Journal of Graph Theory

WILEY

ARTICLE

Modulo Flows and Integer Flows in Signed Graphs

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Received: 23 August 2024 | **Revised:** 18 March 2025 | **Accepted:** 8 May 2025

Funding: Miaomiao Han was partially supported by the National Natural Science Foundation of China (No. 12471333). Jiaao Li was supported by the National Natural Science Foundation of China (Nos. 12222108, 12131013), the Natural Science Foundation of Tianjin (Nos. 24JCJQJC00130, 22JCYBJC01520), and the Fundamental Research Funds for the Central Universities, Nankai University. You Lu was partially supported by National Natural Science Foundation of China (Nos. 12271438, 12431013), Guangdong Basic and Applied Basic Research Foundation (No. 2023A1515012340), Natural Science Foundation of Qinghai Province (No. 2022-ZJ-753), and Shaanxi Fundamental Science Research Project for Mathematics and Physics (No. 22JSZ009). Rong Luo was partially supported by a grant from the Simons Foundation (No. 839830).

Keywords: integer flow | modulo flow | signed graph

ABSTRACT

Modulo flow is a powerful tool in the study of flows in both ordinary graphs and signed graphs. For ordinary graphs, Tutte showed that a graph admits a nowhere-zero k-flow if and only if it admits a nowhere-zero $\mathbb{Z}_k$-flow. However, such equivalence does not hold any more for signed graphs. Mačajova and Škoviera [*SIAM Journal of Discrete Mathematics* 31 (2017) 1937–1952] proved that every flow-admissible signed graph with a nowhere-zero $\mathbb{Z}_2$-flow admits a nowhere-zero 4-flow. DeVos et al. [*Journal of Combinatorial Theory Series B* 149 (2021) 198–221] proved that every flow-admissible signed graph admits a nowhere-zero 11-flow by converting certain special nowhere-zero $\mathbb{Z}_6$-flows into integer flows, and as a key step, they showed that every signed graph with a nowhere-zero $\mathbb{Z}_3$-flow admits a nowhere-zero 5-flow. In this paper, we study how to convert $\mathbb{Z}_4$-flows and $\mathbb{Z}_5$-flows into integer flows by proving the following two results: (1) every flow-admissible signed graph with a nowhere-zero $\mathbb{Z}_4$-flow admits a nowhere-zero 8-flow; (2) every bridgeless signed graph with a nowhere-zero $\mathbb{Z}_5$-flow admits a nowhere-zero 7-flow. Combining known results, it follows that every flow-admissible signed graph with a nowhere-zero $\mathbb{Z}_k$-flow admits a nowhere-zero $2k$-flow for each integer $k \geq 2$.

1 | Introduction

The following fundamental theorem of Tutte provides a powerful tool in the flow theory of ordinary graphs.

Theorem 1.1 (Tutte [1]). *Let $k \geq 2$ be an integer. If a graph G admits a nowhere-zero $\mathbb{Z}_k$-flow (τ, f), then it admits a nowhere-zero k-flow (τ, g) such that $g(e) \equiv f(e) \pmod k$ for each edge $e \in E(G)$.*

Although most landmark results of flow theory are stated as integer flow results, due to Theorem 1.1, they were initially proved for modulo flows, such as, the 8-flow theorem by Jaeger [2], the 6-flow theorem by Seymour [3], the weak 3-flow theorem by Thomassen [4], and the 3-flow theorem by Lovász et al. [5].

Theorem 1.1 is extended to signed graphs without long barbells when k is odd in Ref. [6].

Theorem 1.2 (Lu et al. [6]). *Let (G, σ) be a long-barbell-free signed graph and $k \geq 3$ be an odd integer. If (G, σ) admits a nowhere-zero $\mathbb{Z}_k$-flow (τ, f), then it has a nowhere-zero k-flow (τ, g) such that $g(e) \equiv f(e) \pmod k$ for each edge e.*

Unfortunately, Theorem 1.2 does not hold for signed graphs in general. That is, modulo k-flows and integer k-flows are no longer equivalent for signed graphs. The signed graph in Figure 1-(1) admits a nowhere-zero $\mathbb{Z}_2$-flow but no nowhere-zero k-flow for any integer $k \geq 2$, the signed graph in Figure 1-(2) admits a nowhere-zero $\mathbb{Z}_3$-flow but no nowhere-zero 3-flow [7], and the signed graph in Figure 1-(3) admits a nowhere-zero $\mathbb{Z}_4$-flow but no nowhere-zero 4-flow [7].

