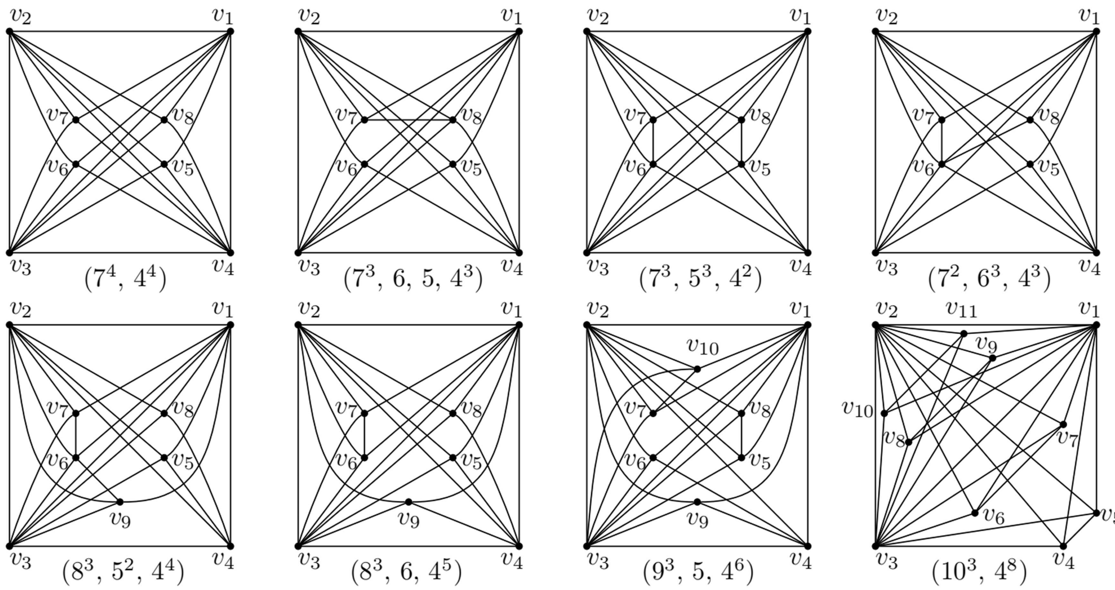


**FIGURE 4** | The $\mathcal{S}_3$-realizations of $(7^4, 4^4)$, $(7^3, 6, 5, 4^3)$, $(7^3, 5^3, 4^2)$, $(7^2, 6^3, 4^3)$, $(8^3, 5^2, 4^4)$, $(8^3, 6, 4^5)$, $(9^3, 5, 4^6)$ and $(10^3, 4^8)$.

*Proof.* Figure 4 shows some realizations of the above mentioned degree sequences. Let $G$ be a realization of $(7^4, 4^4)$ in Figure 4. The lifting process is as follows: $G \to G_{[v_5,v_3v_4]} \to G_{[v_6,v_2v_3]} \to G_{[v_7,v_1v_2]} \to G_{[v_8,v_1v_4]}$. Lastly, the resulting graph is isomorphic to $K_4^*$. By Lemmas 2.1 (iii) and 2.3, $G$ is an $\mathcal{S}_3$-realization of $(7^4, 4^4)$.

Let $G$ be a realization of $(7^3, 6, 5, 4^3)$ in Figure 4. The lifting process is as follows: $G \to G_{[v_5,v_1v_4]} \to G_{[v_6,v_3v_4]} \to G_{[v_7,v_2v_3]} \to G_{[v_8,v_1v_2]}$. Consequently, the resulting graph is isomorphic to $K_4^*$. Based on Lemmas 2.1 (iii) and 2.3, $G$ is an $\mathcal{S}_3$-realization of $(7^3, 6, 5, 4^3)$.

Let $G$ be a realization of $(7^3, 5^3, 4^2)$ in Figure 4. The lifting process is as follows: $G \to G_{[v_7,v_1v_2]} \to G_{[v_8,v_2v_3]} \to G_{[v_5,v_3v_4]} \to G_{[v_6,v_1v_4]}$. Finally, the resulting graph is isomorphic to $K_4^*$. According to Lemmas 2.1 (iii) and 2.3, $G$ is an $\mathcal{S}_3$-realization of $(7^3, 5^3, 4^2)$.

Let $G$ be a realization of $(7^2, 6^3, 4^3)$ in Figure 4. The lifting process is as follows: $G \to G_{[v_5,v_1v_4]} \to G_{[v_7,v_2v_3]} \to G_{[v_8,v_1v_2]} \to G_{[v_6,v_3v_4]}$. Lastly, the resulting graph is isomorphic to $K_4^*$. By Lemmas 2.1 (iii) and 2.3, $G$ is an $\mathcal{S}_3$-realization of $(7^2, 6^3, 4^3)$.

Let $G$ be a realization of $(8^3, 5^2, 4^4)$ in Figure 4. The lifting process is as follows: $G \to G_{[v_9,v_3v_6]} \to G_{[v_5,v_3v_4]} \to G_{[v_8,v_1v_4]} \to G_{[v_7,v_1v_2]} \to G_{[v_6,v_2v_3]}$. Consequently, the resulting graph is isomorphic to $K_4^*$. According to Lemmas 2.1 (iii) and 2.3, $G$ is an $\mathcal{S}_3$-realization of $(8^3, 5^2, 4^4)$.

Let $G$ be a realization of $(8^3, 6, 4^5)$ in Figure 4. The lifting process is as follows: $G \to G_{[v_6,v_3v_7]} \to G_{[v_9,v_3v_4]} \to G_{[v_5,v_1v_4]} \to G_{[v_8,v_1v_2]} \to G_{[v_7,v_2v_3]}$. Lastly, the resulting graph is isomorphic to $K_4^*$. Based on Lemmas 2.1 (iii) and 2.3, $G$ is an $\mathcal{S}_3$-realization of $(8^3, 6, 4^5)$.

Let $G$ be a realization of $(9^3, 5, 4^6)$ in Figure 4. The lifting process is as follows: $G \to G_{[v_{10},v_2v_7]} \to G_{[v_8,v_1v_5]} \to G_{[v_9,v_3v_4]} \to G_{[v_6,v_1v_4]} \to G_{[v_5,v_1v_2]} \to G_{[v_7,v_2v_3]}$. Finally, the resulting graph is isomorphic to $K_4^*$. By Lemmas 2.1 (iii) and 2.3, $G$ is an $\mathcal{S}_3$-realization of $(9^3, 5, 4^6)$.

