


# Odd Induced Subgraphs in Graphs with Treewidth at Most Two

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**Abstract** A long-standing conjecture asserts that there exists a constant  $c > 0$  such that every graph of order  $n$  without isolated vertices contains an induced subgraph of order at least  $cn$  with all degrees odd. Scott (Comb Probab Comput 1:335–349, 1992) proved that every graph  $G$  has an induced subgraph of order at least  $|V(G)|/(2\chi(G))$  with all degrees odd, where  $\chi(G)$  is the chromatic number of  $G$ , this implies the conjecture for graphs with bounded chromatic number. But the factor  $1/(2\chi(G))$  seems to be not best possible, for example, Radcliffe and Scott (Discrete Math 275–279, 1995) proved  $c = \frac{2}{3}$  for trees, Berman et al. (Aust J Comb 81–85, 1997) showed that  $c = \frac{2}{5}$  for graphs with maximum degree 3, so it is interesting to determine the exact value of  $c$  for special family of graphs. In this paper, we further confirm the conjecture for graphs with treewidth at most 2 with  $c = \frac{2}{5}$ , and the bound is best possible.

**Keywords** Odd graph · Treewidth · Induced subgraph

## 1 Introduction

Gallai [5] proved that for every graph  $G$ , the vertex set  $V(G)$  can be partitioned into two sets, each of which induces a subgraph with all degrees even. This implies that every graph of order  $n$  contains an induced subgraph of order at least  $\lceil \frac{n}{2} \rceil$  with all degrees even, and this is best possible by considering paths. This motivates us to consider the

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problem that how large we can find an induced subgraph with all degrees odd. We call a graph with all degrees odd an *odd graph*. Let  $f(G)$  denote the maximum order of an odd induced subgraph in a graph  $G$ . The following long-standing conjecture was cited by Caro in [2] as “part of the graph theory folklore” and the origin is unclear.

**Conjecture 1** *There exists a constant  $c > 0$  such that for every graph  $G$  without isolated vertices,  $f(G) \geq c|V(G)|$ .*

The “without isolated vertices” constraint is natural because an odd graph does not contain isolated vertices. Many results related to Conjecture 1 have been obtained in literatures. In particular, Caro [2] proved that  $f(G) \geq (1 - o(1))\sqrt{|V(G)|/6}$ , later, Scott [7] improved the lower bound to  $\frac{c|V(G)|}{\log|V(G)|}$  for some  $c > 0$ , in the same paper, Scott also proved that every graph  $G$  has an odd induced subgraph of order at least  $|V(G)|/(2\chi(G))$ , where  $\chi(G)$  is the chromatic number of  $G$ , this implies the conjecture for graphs with bounded chromatic number. But the factor  $1/(2\chi(G))$  seems to be not best possible, for example, Radcliffe and Scott [6] confirmed the conjecture for trees (graphs with treewidth one) with  $c = \frac{2}{3}$  and Berman et al. [1] proved the conjecture for graphs with maximum degree 3 with  $c = \frac{2}{5}$ . In this paper, we further confirm Conjecture 1 for graphs with treewidth at most 2 with  $c = \frac{2}{5}$ , and the value of  $c$  is best possible.

A *tree decomposition* of a graph  $G$  is a tree  $T$ , where

1. Each vertex  $i$  of  $T$  is labeled by a subset  $B_i$  of vertices of  $G$ .
2. Each edge of  $G$  is in a subgraph induced by at least one of the  $B_i$ ,
3. For every three vertices  $i, j, k$  in  $T$  with  $j$  lying on the path from  $i$  to  $k$  in  $T$ ,  $B_i \cap B_k \subseteq B_j$ .

The *tree-width*  $tw(G)$  of  $G$  is the minimum integer  $p$  such that there exists a tree decomposition of  $G$  with all subsets of cardinality at most  $p + 1$ . Tree-decomposition is one of the most general and effective techniques for designing efficient algorithms, and a tree-like structure allows us to solve certain difficult problems. It is well-known that a connected graph has treewidth one if and only if it is tree. In terms of treewidth, the result of Radcliffe and Scott [6] can be restated as follows.

**Theorem 2** [6] *For any connected graph  $T$  with  $tw(T) = 1$ ,  $f(T) \geq 2\lfloor \frac{|V(T)|+1}{3} \rfloor$ .*

The following theorem is our main result.

**Theorem 3** *For every graph  $G$  with  $tw(G) \leq 2$  and without isolated vertices,  $f(G) \geq \frac{2}{5}|V(G)|$ .*

The lower bound is sharp by considering the graph of which each component is a cycle of length 5. We remark that graph with treewidth at most two is also known as  *$K_4$ -minor-free graph*, see Proposition 1 in Sect. 3. Some upper and lower bounds on graphs with small treewidth are also discussed in the last section.