However, the modulo flows remain a powerful tool for studying the integer flows in signed graphs. Up to now, almost all results on integer flows in signed graphs heavily rely on how to convert modulo flows into integer flows (see [8–11]), more specifically to attack the 6-flow conjecture proposed by Bouchet [12] in 1983: *every flow-admissible signed graph admits a nowhere-zero 6-flow*. DeVos et al. [8] proved a modulo flow version of Bouchet's conjecture, and by applying the following results on covering/converting a $\mathbb{Z}_2$-flow and a $\mathbb{Z}_3$-flow into integer flows, they further proved the existence of nowhere-zero 11-flows.

Theorem 1.3 (Cheng et al. [13]). *If a signed graph is connected and admits a $\mathbb{Z}_2$-flow (τ, f) such that* $\mathrm{supp}(f)$ *contains an even number of negative edges, then it admits a 3-flow (τ, g) such that* $\mathrm{supp}(f) \subseteq \mathrm{supp}(g)$ *and* $|g(e)| = 2$ *if and only if e is a bridge of $G[\mathrm{supp}(g)]$.*

Theorem 1.4 (DeVos et al. [8]). *Every signed graph with a nowhere-zero $\mathbb{Z}_3$-flow admits a nowhere-zero 5-flow (τ, f) such that $f(e) \neq \pm 3$ for each edge e and $f(e) = \pm 4$ only if e is a bridge of G.*

Theorem 1.5 (DeVos et al. [8]). *Every flow-admissible signed graph admits a nowhere-zero $\mathbb{Z}_6$-flow and a nowhere-zero 11-flow.*

Theorem 1.3 shows that a $\mathbb{Z}_2$-flow whose support contains an even number of negative edges can be covered by an integer 3-flow, while Theorem 1.4 is a conversion of a $\mathbb{Z}_3$-flow into an integer 5-flow, and they together result in a conversion of a special $\mathbb{Z}_6$-flow into an integer 11-flow. Along the line of Theorem 1.4, Mačajova and Škoviera [14] show the following theorem, which is a conversion of a $\mathbb{Z}_2$-flow into an integer 4-flow.

Theorem 1.6 (Mačajova and Škoviera [14]). *Every flow-admissible signed graph with a nowhere-zero $\mathbb{Z}_2$-flow admits a nowhere-zero 4-flow.*

The aforementioned results demonstrate the conversions of $\mathbb{Z}_2$-, $\mathbb{Z}_3$-, and $\mathbb{Z}_6$-flows into integer flows. In this paper, we supplement the above results by proving new results on converting $\mathbb{Z}_4$-flows and $\mathbb{Z}_5$-flows into integer flows as follows.

Theorem 1.7.

1. *Every flow-admissible signed graph with a nowhere-zero $\mathbb{Z}_4$-flow admits a nowhere-zero 8-flow.*
2. *Every bridgeless signed graph with a nowhere-zero $\mathbb{Z}_5$-flow admits a nowhere-zero 7-flow.*

For converting $\mathbb{Z}_k$-flows into integer flows for general k, DeVos [15] proved that, when k is an odd prime, a $\mathbb{Z}_k$-flow can be converted into an integer $2k$-flow.

Theorem 1.8 (DeVos [15]). *Let $k \geq 3$ be a prime. If a signed graph admits a $\mathbb{Z}_k$-flow (τ, f), then it has a $2k$-flow (τ, g) such that $g(e) \equiv f(e) \pmod{k}$ for each edge e.*

In this paper, we generalize Theorem 1.8 to all odd integers k as follows.

Theorem 1.9. *Let $k \geq 3$ be an odd integer. If a signed graph admits a $\mathbb{Z}_k$-flow (τ, f), then it admits a $2k$-flow (τ, g) such that $g(e) \equiv f(e) \pmod{k}$ for each e.*

By Theorem 1.9, together with Theorems 1.5, 1.6, and 1.7-(1) above, we establish the following theorem.

Theorem 1.10. *Let $k \geq 2$ be an integer. If a flow-admissible signed graph admits a nowhere-zero $\mathbb{Z}_k$-flow, then it admits a nowhere-zero $2k$-flow.*

We would like to remark that there is a narrative error in [13] which mistakenly states that Bouchet [12] proved that if a signed graph admits a $\mathbb{Z}_k$-flow (τ, f_1), then it admits a $2k$-flow (τ, f_2) with $\mathrm{supp}(f_1) \subseteq \mathrm{supp}(f_2)$ (see Theorem 1.1 in [13]), but in fact the orginal result of Bouchet [12] is Lemma 3.4 below, which is very different from Theorem 1.10. In addition, we will see below that Lemma 3.4 of Bouchet [12] is very helpful, and we shall apply this lemma and its refinement (Lemma 3.5 below) as a tool to provide a simple proof of Theorem 1.2 different from the one in Ref. [6].