Let $G$ be a realization of $(10^3, 4^8)$ in Figure 4. The lifting process is as follows: $G \to G_{[v_4,v_1v_5]} \to G_{[v_5,v_2v_3]} \to G_{[v_6,v_1v_7]} \to G_{[v_7,v_2v_3]} \to G_{[v_8,v_2v_9]} \to G_{[v_9,v_1v_2]} \to G_{[v_{10},v_2v_{11}]} \to G_{[v_{11},v_1v_2]}$. Consequently, the resulting graph is isomorphic to $K_{(1,3,3)}$. According to Lemmas 2.1 (iii) and 2.3, $G$ is an $\mathcal{S}_3$-realization of $(10^3, 4^8)$. □

**Lemma 2.14.** *If*

$$\pi \in \{(6^3, 5^4), (6^4, 5^4), (7, 6^2, 5^5), (7^2, 5^6), (6^5, 5^4)\},$$

then $\pi$ has an $\mathcal{S}_3$-connected realization.



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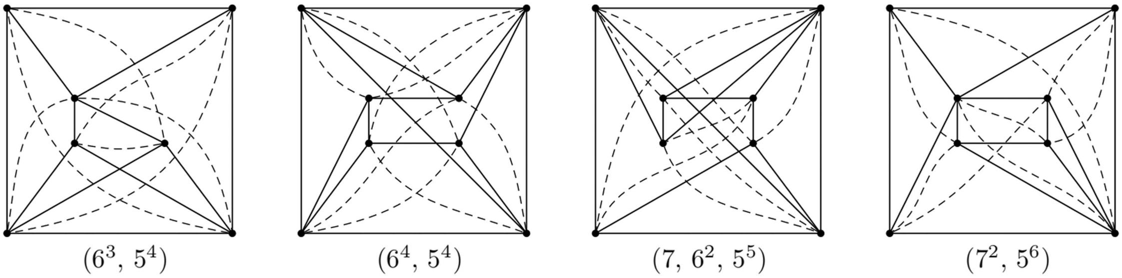


**FIGURE 5** | The $\mathcal{S}_3$-connected realizations of $(6^3, 5^4)$, $(6^4, 5^4)$, $(7, 6^2, 5^5)$ and $(7^2, 5^6)$.

*Proof.* In Figures 2 and 5, we present a realization for each of these sequences. Realizations of the sequences $(6^3, 5^4), (6^4, 5^4), (7, 6^2, 5^5), (7^2, 5^6)$ are shown in Figure 5, and a realization of the sequence $(6^5, 5^4)$ is shown in Figure 2. The dashed cycle in the figures represents a Hamilton cycle; the graphs depicted by solid lines are $\mathbb{Z}_3$-connected.

Let $G$ be such a realization. According to Lemmas 2.11 (iii) and 2.12, each graph is composed of a $\mathbb{Z}_3$-connected graph and a Hamiltonian cycle. Therefore, based on Lemma 2.2, $G$ belongs to $\langle \mathcal{S}_3 \rangle$. □

The above special sequences would serve as inductive bases for the proof of Theorem 1.4.

**Outline of the proofs.** To establish Theorem 1.4, we proceed by induction on $n$. The general strategy is to apply several operations that efficiently reduce the order of graphic sequences. Then, based on the induction hypothesis, we can obtain a simple $\mathcal{S}_3$-connected graph to construct the desired realization. These operations include laying off a term of degree 4 or higher, lifting a term of degree 4 or 5, and contracting a special $K_6$. If the sum of degrees is much larger than $6n - 4$, then we lay off a minimum degree term directly. On the other hand, if the sum of all degrees is close to $6n - 4$, then we perform the lifting operation on a minimum degree term. After performing this operation, we need to carefully verify whether the sum of degrees is at least $6(n - 1) - 4$. And in cases where the sum of degrees equals $6n - 4$ and the minimum degree is 5, laying and lifting operations are ineffective. However, we can reduce the order of sequences by contracting a special $K_6$, after which the sum of degrees in the resulting reduced sequence is large enough to satisfy the induction hypothesis. Additionally, for the other degree sequences in which the above operations are not applicable, their realizations are obtained by adding Hamiltonian cycles to certain $\mathbb{Z}_3$-connected graphs.

## 3 | Graphic Sequences With at Least Two Large Degree Terms

In this section, we focus on graphic sequences that contain at least two terms of high degree. Specifically, we present $\mathcal{S}_3$-connected realizations for these sequences.

**Theorem 3.1.** *Let $n \geq 7$, and let $\pi = (d_1, \ldots, d_n) \in \mathcal{GS}$ with $d_1 = d_2 = n - 1$ and $d_n \geq 4$. If $\sum_{i=1}^{n} d_i \geq 6n - 4$, then $\pi \in \mathcal{GS}(\mathcal{S}_3)$.*

Before providing the complete proof of Theorem 3.1, we first need to establish the cases where $n$ is small, as these will form the foundation for our induction.

**Lemma 3.2.** *Let $n \in \{7, 8\}$, and let $\pi = (d_1, \ldots, d_n) \in \mathcal{GS}$ with $d_1 = d_2 = n - 1$ and $d_n \geq 4$. If $\sum_{i=1}^{n} d_i \geq 6n - 4$, then $\pi \in \mathcal{GS}(\mathcal{S}_3)$.*

*Proof.* If $n = 7$, then, by the condition above, we have $38 \leq \sum_{i=1}^{7} d_i \leq 42$, and hence

$$\pi \in \{(6^3, 5^4), (6^4, 5^2, 4), (6^5, 5^2), (6^7)\}.$$

The sequence $\pi = (6^3, 5^4)$ belongs to $\mathcal{GS}(\mathcal{S}_3)$ by Lemma 2.14. Now we give proofs for the other sequences. For $\pi = (6^4, 5^2, 4)$, let $G$ be the graph obtained by deleting two adjacent edges from a $K_7$. Clearly, $G$ is a realization of $\pi$ and contains $K_6$ as a proper subgraph. Additionally, $G/K_6 = 4K_2$. According to Lemmas 2.1 (ii) and 2.5, $G$ is an $\mathcal{S}_3$-connected realization of $\pi$, i.e., $\pi \in \mathcal{GS}(\mathcal{S}_3)$. For $\pi = (6^5, 5^2)$, the sequence has a realization that is isomorphic to $K_7$ with one edge deleted, denoted by $G$. This graph $G$ contains $K_6$ as a proper subgraph and



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$G/K_6 = 5K_2$. Consequently, $G$ is $\mathcal{S}_3$-connected by Lemmas 2.1 (ii) and 2.5. Therefore, $G$ is an $\mathcal{S}_3$-connected realization of $\pi$, indicating that $\pi \in \mathcal{GS}(\mathcal{S}_3)$. For $\pi = (6^7)$, $K_7$ is a realization of $\pi$. Therefore, by Lemma 2.1 (i), we have $\pi \in \mathcal{GS}(\mathcal{S}_3)$.