In this paper, standard notation follows from [3]. In particular, for a graph  $G$  and a set  $S \subseteq V(G)$ , let  $G[S]$  be the subgraph induced by  $S$  and let  $N_G(S)$  be the union of neighbors of vertices in  $S$ , for a vertex  $u \in V(G)$ , let  $N_G^1(u) = \{x \mid x \in N_G(u) \text{ and } d_G(x) = 1\}$  and  $N_G^2(u) = \{x \mid x \in N_G(u) \text{ and } d_G(x) = 2\}$ , and denote

$N_G^2(u, v) = N_G^2(u) \cap N_G^2(v)$ . A vertex of degree  $k$  is called a  $k$ -vertex. Define  $S_G(u) = \{x \mid x \in N_G(u) \text{ with } d_G(x) \geq 3 \text{ or there exists a vertex } z \in N_G^2(u, x)\}$ . Let  $D_G(u) = |S_G(u)|$ . For two sets  $S, T$ , we use  $S \setminus T$  denote  $S - (S \cap T)$ .

The rest of the paper is arranged as follows. In Sect. 2, we establish structural properties of minimum counterexample of Theorem 3. Then the proof of Theorem 3 is presented in Sect. 3, and in the last section, we give some discussions.

## 2 Properties of Minimal Counterexample

Let  $G$  be a minimum counterexample of Theorem 3 with respect to the order of  $G$ . The main idea of the proof is as the following. We first pick some set  $V_0 \subset V(G)$  so that  $G' = G - V_0$  has no isolated vertex, by the minimality of  $G$ ,  $G'$  has an odd induced subgraph  $H'$  with  $|V(H')|/|V(G')| \geq 2/5$ . We will find a set  $S_0 \subset V_0$  with  $|S_0| \geq \frac{2}{5}|V_0|$  such that  $S_0 \cup V(H')$  induces an odd induced subgraph  $H$  of  $G$ . We should be careful to remain the parity of the degrees of the vertices in  $N_G(S_0) \cap V(H')$  and  $S_0 \cap N_G(V(H'))$ . Here we allow  $V(G') = \emptyset$ .

**Lemma 4** *Let  $u$  be a vertex of  $G$  with  $D_G(u) = 1$  and let  $S_G(u) = \{v\}$ . Then  $N_G^1(u) \cup N_G^2(u, v) = \emptyset$ .*

*Proof* Suppose to the contrary that  $G$  has a vertex  $u$  with  $D_G(u) = 1$  and  $N_G^1(u) \cup N_G^2(u, v) \neq \emptyset$ . Let  $t_1 = |N_G^1(u)|$  and  $t_2 = |N_G^2(u, v)|$ . Then  $t_1 + t_2 > 0$ .

*Case 1*  $|N_G^1(v)| \leq 1$ .

Set  $V_0 = N_G^1(u) \cup N_G^2(u, v) \cup \{u, v\} \cup N_G^1(v)$  and  $G' = G - V_0$ . Then  $G'$  has no isolated vertex, so, by the minimality of  $G$ ,  $G'$  has an odd induced subgraph  $H'$  with  $|V(H')| \geq \frac{2}{5}|V(G')|$ . Now let  $S_0 = V_0 \setminus (N_G^1(v) \cup \{v\})$ . Then  $G[S_0] \cong K_{1, t_1+t_2}$ , and so  $G[S_0]$  contains an odd induced subgraph  $K = K_{1, t}$  with  $t = t_1 + t_2$  if  $t_1 + t_2$  is odd or  $t = t_1 + t_2 - 1$  if  $t_1 + t_2$  is even. So  $(t + 1)/|V_0| \geq (t + 1)/(t_1 + t_2 + 3) \geq 2/5$ . Furthermore, we have  $N_G(V(K)) \cap V(H') = \emptyset$  and  $V(K) \cap N_G(V(H')) = \emptyset$ . Hence  $H = H' \cup K$  is an odd induced subgraph of  $G$  with  $|V(H)| \geq \frac{2}{5}|V(G)|$ , a contradiction.

*Case 2*  $|N_G^1(v)| \geq 2$ .

Choose a vertex  $x \in N_G^1(v)$  and set  $V_0 = N_G^1(u) \cup N_G^2(u, v) \cup \{u, x\}$  and  $G' = G - V_0$ . Then  $G'$  has no isolated vertex and so, by the minimality of  $G$ ,  $G'$  has an odd induced subgraph  $H'$  with  $|V(H')|/|V(G')| \geq 2/5$ .