The rest of the paper is organized as follows. Section 2 introduces some basic notations related to flows. Section 3 completes the proofs of Theorems 1.2 and 1.9. Sections 4 and 5 complete the proofs of (1) and (2) in Theorem 1.7, respectively.

2 | Notation and Terminology

Graphs considered in this paper are finite and may have multiple edges or loops. For most standard notation and terminology, we follow Bondy and Murty [16] and West [17].

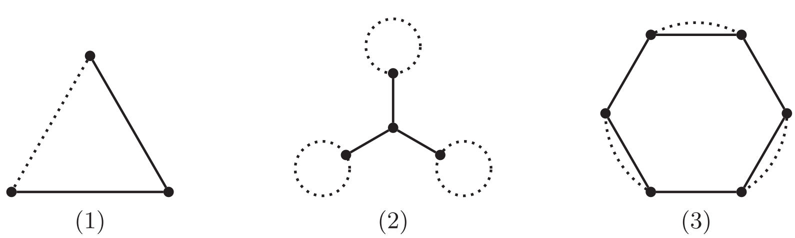

FIGURE 1 | Three specific signed graphs. Dotted edges are negative.

A *signed graph* (G, σ) is an *underlying graph* G together with a *signature* $\sigma: E(G) \to \{1, -1\}$. For convenience, the signature σ is usually omitted if no confusion arises and is written as σ_G if it needs to emphasize G. An edge e is *positive* if $\sigma(e) = 1$ and *negative* if $\sigma(e) = -1$. For a subgraph H of G, denoted by $E_N(H)$, the set of negative edges in H. A signed graph is *all-positive* or *ordinary* if it contains no negative edges.

A circuit is *balanced* if it contains an even number of negative edges and is *unbalanced* otherwise. A signed graph is called *balanced* if it contains no unbalanced circuit, and is *unbalanced* otherwise. A *signed circuit* is a signed graph of any of the following three types: (1) a balanced circuit; (2) a short barbell, the union of two unbalanced circuits that meet at a single vertex; (3) a long barbell, the union of two vertex disjoint unbalanced circuits connected by a path.

In a signed graph, *switching* at a vertex v means interchanging the signs of all edges (except loops) incident on v. Two signed graphs are said to be *equivalent* if one can be obtained from the other by making a sequence of switch operations. In particular, a signed graph is balanced (resp., *antibalanced*) if and only if it is equivalent to an all-positive (resp., all-negative) signed graph.

We regard an edge $e = uv$ of a signed graph G as two half edges h_e^u and h_e^v, where h_e^u is incident with u and h_e^v is incident with v. Let $H_G(v)$ (or simply $H(v)$ if no confusion occurs) be the set of all half edges incident with v, and $H(G)$ be the set of all half edges of G. An *orientation* of G is a mapping $\tau: H(G) \to \{1, -1\}$ such that for each $e = uv \in E(G)$, $\tau(h_e^u)\tau(h_e^v) = -\sigma_G(e)$. For $h_e^u \in H(G)$, h_e^u is *oriented away from* u if $\tau(h_e^u) = 1$ and h_e^u is *oriented toward* u if $\tau(h_e^u) = -1$. A signed graph G together with an orientation τ is called an *oriented signed graph*, denoted by (G, τ).

Definition 2.1. Let (G, τ) be an oriented signed graph. Let A be an abelian group and $f: E(G) \to A$ be a mapping.

1. The *boundary* of f under τ is the mapping $\partial(\tau, f): V(G) \to A$ defined as
$$\partial(\tau, f)(v) = \sum_{h \in H(v)} \tau(h) f(e_h)$$
for each vertex v, where e_h is the edge of G containing h.
2. (τ, f) is called an A-flow in G if $\partial(\tau, f)(v) = 0$ for each vertex v.
3. The *support* of f, denoted by $\mathrm{supp}(f)$, is the set of edges e with $f(e) \neq 0$.
4. An A-flow (τ, f) is said to be *nowhere-zero* if $\mathrm{supp}(f) = E(G)$.
5. An A-flow (τ, f) is called a *k-flow* if $A = \mathbb{Z}$ and $|f(e)| < k$ for each edge e.

For convenience, we shorten the notation of nowhere-zero k-flow and nowhere-zero A-flow as k-NZF and A-NZF, respectively.