If $n = 8$, then $44 \leq \sum_{i=1}^{8} d_i \leq 56$. Now we divide our discussion according to the value of $\sum_{i=1}^{8} d_i$.

We may suppose $\sum_{i=1}^{8} d_i = 44$ first. Since $\pi \in \mathcal{GS}$, we have

$$\pi \in \{(7^4, 4^4), (7^3, 6, 5, 4^3), (7^3, 5^3, 4^2), (7^2, 6^3, 4^3), (7^2, 5^6), (7^2, 6^2, 5^2, 4^2), (7^2, 6, 5^4, 4)\}.$$

By Lemmas 2.13 and 2.14, it only remains to consider the last two cases:

$$\pi \in \{(7^2, 6^2, 5^2, 4^2), (7^2, 6, 5^4, 4)\}.$$

For these two cases, we have $\pi_l \in \{(6^4, 5^2, 4), (6^3, 5^4)\}$. According to the proof of case where $n = 7$ above, we find that $\pi_l \in \mathcal{GS}(\mathcal{S}_3)$. Therefore, by Lemma 2.9, $\pi \in \mathcal{GS}(\mathcal{S}_3)$.

Assume instead that $46 \leq \sum_{i=1}^{8} d_i \leq 56$. We consider the laying sequence $\pi'$. If $\pi' \in \mathcal{GS}$ satisfies $\sum_{i=1}^{7} d_i' \geq 38$ and $d_7' \geq 4$, then, by the discussion of the case where $n = 7$ above, $\pi'$ has an $\mathcal{S}_3$-connected realization, so does $\pi$ by Lemma 2.10. Note that

$$\sum_{i=1}^{7} d_i' = \sum_{i=1}^{7} d_i - 2d_8.$$

For $d_8 \in \{4, 6, 7\}$, we have $\sum_{i=1}^{7} d_i' \geq \min\{46 - 2 \times 4, 50 - 2 \times 6, 56 - 2 \times 7\} = 38$. Thus, we are done when $d_8 \in \{4, 6, 7\}$. For $d_8 = 5$, we can also get that $\sum_{i=1}^{7} d_i' \geq 38$ and obtain the $\mathcal{S}_3$-connected realization of $\pi$ expect the situation that $d_8 = 5$ and $\sum_{i=1}^{n} d_i = 46$. There are only two remaining cases: $\pi \in \{(7^3, 5^5), (7^2, 6^2, 5^4)\}$, which have the same corresponding lifting sequence $\pi_l = (6^3, 5^4)$. According to Lemma 2.9 and the discussion of the case where $n = 7$ above, we get $\pi \in \mathcal{GS}(\mathcal{S}_3)$. This completes the proof of Lemma 3.2. □

In the cases when there are many low degree terms, the induction may not be an effective method for our purpose. Instead, a construction method is employed to prove those cases. Given any two graphs $G$ and $H$, the graph $G \vee H$ is defined as $V(G \vee H) = V(G) \cup V(H)$ and

$$E(G \vee H) = E(G) \cup E(H) \cup \{uv | u \in V(G), v \in V(H)\}.$$

**Lemma 3.3.** *If $\pi = ((n-1)^2, d_3, 4^{n-3}) \in \mathcal{GS}$ with $d_3 \geq 10$, then $\pi \in \mathcal{GS}(\mathcal{S}_3)$.*

*Proof.* Let $d = \frac{d_3 - 2}{2}$. For $1 \leq i \leq d$, let $C_i$ be a cycle of length $n_i + 1$ such that $V(C_i) = \{u, v_{(i,1)}, \ldots, v_{(i,n_i)}\}$ and $E(C_i) = \{uv_{(i,1)}, v_{(i,1)}v_{(i,2)}, \ldots, v_{(i,n_i-1)}v_{(i,n_i)}, v_{(i,n_i)}u\}$, where the integer $n_i \geq 2$. Now we suppose $\sum_{i=1}^{d} n_i = n - 3$ and $V(C_j) \cap V(C_k) = \{u\}$ for any $j \neq k$. Then we construct a new graph $C^*$ such that $V(C^*) = \cup_{i=1}^{d} V(C_i)$ and $E(C^*) = \cup_{i=1}^{d} E(C_i)$. Clearly, $C^*$ is a simple graph, and its degree sequence is $(d_3 - 2, 2^{n-3})$. Next we construct a new graph $G$ such that $G = K_2 \vee C^*$ as depicted in Figure 6, where $V(K_2) = \{u_1, u_2\}$. Note that it is a realization of $\pi$.

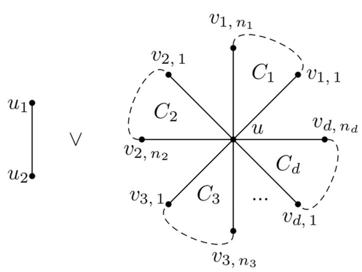


**FIGURE 6** | The $\mathcal{S}_3$-connected realization of $\pi = ((n-1)^2, d_3, 4^{n-3})$ with $n - 1 \geq d_3 \geq 10$.



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Furthermore, by a series of lifting operations, the graph containing $K_{(1,3,3)}$ as a spanning subgraph with the vertex set $\{u, u_1, u_2\}$ can be obtained from $G$. The lifting process is as follows:

$$\begin{aligned} G &\to G_{[v_{(1,1)}, uv_{(1,2)}]} \to G_{[v_{(1,2)}, uv_{(1,3)}]} \to \cdots \to G_{[v_{(1,n_1-1)}, uv_{(1,n_1)}]} \to G_{[v_{(1,n_1)}, uu_1]} \\ &\to G_{[v_{(2,1)}, uv_{(2,2)}]} \to \cdots \to G_{[v_{(2,n_2-1)}, uv_{(2,n_2)}]} \to G_{[v_{(2,n_2)}, uu_1]} \\ &\to \cdots \\ &\to G_{[v_{(p,1)}, uv_{(p,2)}]} \to \cdots \to G_{[v_{(p,n_p-1)}, uv_{(p,n_p)}]} \to G_{[v_{(p,n_p)}, uu_1]} \\ &\to G_{[v_{(p+1,1)}, uv_{(p+1,2)}]} \to \cdots \to G_{[v_{(p+1,n_{p+1}-1)}, uv_{(p+1,n_{p+1})}]} \to G_{[v_{(p+1,n_{p+1})}, uu_2]} \\ &\to \cdots \\ &\to G_{[v_{(d,1)}, uv_{(d,2)}]} \to \cdots \to G_{[v_{(d,n_d-1)}, uv_{(d,n_d)}]} \to G_{[v_{(d,n_d)}, uu_2]}, \end{aligned}$$

where $p = \left\lfloor \frac{d_3 - 2}{4} \right\rfloor$ and $d = \frac{d_3 - 2}{2}$. The resulting graph, denoted by $G'$, satisfies $V(G') = \{u, u_1, u_2\}$, $|E_{G'}(u_1, u_2)| = 1$,

$$|E_{G'}(u, u_1)| = \left\lfloor \frac{d_3 - 2}{4} \right\rfloor + 1, \text{ and } \quad |E_{G'}(u, u_2)| = \frac{d_3 - 2}{2} - \left\lfloor \frac{d_3 - 2}{4} \right\rfloor + 1.$$

Since $d_3 \geq 10$, we get

$$|E_{G'}(u, u_1)| \geq 3 \quad \text{and} \quad |E_{G'}(u, u_2)| \geq 3,$$

so $K_{(1,3,3)}$ is a spanning subgraph of $G'$. Hence, by Lemma 2.1 (iii) and the definition of $\mathcal{S}_3$-connectivity, $G'$ is $\mathcal{S}_3$-connected. Therefore, by Lemma 2.3 recursively, $G$ is an $\mathcal{S}_3$-connected realization of $\pi$, which means that $\pi \in \mathcal{GS}(\mathcal{S}_3)$. □

Now we prove Theorem 3.1.

*Proof of Theorem* 3.1. We argue by induction on $n$. The cases of $7 \leq n \leq 8$ are proved by Lemma 3.2. Now we focus on $n \geq 9$ and divide our discussion according to the value of $d_n$.

We may suppose $d_n = 4$ first. If $\sum_{i=1}^{n} d_i = 6n - 4$ and $d_3 \leq n - 2$, then we consider the lifting sequence $\pi_l$. Note that, we have $\sum_{i=1}^{n-1} d_i^l = 6n - 10$ and $n - 2 = d_1^l = d_2^l \geq d_3^l \geq \cdots \geq d_{n-1}^l \geq 4$. We also have $f(\pi) = \max\{i | d_i \geq i, 1 \leq i \leq n\} \leq 5$, as otherwise $\sum_{i=1}^{n-1} d_i^l > 6n - 10$, which leads to a contradiction. Since $d_{n-1}^l \geq 4$, we have $\sum_{i=1}^{k} d_i^l \leq k(n-2) \leq k(k-1) + \sum_{i=k+1}^{n-1} \min\{k, d_i^l\}$ for each integer $k$ with $1 \leq k \leq 4$, so we only need to consider the case of $k = 5$. By $\sum_{i=1}^{n-1} d_i^l - (d_1^l + d_2^l + 4(n-3)) = 6$, we know that after assigning 4 to each of $d_3^l, \ldots, d_{n-1}^l$, there are a total of 6 remaining. Consequently, we have

$$\sum_{i=1}^{5} d_i^l \leq d_1^l + d_2^l + 4 \times 3 + 6 = 2n + 14.$$

Since $d_{n-1}^l \geq 4$, we get

$$5 \times 4 + \sum_{i=6}^{n-1} \min\{5, d_i^l\} \geq 20 + 4(n - 1 - 5) = 4n - 4.$$

According to $n \geq 9$, we can verify that

$$\sum_{i=1}^{5} d_i^l \leq 5 \times 4 + \sum_{i=6}^{n-1} \min\{5, d_i^l\}.$$



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Hence Theorem 2.6 says that $\pi_l \in \mathcal{GS}$. Then, by the induction hypothesis, $\pi_l$ has an $\mathcal{S}_3$-connected realization. Since $2(n-1)-2 < 6(n-1)-4-2(n-2)$ and by Lemma 2.9, we conclude that $\pi \in \mathcal{GS}(\mathcal{S}_3)$.

If $\sum_{i=1}^{n} d_i = 6n-4$ and $d_3 = n-1$, then we have $6n-4 = \sum_{i=1}^{n} d_i \geq 3(n-1)+4(n-3)$. This implies $n \leq 11$. According to the conditions above, a simple calculation shows that

$$\pi \in \{(8^3, 5^2, 4^4), (8^3, 6, 4^5), (9^3, 5, 4^6), (10^3, 4^8)\}.$$

It follows from Lemma 2.13 that $\pi$ has an $\mathcal{S}_3$-connected realization.

If $\sum_{i=1}^{n} d_i \geq 6n-2$, then we consider the laying sequence $\pi'$. Note that $\sum_{i=1}^{n-1} d_i' \geq 6(n-1)-4$. The case of $d_4 = 4$ follows by Lemma 3.3, and we only need to consider $d_4 \geq 5$. Now $\pi' \in \mathcal{GS}$ by Theorem 2.7. Since $\sum_{i=1}^{n-1} d_i' \geq 6(n-1)-4$ and $d_{n-1}' \geq 4$, by the induction hypothesis $\pi'$ has an $\mathcal{S}_3$-connected realization. Therefore, by Lemma 2.10, $\pi$ has an $\mathcal{S}_3$-connected realization as well.

Assume instead that $d_n \geq 5$. If $\sum_{i=1}^{n} d_i \geq 6n$, then $\sum_{i=3}^{n} d_i \geq 6n-2(n-1) \geq 5(n-2)$, which implies that $9 \leq n \leq 12$. For the laying sequence $\pi'$, if $d_n = 5$, then $\sum_{i=1}^{n-1} d_i' \geq 6(n-1)-4+(10-2d_n) = 6(n-1)-4$. If $d_n \geq 6$, then $\sum_{i=1}^{n-1} d_i' \geq 2(n-1)+(n-4)d_n \geq 6(n-1)-4+(2n-16) > 6(n-1)-4$. Thus $\pi'$ has an $\mathcal{S}_3$-connected realization by the induction hypothesis, so does $\pi$ by Lemma 2.10. When $n \geq 13$, we get $\sum_{i=1}^{n} d_i \geq 2(n-1)+5(n-2) \geq 6n+2$, which completes the proof of this case. Hence we only need to consider the situation where

$$9 \leq n \leq 11,\ d_n = 5,\ d_1 = d_2 = n-1, \text{ and } \sum_{i=1}^{n} d_i \leq 6n-2.$$

Then a simple calculation shows that $\pi \in \{(8^2, 6, 5^6), (9^2, 5^8)\}$. Thus the lifting sequence $\pi_l$ satisfies $\pi_l \in \{(7^2, 5^6), (8^2, 5^6, 4)\}$, which is graphic. Notice that $(7^2, 5^6) \in \mathcal{GS}(\mathcal{S}_3)$ by Lemma 3.2 and $(8^2, 5^6, 4) \in \mathcal{GS}(\mathcal{S}_3)$ by the induction hypothesis. Therefore, $\pi \in \mathcal{GS}(\mathcal{S}_3)$ by Lemma 2.9. This completes the proof of Theorem 3.1. □

## 4 | Some Critical Cases

In this section, we address critical cases where laying and lifting operations are ineffective. To do this, we first need to verify whether the given degree sequence is a graphic sequence. The following lemma is useful for this purpose.