A signed graph is *flow-admissible* if it admits a k-NZF for some integer k. Note that switching at a vertex does not change the parity of the number of negative edges in a circuit and although technically it changes the flows, it only reverses the directions of the half edges incident with the vertex, but the directions of other half edges and the flow values of all edges remain the same. Bouchet [12] provided a well-known characterization for flow-admissible signed graphs: *a signed graph G is flow-admissible if and only if every edge of G belongs to a signed circuit*.

3 | Proofs of Theorems 1.2 and 1.9

In an oriented signed graph (G, τ), for a mapping $f: E(G) \to \mathbb{Z}$ and an odd integer k, we have the following property:

$$\sum_{v \in V(G)} \partial(\tau, f)(v) = 2 \sum_{e \in E_N(G)} \tau(h_e) f(e),$$

where h_e is a half edge in e, and thus is an even multiple of k when $\partial(\tau, f)(v) \equiv 0 \pmod{k}$ for each $v \in V(G)$. Therefore, Theorems 1.2 and 1.9 follow from the following slightly stronger result.

Theorem 3.1. *Let (G, τ) be an oriented signed graph and $k \geq 1$ be an integer. Let $f: E(G) \to \mathbb{Z}$ such that $\partial(\tau, f)(v) \equiv 0 \pmod{k}$ for each $v \in V(G)$ and $\sum_{v \in V(G)} \partial(\tau, f)(v) = (2p)k$ for some integer p. Then (G, σ) has an ℓ-flow (τ, g) such that $g(e) \equiv f(e) \pmod{k}$ for every $e \in E(G)$, where $\ell = 2k$. Moreover, when (G, σ) is long-barbell-free, then $\ell = k$.*

To prove Theorem 3.1, we need the following four lemmas.

Lemma 3.2 (Li et al. [18]). *Let (G, τ) be an oriented and unbalanced signed graph and A be an abelian group. Then for each mapping $b: V(G) \to A$ with $\sum_{v \in V(G)} b(v) = 2\alpha$ for some $\alpha \in A$, there is a mapping $f: E(G) \to A$ such that $\partial(\tau, f)(v) = b(v)$ for each $v \in V(G)$.*

For a signed circuit C in a signed graph G, the *characteristic flow* of C is a k-flow (τ, χ) of G satisfying that $\mathrm{supp}(\chi) = E(C)$ and $k = 3$ if C is a long barbell and $k = 2$ otherwise. The following lemma is a corollary of Theorem 3.1 in [19].

Lemma 3.3 (Mačajova and Škoviera [19]). *Let (τ, f) be a $\mathbb{Z}$-flow of a signed graph G. For every $e_0 \in E(G)$ with $f(e_0) \neq 0$, there is a signed circuit C containing e_0 such that C has a characteristic flow (τ, χ) satisfying $f(e)\chi(e) > 0$ for every edge $e \in E(C)$.*

Lemma 3.4 (Bouchet [12]). *Let $k \geq 1$ be an integer. For each $\mathbb{Z}$-flow (τ, f) of a signed graph G, there exists a $2k$-flow (τ, g) in G such that $g(e) \equiv f(e) \pmod{k}$ for every edge $e \in E(G)$.*

Lemma 3.4 can be improved from $2k$-flow to k-flow for long-barbell-free signed graphs.

Lemma 3.5. *Let G be a long-barbell-free signed graph and $k \geq 1$ be an integer. For each $\mathbb{Z}$-flow (τ, f) in G, there is a k-flow (τ, g) in G such that $f(e) \equiv g(e) \pmod{k}$ for every edge $e \in E(G)$.*

Proof. We choose a $\mathbb{Z}$-flow (τ, g) in G such that $g(e) \equiv f(e) \pmod{k}$ for any $e \in E(G)$ and

$$\eta(g) = \sum_{e \in E(G)} \tilde{g}(e)$$

is as small as possible, where $\tilde{g}(e) = \max\{0, |g(e)| + 1 - k\}$.

If $\eta(g) = 0$, then (τ, g) is a desired k-flow. Assume that $\eta(g) > 0$ below.

Since $\eta(g) > 0$, there is an edge $e_0 \in E(G)$ such that $|g(e_0)| \geq k$. By Lemma 3.3, there is a signed circuit C containing e_0 such that there is a characteristic flow (τ, χ) satisfying $g(e)\chi(e) > 0$ for each $e \in E(C)$. Without loss of generality, we may assume that $g(e) \geq 0$ and $\chi(e) \geq 0$ for each $e \in E(G)$. Since G does not contain long barbells, C is either a balanced circuit or a short barbell. Thus χ is a 2-flow and $\chi(e) = 1$ for each edge $e \in E(C)$.

Let $h = g - k\chi$. Then (τ, h) is also a $\mathbb{Z}$-flow in G and $h(e) \equiv g(e) \equiv f(e) \pmod{k}$ for each $e \in E(G)$.

We claim that $\tilde{h}(e) \leq \tilde{g}(e)$ for each edge e and $\tilde{h}(e_0) < \tilde{g}(e_0)$. It is clear that $\tilde{h}(e_0) < \tilde{g}(e_0)$ since $e_0 \in E(C)$ and $g(e_0) \geq k$. If $e \notin E(C)$, then $h(e) = g(e)$ and thus $\tilde{h}(e) = \tilde{g}(e)$. If $e \in E(C)$ and $g(e) < k$, then $|h(e)| = k - g(e) < k$ and thus $\tilde{h}(e) = 0 = \tilde{g}(e)$. If $e \in E(C)$ and $g(e) \geq k$, then $g(e) > h(e) = g(e) - k \geq 0$ and thus $\tilde{h}(e) = \max\{0, 1 + h(e) - k\} < 1 + g(e) - k = \tilde{g}(e)$. This proves the claim.

The above claim implies that $\eta(h) < \eta(g)$, a contradiction to the minimality of $\eta(g)$. Thus the proof of the lemma is complete. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. It is trivial for $k = 1$ and assume that $k \geq 2$. If G is balanced, then the theorem follows from Theorem 1.1 since (τ, f) is a $\mathbb{Z}_k$-flow in G. Assume that G is unbalanced below.

Note that $\frac{1}{k}\partial(\tau, f)(v) \in \mathbb{Z}$ for each $v \in V(G)$ and $\sum_{v \in V(G)} \left[\frac{1}{k}\partial(\tau, f)(v)\right] = 2p$. By Lemma 3.2, there is a mapping $f_1 : E(G) \to \mathbb{Z}$ such that $\partial(\tau, f_1)(v) = \frac{1}{k}\partial(\tau, f)(v)$ for each $v \in V(G)$. Then $(\tau, f - kf_1)$ is a $\mathbb{Z}$-flow in G. Thus by Lemmas 3.4 and 3.5, there exists an ℓ-flow (τ, g) in G such that $g(e) \equiv (f - kf_1)(e) \equiv f(e) \pmod{k}$ for each edge $e \in E(G)$, which is a desired flow. □

4 | Proof of Theorem 1.7-(1)

In a signed graph G, *contract* a positive edge e means deleting the edge and then identifying its ends. The resulting signed graph is denoted by G/e. For a positive edge subset or an all-positive subgraph H, let G/H denote the signed graph obtained from G by contracting all edges of H.

Lemma 4.1 (Luo et al. [20]). *Let H be a bridgeless and all-positive subgraph of a signed graph G. Then G/H is flow-admissible if and only if so is G.*

Theorem 4.2 (Luo et al. [20]). *Let G be a flow-admissible signed cubic graph. If the underlying graph of G is 3-edge-colorable, then G admits an 8-NZF.*

Let k be a positive integer. A *k-factor* of a graph G is a k-regular spanning subgraph of G. For a mapping $f : E(G) \to \mathbb{Z}$ of a signed graph G, denote $E_{f=\pm i} = \{e \in E(G) : |f(e)| = i\}$. From the proof of Theorem 4.2 in Ref. [7], we obtained the following result.

Lemma 4.3 (Mačajova and Škoviera [7]). *Let G be a signed cubic graph. If (τ, f) is a $\mathbb{Z}_4$-NZF of G, then $G[E_{f=1} \cup E_{f=3}]$ is an antibalanced 2-factor of G.*

Now we are ready to prove Theorem 1.7-(1).

Proof of Theorem 1.7-(1). Let G be a counterexample with $\beta(G) = \sum_{x \in V(G)} |d_G(x) - 3|$ as small as possible. Clearly, $\delta(G) \geq 3$. Let (τ, f) be a $\mathbb{Z}_4$-NZF in G. Consider f as an integer-valued mapping where $1 \leq f(e) \leq 3$ for each edge e. By Theorem 3.1, we have the following claim. □

Claim 4.1. $\sum_{v \in V(G)} \partial(\tau, f)(v) = (2p + 1)4$ for some integer p.

Claim 4.2. The subgraph induced by $E_{f=1} \cup E_{f=3}$ is an even subgraph containing no unbalanced circuit.