**Lemma 4.1.** *Let $n \geq 7$, and let $\pi = (d_1, \ldots, d_n)$ be an integer-valued sequence with $n-1 \geq d_1 \geq \cdots \geq d_n \geq 5$. If $\sum_{i=1}^{n} d_i = 6n-4$, then $\pi \in \mathcal{GS}$.*

*Proof.* Since $d_n \geq 5$, we have $\sum_{i=1}^{k} d_i \leq k(n-1) = k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$ for each integer $k$ with $1 \leq k \leq 5$. Thus when $f(\pi) \leq 5$, we have $\pi \in \mathcal{GS}$ by Theorem 2.6. Now we consider the case of $f(\pi) \geq 6$.

Suppose by contradiction that $\pi \notin \mathcal{GS}$. Then, according to Theorem 2.6, there exists an integer $k_0$ with $6 \leq k_0 \leq f(\pi)$ such that $\sum_{i=1}^{k_0} d_i > k_0(k_0-1) + \sum_{i=k_0+1}^{n} \min\{k_0, d_i\}$. Then it follows that

$$\begin{aligned} 6n-4 = \sum_{i=1}^{k_0} d_i + \sum_{i=k_0+1}^{n} d_i &> k_0(k_0-1) + \sum_{i=k_0+1}^{n} \min\{k_0, d_i\} + \sum_{i=k_0+1}^{n} d_i \\ &\geq k_0(k_0-1) + 5(n-k_0) + 5(n-k_0) \\ &= 10n + \left(k_0 - \frac{11}{2}\right)^2 - \frac{121}{4} \\ &> 6n-4, \end{aligned}$$

where the last inequality is obtained from $n \geq 7$ and $k_0 \geq 6$. This contradiction implies the truth of Lemma 4.1. □

In some cases where degree sequences contain many low degrees, we use Lemma 2.2 and some Hamiltonian properties to obtain $\mathcal{S}_3$-connected realizations. We need the following closure concept of Hamiltonian graphs



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due to Bondy and Chvátal [18]. To obtain the closure of a graph $G$, we repeatedly connect pairs of nonadjacent vertices whose degree sum is at least $|V(G)|$ until no such pair remains in the graph. Additionally, the following sufficient condition for the existence of Hamiltonian cycle by Bondy and Chvátal [18] is essential for the concept of closure.

**Lemma 4.2** ([18]). *A simple graph is Hamiltonian if and only if its closure is Hamiltonian.*

For an integer-valued sequence $\pi = (d_1, \ldots, d_n)$ with $n - 1 \ge d_1 \ge \cdots \ge d_n \ge 2$, we define $\pi_{-2} = (d_1'', \ldots, d_n'') = (d_1 - 2, \ldots, d_n - 2)$. The following observation follows from Lemma 2.2 immediately.

*Observation* 4.3. Let $\pi \in \mathcal{GS}$. If $\pi_{-2} \in \mathcal{GS}(\mathbb{Z}_3)$ has a $\mathbb{Z}_3$-connected realization $G$ such that the complement of $G$ is Hamiltonian, then $\pi \in \mathcal{GS}(\mathcal{S}_3)$.

*Proof.* Let $C$ be a Hamiltonian cycle in the complement of $G$. Then we construct a new graph $G_1$ from $G$ and $C$ by setting $V(G_1) = V(G)$ and $E(G_1) = E(G) \cup E(C)$. Then, by Lemma 2.2, $G_1$ constitutes an $\mathcal{S}_3$-connected realization of $\pi$, i.e., $\pi \in \mathcal{GS}(\mathcal{S}_3)$. □

**Lemma 4.4.** *Let $n \ge 8$, and let $\pi = (d_1, d_2, d_3, 5^{n-3}) \in \mathcal{GS}$ with $d_2 \le n - 2$, and $d_3 \le n - 3$. If $\pi_{-2} \in \mathcal{GS}(\mathbb{Z}_3)$, then $\pi \in \mathcal{GS}(\mathcal{S}_3)$.*

*Proof.* Let $G'$ be a $\mathbb{Z}_3$-connected realization of $\pi_{-2}$. The complement of graph $G'$, denoted as $\overline{G'}$, has degree sequence $\overline{\pi_{-2}} = ((n-4)^{n-3}, n+1-d_3, n+1-d_2, n+1-d_1)$. Since $n + 1 - d_3 + (n-4) \ge n, n + 1 - d_2 + (n-3) \ge n$, and $n + 1 - d_1 + (n-2) \ge n$, we can form the closure of $\overline{G'}$ by connecting all vertices of degree $(n-4)$ in $\overline{G'}$, connecting the vertex of degree $(n + 1 - d_3)$ to the vertices of degree $(n-4)$ in $\overline{G'}$, and connecting the remaining vertices to the large ones. Then the closure of $\overline{G'}$ is $K_n$. Hence, by Lemma 4.2, $\overline{G'}$ is Hamiltonian. By Observation 4.3, we have $\pi \in \mathcal{GS}(\mathcal{S}_3)$. □

Now, we are ready to establish the main result of this section.

**Theorem 4.5.** *Let $n \ge 7$, and let $\pi = (d_1, \ldots, d_n) \in \mathcal{GS}$ with $d_n \ge 5$. If $\sum_{i=1}^{n} d_i = 6n - 4$, then $\pi \in \mathcal{GS}(\mathcal{S}_3)$.*

*Proof.* We argue by induction on $n$. If $n = 7$, then $d_1 = d_2 = 6$, and thus $\pi \in \mathcal{GS}(\mathcal{S}_3)$ by Lemma 3.2. According to Theorem 3.1, we only need to consider the case where $n \ge 8$ and $d_2 \le n - 2$.

We may suppose $d_1 = 6$ first. This means that $\pi = (6^{n-4}, 5^4)$. If $n \in \{8, 9\}$, then $\pi \in \{(6^4, 5^4), (6^5, 5^4)\}$. Hence $\pi \in \mathcal{GS}(\mathcal{S}_3)$ by Lemma 2.14. If $n \ge 10$, then we consider $\pi_{-2} = (4^{n-4}, 3^4)$ and the sum of its degrees is $4n - 4$. Since $10 \ge \frac{1}{3}\lfloor \frac{(4+3+1)^2}{4} \rfloor$ and by Lemma 2.8, $\pi_{-2}$ is graphic. Furthermore, according to Theorem 1.3, we have $\pi_{-2} \in \mathcal{GS}(\mathbb{Z}_3)$. Let $G'$ be a $\mathbb{Z}_3$-connected realization of $\pi_{-2}$, and let $\overline{G'}$ be the complement of $G'$. Then the degree sequence of $\overline{G'}$ is $((n-4)^4, (n-5)^{n-4})$. Since $n \ge 10$, the closure of $\overline{G'}$ is $K_n$, so $\overline{G'}$ is Hamiltonian by Lemma 4.2. Therefore, by Observation 4.3, we have $\pi \in \mathcal{GS}(\mathcal{S}_3)$.

Assume instead that $d_1 \ge 7$. If $n = 8$, then we have $\pi = (7, 6^2, 5^5) \in \mathcal{GS}(\mathcal{S}_3)$ by Lemma 2.14. If $n \in \{9, 10, 11\}$, then $d_3 \le n - 3$ by calculating the sum of degrees. Then we consider $\pi_{-2} = (d_1'', \ldots, d_n'')$. Note that the sum of its degrees is $4n - 4$. Since $\sum_{i=1}^{n} d_i = 6n - 4$ and $d_1 \ge 7$, we have $d_4 \le 6$. Thus $f(\pi_{-2}) \le 4$. Since $\sum_{i=1}^{k} d_i'' \le k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i''\}$ for $1 \le k \le f(\pi_{-2})$, by Theorem 2.6 we conclude that $\pi_{-2} \in \mathcal{GS}$. Furthermore, according to Theorem 1.3, $\pi_{-2} \in \mathcal{GS}(\mathbb{Z}_3)$. For $d_4 = 5$, we have $\pi \in \mathcal{GS}(\mathcal{S}_3)$ by Lemma 4.4. For $d_4 = 6$, we consider $\pi_{-2} = (d_1 - 2, d_2 - 2, d_3 - 2, 4, \ldots, 3)$, where the dots imply that the omitted degrees take values between 3 and 4. Since $d_4''$ is small, it is easy to check that $\pi_{-2} \in \mathcal{GS}(\mathbb{Z}_3)$ by Theorems 2.6 and 1.3. Let $G'$ be a $\mathbb{Z}_3$-realization of $\pi_{-2}$. Then the degree sequence of $\overline{G'}$ is $(n + 1 - d_1, n + 1 - d_2, n + 1 - d_3, n - 5, \ldots, n - 4)$. Given that $n \ge 9$, the closure of $\overline{G'}$ is $K_n$, so $\overline{G'}$ is Hamiltonian by Lemma 4.2. Then it follows from Observation 4.3 that $\pi \in \mathcal{GS}(\mathcal{S}_3)$.

If $n \ge 12$ and $d_3 \ge n - 5$, then $n \le 13$ as $3(n-5) - 15 \le d_1 + d_2 + d_3 - 15 \le 6n - 4 - 5n$. Hence, by $\pi \in \mathcal{GS}$ and $\sum_{i=1}^{n} d_i = 6n - 4$, we have

$$\pi \in \{(9, 7^2, 5^9), (8^2, 7, 5^9), (8, 7^2, 6, 5^8), (7^4, 5^8), (7^3, 6^2, 5^7), (8^3, 5^{10})\}.$$



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Now we consider $\pi_{-2}$. Since $12 \geq \frac{1}{3}\left\lfloor\frac{(7+3+1)^2}{4}\right\rfloor$ and by Lemma 2.8, we have $\pi_{-2} \in \mathcal{GS}$. Hence, by Theorem 1.3, $\pi_{-2} \in \mathcal{GS}(\mathbb{Z}_3)$. Let $G'$ be a $\mathbb{Z}_3$-realization of $\pi_{-2}$. From its degree sequence, we can determine that the closure of $\overline{G'}$ is $K_n$, which means that $\overline{G'}$ is Hamiltonian by Lemma 4.2. Hence $\pi \in \mathcal{GS}(\mathcal{S}_3)$ by Observation 4.3.

If $n \geq 12$ and $d_3 \leq n-6$, then we consider three cases according to the value of $d_1 + d_2 - 10$.

- For $d_1 + d_2 - 10 \geq n - 5$, we have $\pi \in \{(d_1, d_2, 6, 5^{n-3}), (d_1, d_2, 5^{n-2})\}$, since $d_i \geq 5$, $\sum_{i=1}^{n} d_i = 6n - 4$ and $d_1 + d_2 - 10 \geq n - 5$. Now we consider $\pi_{-2}$. Since $f(\pi_{-2}) \leq 3$ and by Theorem 2.6, a simple calculation shows $\pi_{-2} \in \mathcal{GS}$. Hence, by Theorem 1.3, $\pi_{-2} \in \mathcal{GS}(\mathbb{Z}_3)$. Therefore, based on Lemma 4.4, $\pi \in \mathcal{GS}(\mathcal{S}_3)$.
- For $5 \leq d_1 + d_2 - 10 \leq n - 6$, we have $\pi = (d_1, \ldots, d_{n-4}, 5^4)$ since the sum of the degrees is $6n - 4$. Now let $\pi^*$ be the non-increasing reordered sequence of $(d_1 + d_2 - 10, d_3, \ldots, d_{n-4})$. Since $\sum_{i=1}^{n-4} d_i - 10 = 6(n-5) - 4$ and $d_3 \leq n - 6$, by Lemma 4.1 we have $\pi^* \in \mathcal{GS}$. Then, by the induction hypothesis, $\pi^*$ has an $\mathcal{S}_3$-connected realization, denoted by $G'$. Let $u$ be the vertex of $G'$ with degree $d_1 + d_2 - 10$, and let $\{u_1, \ldots, u_{d_1+d_2-10}\}$ be the set of neighbors of $u$. We also consider the graph $K_6$ with vertex set $V(K_6) = \{v_1, \ldots, v_6\}$. Next we construct a new graph $G$ from $G'$ and $K_6$ by setting $V(G) = V(G') \cup V(K_6) - \{u\}$ and

  $$E(G) = E(G') \cup E(K_6) \cup \{v_1u_1, \ldots, v_1u_{d_1-5}\} \cup \{v_2u_{d_1-4}, \ldots, v_2u_{d_1+d_2-10}\} - E(u).$$