Proof of Claim 4.2. Let $H = G[E_{f=1} \cup E_{f=3}]$. It is obvious that H is even since $\partial(\tau, f)(v) \equiv 0 \pmod{4}$ and $f(e) \in \{1, 2, 3\}$ for each $v \in V(G)$ and $e \in E_G(v)$, where $E_G(v)$ is the set of edges incident with v.

Suppose by contradiction that C is an unbalanced circuit of H. Define a mapping $f_1 : E(G) \to \mathbb{Z}$ such that $f_1(e) = 2$ if $e \in E(C)$ and $f_1(e) = 0$ otherwise. Then (τ, f_1) is a $\mathbb{Z}_4$-flow of G and $(\tau, f + f_1)$ is also a $\mathbb{Z}_4$-NZF of G. Since C is unbalanced, it contains an odd number of negative edges. Thus there is an integer q such that

$$\sum_{v \in V(G)} \partial(\tau, f_1)(v) = 2 \sum_{e \in E_N(C)} \tau(h_e) f_1(e) = 4 \sum_{e \in E_N(C)} \tau(h_e)$$
$$= (2q + 1)4.$$

Therefore,

$$\sum_{v \in V(G)} \partial(\tau, f + f_1)(v) = \sum_{v \in V(G)} \partial(\tau, f)(v) + \sum_{v \in V(G)} \partial(\tau, f_1)(v)$$
$$= [2(p + q) + 2]4.$$

By Theorem 3.1, G has an 8-NZF (τ, g) with $g(e) \equiv (f + f_1)(e) \pmod{k}$ for every $e \in E(G)$. This contradiction proves the claim. □

Claim 4.3. G is cubic.

Proof of Claim 4.3. Suppose to the contrary that there is a vertex v with degree $t = d_G(v) \geq 4$. Without loss of generality, assume that v is the head of every half-edge incident with v.

By the pigeonhole principle, there must be two edges incident with v having the same flow value. Denote by $h_1, h_2, h_3, h_4, \ldots, h_t$ the half edges incident with v and $f(e_{h_1}) = f(e_{h_2}) = a$, where e_{h_i} is the edge containing h_i. □

We construct a new signed graph G' and extend the $\mathbb{Z}_4$-NZF (τ, f) to be a $\mathbb{Z}_4$-NZF (τ', g) in G' according to the following two cases.

Case 1. $a = 2$ or $f(e_{h_3}) \neq a$.

Let G' be the signed graph obtained from G by replacing v with an all-positive cycle $H_v = x_1x_2x_3x_4x_1$ such that x_i is the end of h_i for $i \in \{1, 2, 3\}$ and x_4 is the end of $h_4, \ldots, h_t$. Fix an orientation τ' of G' such that $\tau'|_{H(G)} = \tau$ and H_v is a directed circuit (see Figure 2).

Denote $f(e_{h_3}) = b$ and choose $c \in \mathbb{Z}_4$. Define a mapping g: $E(G') \to \mathbb{Z}_4$ such that $g(e) = f(e)$ for each edge $e \in E(G)$, $g(x_1x_2) = c, g(x_2x_3) = a + c, g(x_3x_4) = a + b + c$ and $g(x_4x_1) = c - a$ (see Figure 2). Since (τ, f) is a $\mathbb{Z}_4$-NZF in G, $\partial(\tau, f)(v) = \sum_{i=1}^{t} f(e_{h_i}) = 0$, and hence (τ', g) is a $\mathbb{Z}_4$-flow in G'.

If $a = 2$, then choose $c \in \{1, 3\} \setminus \{b\}$. If $a \in \{1, 3\}$, then $b = f(e_{h_3}) \neq a$ and choose $c = 2$. In both cases, $\{a, b, c\} = \{1, 2, 3\}$ or $\{2, 2, 1\}$ or $\{2, 2, 3\}$, and thus $g(x_1x_2) = c \neq 0$, $g(x_2x_3) = a + c \neq 0, g(x_3x_4) = a + b + c \neq 0$ and $g(x_4x_1) = c - a \neq 0$. Therefore (τ', g) is a $\mathbb{Z}_4$-NZF in G'.

Case 2. $a \in \{1, 3\}$ and $f(e_{h_3}) = a$.