  Obviously, $\pi$ is the degree sequence of $G$. Since $\sum_{i=1}^{n} d_i = 6n - 4$ and $d_1 \leq n - 1$, we have $d_2 - 5 > 0$, so $K_6$ is a proper subgraph of $G$. Therefore, based on $G/K_6 = G' \in \langle \mathcal{S}_3 \rangle$ and by Lemma 2.5, $G$ is an $\mathcal{S}_3$-connected realization of $\pi$.
- For $d_1 + d_2 - 10 \leq 4$, let $\pi^*$ be the non-increasing reordered sequence of $(d_1 + \cdots + d_k - 5k, d_{k+1}, \ldots, d_{n-6+k})$. Here, $k$ is the minimum value of $i$ such that $3 \leq i \leq 5$ and $d_1 + \cdots + d_i - 5i \geq 5$. Since $\sum_{i=1}^{n-6+k} d_i - 5k = 6(n-5) - 4$ and by Lemma 4.1, we have $\pi^* \in \mathcal{GS}$. Hence, by the induction hypothesis, $\pi^*$ has an $\mathcal{S}_3$-connected realization, denoted by $G'$. Let $u$ be the vertex of $G'$ with degree $d_1 + \cdots + d_k - 5k$, and let $\{u_1, \ldots, u_{d_1+\cdots+d_k-5k}\}$ be the set of neighbors of $u$. Furthermore, we consider the graph $K_6$ with vertex set $V(K_6) = \{v_1, \ldots, v_6\}$. Next we construct a new graph $G$ based on $G'$ and $K_6$ by setting $V(G) = V(G') \cup V(K_6) - \{u\}$ and

  $$E(G) = E(G') \bigcup E(K_6) \bigcup_{i=1}^{k} \{v_iu_{d_1+\cdots+d_{i-1}-5(i-1)+1}, \ldots, v_iu_{d_1+\cdots+d_i-5i}\} - E(u).$$

  Here we agree that when $i = 1$, $d_1 + \cdots + d_{i-1} - 5(i-1) + 1 = 1$. Clearly, $G$ is a realization of $\pi$. By the definition of $k$, we have $d_k - 5 > 0$, which implies that $K_6$ is a proper subgraph of $G$. Therefore, as $G/K_6 = G' \in \langle \mathcal{S}_3 \rangle$ and by Lemma 2.5, $G$ is an $\mathcal{S}_3$-connected realization of $\pi$.

*This completes the proof of Theorem 4.5.* □

## 5 | Proof of Theorem 1.4

The degree sum condition in Theorem 1.4 can be illustrated by the following lemma, which is a necessary condition for $\mathcal{S}_3$-connectivity established in [8].

**Lemma 5.1** ([8]). *Let $G$ be a nontrivial graph. If $G \in \langle \mathcal{S}_3 \rangle$, then $|E(G)| \geq 3|V(G)| - 2$.*

Now we are ready to prove Theorem 1.4.

*Proof of Theorem* 1.4. We first show the necessity.

If $\pi$ has an $\mathcal{S}_3$-connected realization $G$, then, according to Lemma 5.1, we have $\sum_{i=1}^{n} d_i \geq 6n - 4$. It is also straightforward to observe that $\min\{d_1, \ldots, d_n\} \geq 4$. For otherwise, if $G$ contains a vertex $v$ with $d(v) \leq 3$, then we assign $\beta(v) = 3 - d(v)$. Under this boundary condition, all edges incident to $v$ are oriented either all-in or all-out, which means the orientation cannot be strongly connected. This proves the necessity of Theorem 1.4.

Now, we are ready to complete the proof of sufficiency for Theorem 1.4.



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We argue by induction on $n$. Assume that $\sum_{i=1}^{n} d_i \geq 6n - 4$ and $\min\{d_1, \ldots, d_n\} \geq 4$. This implies that $n \geq 7$ since $\pi \in \mathcal{GS}$. If $n = 7$, then $d_1 = d_2 = 6$. Hence, by Theorem 3.1, we have $\pi \in \mathcal{GS}(\mathcal{S}_3)$ and we only need consider that $n \geq 8$ with $d_2 \leq n - 2$. We divide our discussion according to the value of $d_n$.

If $d_n \geq 7$, then we consider the laying sequence $\pi'$. Then, by Theorem 2.7, $\pi' \in \mathcal{GS}$. Since $\sum_{i=1}^{n} d_i \geq nd_n \geq 7n$, We also have

$$\sum_{i=1}^{n-1} d_i' \geq nd_n - 2d_n = (n-2)d_n \geq 6(n-1) - 4.$$

Hence, by the induction hypothesis, $\pi'$ has an $\mathcal{S}_3$-connected realization. Therefore, by Lemma 2.10, we have $\pi \in \mathcal{GS}(\mathcal{S}_3)$.

If $d_n = 6$ and $\sum_{i=1}^{n} d_i \geq 6n + 2$, then the case can be completed by considering the laying sequence $\pi'$ and using the induction hypothesis. Assume instead that $d_n = 6$ and $\sum_{i=1}^{n} d_i \leq 6n$. Then we have $\pi = (6^n)$. Now we consider the lifting sequence $\pi_l = (6^{n-5}, 5^4)$. Using Lemma 4.1 and by the induction hypothesis, we obtain that $\pi_l$ is graphic and belongs to $\mathcal{GS}(\mathcal{S}_3)$. For the case of $n = 8$, we have $\pi_l = (6^3, 5^4)$. We denote $G'$ as the $\mathcal{S}_3$-connected realization of $(6^3, 5^4)$ shown in Figure 5. Let $u_1$, $u_2$, and $u_3$ be the vertices of $G'$ with degree 6, and let $v_1, \ldots, v_4$ be the vertices of $G'$ with degree 5. We introduce a new vertex $w$ and construct a new graph $G$ by setting

$$V(G) = V(G') \cup \{w\} \quad \text{and} \quad E(G) = E(G') \cup \{wu_1, wu_2, wv_1, \ldots, wv_4\} - \{u_1u_2\}.$$

Therefore, based on Lemma 2.3 and the fact that $G_{[w,u_1u_2]} = G' \in \langle \mathcal{S}_3 \rangle$, $G$ is an $\mathcal{S}_3$-connected realization of $\pi$. This implies that $\pi \in \mathcal{GS}(\mathcal{S}_3)$. For the cases of $n \geq 9$, as $4 \times 6 - (6-2) < (n-5) \times 6$ and by Lemma 2.9, we conclude that $\pi \in \mathcal{GS}(\mathcal{S}_3)$.

If $d_n = 5$, then we consider the value of $\sum_{i=1}^{n} d_i$. For $\sum_{i=1}^{n} d_i \geq 6n$, we consider the laying sequence $\pi'$. Then, by the induction hypothesis and Lemma 2.10, we have $\pi \in \mathcal{GS}(\mathcal{S}_3)$. For $\sum_{i=1}^{n} d_i = 6n - 2$ and $d_3 = 5$, we determine $\pi = (d_1, d_2, 5^{n-2})$ and $\pi_{-2} = (d_1 - 2, d_2 - 2, 3^{n-2})$. Using Theorems 1.3 and 2.6, we can obtain that $\pi_{-2}$ is graphic and belongs to $\mathcal{GS}(\mathbb{Z}_3)$. Therefore, by Lemma 4.4, we obtain that $\pi \in \mathcal{GS}(\mathcal{S}_3)$. For $\sum_{i=1}^{n} d_i = 6n - 2$ and $d_3 \geq 6$, we consider the lifting sequence $\pi_l$, which satisfies $\sum_{i=1}^{n-1} d_i^l = 6(n-1) - 4$ and $\min\{d_1^l, d_2^l, \ldots, d_{n-1}^l\} \geq 5$. Based on Lemma 4.1, we have $\pi_l \in \mathcal{GS}$. Then, by the induction hypothesis, we can conclude that $\pi_l \in \mathcal{GS}(\mathcal{S}_3)$. Since $d_1 + d_2 + d_3 - (5-2) \leq 3n - 8 \leq 3n + 1 \leq d_4 + d_5 + \cdots + d_{n-1}$, we have $\pi \in \mathcal{GS}(\mathcal{S}_3)$ by Lemma 2.9. For $\sum_{i=1}^{n} d_i = 6n - 4$, we have $\pi \in \mathcal{GS}(\mathcal{S}_3)$ based on Theorem 4.5.

If $d_n = 4$, then $d_2 \geq 5$, otherwise $\sum_{i=1}^{n} d_i \leq (n-1) + 4 \times (n-1) = 5n - 5 < 6n - 4$, which leads to a contradiction. The following enumerates all possible cases of $\pi$.

- For $\sum_{i=1}^{n} d_i \geq 6n - 2$ and $d_4 \geq 5$, we consider the laying sequence $\pi'$. Then, by the induction hypothesis and as Lemma 2.10, we conclude that $\pi \in \mathcal{GS}(\mathcal{S}_3)$.
- For $\sum_{i=1}^{n} d_i \geq 6n - 2$ and $d_4 = 4$, we consider the lifting sequence $\pi_l$. According to $d_2 \geq 5$, we have $\min\{d_1^l, \ldots, d_{n-1}^l\} \geq 4$. Since $f(\pi_l) = 4$, a simple calculation shows that $\pi_l$ is graphic by Theorem 2.6. Furthermore, we obtain $\pi_l \in \mathcal{GS}(\mathcal{S}_3)$ based on the fact that $\sum_{i=1}^{n-1} d_i^l \geq 6(n-1) - 4$ and by the induction hypothesis. Therefore, as $d_1 + d_2 - (4-2) \leq 2(n-2) < 4n - 3 \leq d_3 + d_4 + \cdots + d_{n-1}$ and by Lemma 2.9, we have $\pi \in \mathcal{GS}(\mathcal{S}_3)$.
- For $\sum_{i=1}^{n} d_i = 6n - 4$, we consider the lifting sequence $\pi_l$, because $d_n = 4$, then $\pi_l = (d_1 - 1, d_2 - 1, d_3, \ldots, d_{n-1})$ and $\sum_{i=1}^{n-1} d_i^l = 6(n-1) - 4$. Since $d_2 - 1 \geq 5 - 1$, we have $\min\{d_1^l, \ldots, d_{n-1}^l\} \geq 4$. If $\pi_l$ is graphic, then, by the induction hypothesis and using Lemma 2.9, we have $\pi \in \mathcal{GS}(\mathcal{S}_3)$. Assume instead that $\pi_l \notin \mathcal{GS}$. Since the sequence $(d_1, \ldots, d_{n-1}, 2)$ is the laying sequence of $\pi_l$, then according to Theorem 2.7, $\pi_l \notin \mathcal{GS}$ if and only if $(d_1, \ldots, d_{n-1}, 2) \notin \mathcal{GS}$. For convenience, we denote $\pi'' = (d_1, \ldots, d_{n-1}, 2)$. Furthermore, Theorem 2.6 states that there exists an integer $k$ such that $1 \leq k \leq f(\pi'')$ and $\sum_{i=1}^{k} d_i > k(k-1) + \sum_{i=k+1}^{n-1} \min\{k, d_i\} + \min\{k, 2\}$. Clearly, by the fact that $d_1 \leq n - 1$ and the smallest term of $\pi''$ is 2, it is evident that $k \notin \{1, 2\}$. Additionally,



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since $d_3, d_4 \le n-2$, it can be concluded that $k \notin \{3,4\}$. Thus, our attention now turns to the cases where $k \ge 5$. Now

$$\begin{aligned} 6n-4 = \sum_{i=1}^{n} d_i = \sum_{i=1}^{k} d_i + \sum_{i=k+1}^{n} d_i \\ > k(k-1) + 4(n-1-k) + 2 + 4(n-k) \\ = (6n-4) + (2n-18) + (k-4)(k-5). \end{aligned}$$

Note that the last inequality holds only if $k = 5$ and $n = 8$. Hence we have $\sum_{i=1}^{8} d_i = 44$ and $d_5 \ge 5$. Furthermore, we get $d_6 = 4$, otherwise

$$\begin{aligned} \sum_{i=1}^{8} d_i = \sum_{i=1}^{5} d_i + \sum_{i=6}^{8} d_i > 5 \times 4 + \sum_{i=6}^{7} \min\{5, d_i\} + 2 + \sum_{i=6}^{8} d_i \\ \ge (20 + 5 + 4 + 2) + (5 + 2 \times 4) \\ = 44, \end{aligned}$$

which is a contradiction to $\sum_{i=1}^{8} d_i = 44$. Thus $\pi = (d_1, \ldots, d_5, 4^3)$ with $d_1 + \cdots + d_5 = 32$. This implies that $d_1 = d_2 = 7$. However, this contradicts the fact that $d_2 \le n-2 = 6$.

*This completes the proof of Theorem 1.4.* □


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## Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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