Let G' be the signed graph obtained from G by replacing v with an all-positive graph $H_v = x_1x_2x_3x_4x_5x_6x_1 + x_3x_6$ such that x_i is the end of h_i for $i \in \{1, 2\}$, x_4 is the end of h_3, and x_5 is the end of $h_4, \ldots, h_t$. Fix an orientation τ' of G' such that $\tau'|_{H(G)} = \tau$, $x_1x_2x_3x_4x_5x_6x_1$ is a directed circuit, and x_3x_6 is oriented from x_3 to x_6 (see Figure 3).

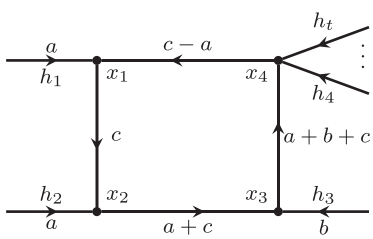

FIGURE 2 | The all-positive circuit $H_v = x_1x_2x_3x_4x_1$.

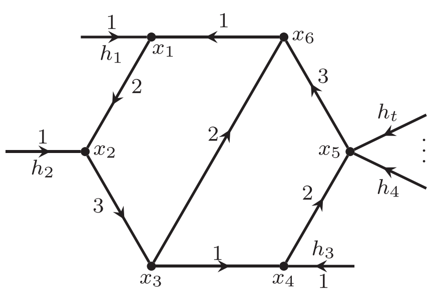

FIGURE 3 | The all-positive signed graph $H_v = x_1x_2x_3x_4x_5x_6x_1 + x_3x_6$.

Without loss of generality, assume that $a = 1$. Let $g : E(G) \to \mathbb{Z}_4$ such that $g(e) = f(e)$ for each edge $e \in E(G), g(x_1x_2) = g(x_3x_6) = g(x_4x_5) = 2, g(x_2x_3) = g(x_5x_6) = 3$, and $g(x_3x_4) = g(x_6x_1) = 1$ (see Figure 3). Therefore, (τ', g) is a $\mathbb{Z}_4$-NZF in G'.

By the above two cases, we have that H_v is bridgeless and all-positive, $G = G'/H_v$ and G' admits a $\mathbb{Z}_4$-NZF. Since G is flow-admissible, by Lemma 4.1, G' is also flow-admissible. Note that $\beta(G') = \beta(G) - 1 < \beta(G)$. By the minimality of $\beta(G)$, G' admits an 8-NZF. Since $G = G'/H_v$ and H_v is all-positive, G admits an 8-NZF as well, a contradiction. This proves the claim. □

The final step. Since G is cubic by Claim 4.3, the induced subgraph $H = G[E_{f=1} \cup E_{f=3}]$ is a 2-factor of G. By Claim 4.2, each component of H is a balanced circuit. By Lemma 4.3, each component of H is also antibalanced and thus is an even circuit. This implies that the underlying graph G is 3-edge-colorable. By Theorem 4.2, G admits an 8-NZF, a contradiction to the choice of G. This contradiction completes the proof of Theorem 1.7-(1). □

5 | Proof of Theorem 1.7-(2)

To prove Theorem 1.7-(2), we need a couple of earlier results. The first one is the reduction lemma of an original signed graph with modulo orientation (modulo flow) to a new regular signed graph. For a signed graph G and a positive integer p, an orientation τ of G is called a *modulo* $(2p + 1)$*-orientation* if for every vertex $v \in V(G)$,

$$\sum_{h \in H_G(v)} \tau(h) \equiv 0 \pmod{2p + 1}.$$

Lemma 5.1 (Han et al. [21]). *Let G be a bridgeless signed graph with a modulo 5-orientation τ. Then there exists a bridgeless 5-regular signed graph G' and a set T of positive edges of G' such that $G = G'/T$ and G' admits a modulo 5-orientation τ' which agrees with τ on all edges in $E(G)$.*

The next one is a special case of a theorem due to Bäbler [22].

Lemma 5.2 (Bäbler [22]). *Every bridgeless 5-regular graph has a 2-factor.*

Now we are ready to prove Theorem 1.7-(2).

Proof of Theorem 1.7-(2). Let (τ, f) be a $\mathbb{Z}_5$-NZF of G such that $f(e) \in \{1, 2\}$ for each edge $e \in E(G)$. For each edge e with $f(e) = 2$, replace it with two parallel edges with the same orientation as e. Let G_1 be the resulting signed graph and τ_1 be the resulting orientation of G_1. Then G_1 is bridgeless since G is bridgeless, and τ_1 is a modulo 5-orientation of G_1.

By Lemma 5.1, let G_2 be a bridgeless 5-regular signed graph and T be a set of positive edges of G_2 such that $G_1 = G_2/T$ and G_2 admits a modulo 5-orientation τ_2, which agrees with τ_1 on all edges in $E(G_1)$.

By Lemma 5.2, G_2 has a 2-factor M_1 and a 3-factor M_2 such that $M_2 = G_2 - E(M_1)$. Define the mapping $f_1 : E(G_2) \to \mathbb{Z}$ as follows:

$$f_1(e) = \begin{cases} 3 & \text{if } e \in E(M_1); \\ -2 & \text{if } e \in E(M_2). \end{cases}$$

Since G_2 is 5-regular and τ_2 is a modulo 5-orientation of G_2, we have $\tau_2(h_1) = \tau_2(h_2)$ for any $v \in V(G_2)$ and $h_1, h_2 \in H_{G_2}(v)$, and thus (τ_2, f_1) is a 4-NZF. By contracting all positive edges in T, $G_1 = G_2/T$ admits a 4-NZF (τ_1, f_2) with $f_2 = f_1|_{G_2-T}$. Thus $f_2(e) \in \{3, -2\}$ for each edge $e \in E(G_1)$.

For each edge $e \in E(G)$, either it corresponds to the two parallel edges e_1, e_2 in G_1 with the same orientations as e in τ (when $f(e) = 2$), or it corresponds to a single edge e_3 in G_1 with the same orientation as e in τ (when $f(e) = 1$). For each edge $e \in E(G)$ and its corresponding edge e_1, e_2, or e_3 in $E(G_1)$, we define a mapping $f_3 : E(G) \to \mathbb{Z}$ as follows:

$$f_3(e) = \begin{cases} f_2(e_1) + f_2(e_2) & \text{for each edge } e \in E(G) \text{ with } f(e) = 2 \\ f_2(e_3) & \text{for each edge } e \in E(G) \text{ with } f(e) = 1. \end{cases}$$

Then $f_3(e) \in \{3, -2, 3 + 3, -2 - 2, 3 - 2\} = \{-4, -2, 1, 3, 6\}$ for each edge $e \in E(G)$. Therefore (τ, f_3) is a desired 7-NZF of G. This proves Theorem 1.7-(2). □

6 | Conclusions

As stated in Theorem 1.10, every flow-admissible signed graph with a $\mathbb{Z}_k$-NZF admits a $2k$-NZF. We conjecture that this bound $2k$ can be reduced to $k + 2$,

Conjecture 6.1. *Let $k \geq 2$ be an integer. If a flow-admissible signed graph (G, σ) admits a $\mathbb{Z}_k$-NZF, then it also admits a $(k + 2)$-NZF.*

As indicated by Theorems 1.6, 1.4 and 1.7-(2), Conjecture 6.1 holds for $k = 2$, $k = 3$, and $k = 5$ (G is bridgeless). It is even plausible that every signed graph with a $\mathbb{Z}_5$-NZF admits a 5-NZF.

It has been shown in Ref. [6] that every flow-admissible long-barbell-free signed graph admits a 6-NZF. Thus, together with Theorem 1.2, for any flow-admissible long-barbell-free signed graph (G, σ), if (G, σ) admits a $\mathbb{Z}_k$-NZF, then it also admits a k-NZF, except when $k = 4$. As noted in Ref.[6], there is a signed W_5 (the wheel with six vertices) which has a $\mathbb{Z}_4$-NZF but does not have a 4-NZF (see [23]). While such graphs admit a 6-NZF, we believe they admit a 5-NZF.

Conjecture 6.2. *Let (G, σ) be a flow-admissible long-barbell-free signed graph. If (G, σ) admits a $\mathbb{Z}_4$-NZF, then it admits a 5-NZF.*

The main focus of this paper is to explore the relationship between modulo flows and integer flows in given signed graphs. Another avenue of research is to study the relationship between the flow indices of a signed graph and its underlying (ordinary) graph.

Theorem 1.6 is equivalent to the statement that if a graph G admits a nowhere-zero 2-flow, then any flow-admissible signed graph (G, σ) admits a 4-NZF. It is proved in Ref. [20] that a flow-admissible signed graph admits a nowhere-zero 8-flow provided that its underlying (ordinary) graph admits a nowhere-zero 4-flow. We conclude the paper with the following problem.

Problem 6.3. Is it true that for every integer $k \geq 2$, if a graph G admits a k-NZF, then any flow-admissible signed graph (G, σ) admits a $2k$-NZF?

The answer to the above problem is affirmative for $k = 2, 4$ and $k \geq 6$ (by the 11-flow theorem), but it remains open for $k = 3$ and $k = 5$.

Data Availability Statement

The authors have nothing to report.

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