**Claim 1**  $v$  must be in  $V(H')$ .

Suppose to the contrary that  $v \notin V(H')$ . Set  $S_0 = V_0 \setminus \{x\}$ , then  $G[S_0] \cong K_{1, t_1+t_2}$ , and so  $G[S_0]$  contains an odd induced subgraph  $K = K_{1, t}$  with  $t = t_1 + t_2$  or  $t = t_1 + t_2 - 1$  with respect to the parity of  $t_1 + t_2$ . Note that  $(t + 1)/|V_0| = (t + 1)/(t_1 + t_2 + 2) > 2/5$ ,  $N_G(V(K)) \cap V(H') = \emptyset$  and  $V(K) \cap N_G(V(H')) = \emptyset$ . Therefore,  $H = K \cup H'$  is an odd induced subgraph of  $G$  with  $|V(H)|/|V(G)| > 2/5$ , a contradiction. The claim is true.

Now suppose  $v \in V(H')$ .

**Claim 2** We have  $t_2 \leq t_1$ .

Suppose to the contrary that  $t_2 \geq t_1 + 1$ . Set  $S_0 = N_G^2(u, v) \cup \{x\}$  if  $t_2$  is odd or  $S_0 = N_G^2(u, v)$  if  $t_2$  is even, then  $S_0 \cup V(H')$  still induces an odd subgraph  $H$  of  $G$  with  $|V(H)| = |S_0| + |V(H')| \geq \frac{2}{5}|V_0| + \frac{2}{5}|V(G')| = \frac{2}{5}|V(G)|$ , a contradiction, where the second inequality holds since  $|S_0|/|V_0| \geq |S_0|/(t_1 + t_2 + 2) \geq |S_0|/(2t_2 + 1) \geq 2/5$ . Hence the claim holds.

Now suppose  $t_2 \leq t_1$  and let  $T_1$  (resp.  $T_2$ ) be a maximum subset of odd (resp. even) order in  $N_G^1(u)$ . Set  $S_0 = T_1 \cup \{u\}$  if  $uv \notin E(G)$  or  $S_0 = T_2 \cup \{u, x\}$  if  $uv \in E(G)$ . In both cases,  $|S_0|/|V_0| = |S_0|/(t_1 + t_2 + 2) \geq |S_0|/(2t_1 + 2) \geq 2/5$  unless  $t_1 = t_2 = 2$  and  $uv \notin E(G)$ . Then  $S_0 \cup V(H')$  induces an odd subgraph  $H$  of  $G$  with  $|V(H)| = |S_0| + |V(H')| \geq 2/5|V_0| + 2/5|V(G')| = 2/5|V(G)|$ , a contradiction. For  $t_1 = t_2 = 2$  and  $uv \notin E(G)$ , reset

$V_0 = N_G^1(u) \cup N_G^2(u, v) \cup \{u\} = N_G(u) \cup \{u\}$  and let  $G' = G - V_0$ , then  $G'$  has no isolated vertex and so, by the minimality of  $G$ ,  $G'$  has an odd induced subgraph  $H'$  with  $|V(H')|/|V(G')| \geq 2/5$ . Let  $N_G^1(u) = \{a, b\}$  and set  $S_0 = \{a, u\}$ . Then  $S_0 \cup V(H')$  induces an odd subgraph  $H$  of  $G$  with  $|V(H)| \geq 2 + \frac{2}{5}|V(G')| = \frac{2}{5}|V(G)|$ , a contradiction again.

This completes the proof of the lemma. □

**Lemma 5** *Let  $u$  be a vertex of  $G$  with  $D_G(u) = 2$  and let  $S_G(u) = \{v, w\}$ . Then  $N_G^1(u) \cup N_G^2(u, v) \cup N_G^2(u, w) = \emptyset$ .*

*Proof* Suppose to the contrary that  $G$  has a vertex  $u$  with  $S_G(u) = \{v, w\}$  and  $N_G^1(u) \cup N_G^2(u, v) \cup N_G^2(u, w) \neq \emptyset$ . Let  $t_1 = |N_G^1(u)|$ ,  $t_2 = |N_G^2(u, v)|$  and  $t_3 = |N_G^2(u, w)|$ . Then  $t_1 + t_2 + t_3 > 0$ . Let  $\bar{N}_G^2(v, w) = N_G^2(v, w) \setminus \{u\}$ .

**Claim 3** *If  $N_G^1(v) = \emptyset$  then  $N_G^1(w) \cup \bar{N}_G^2(v, w) \neq \emptyset$ ; symmetrically, if  $N_G^1(w) = \emptyset$  then  $N_G^1(v) \cup \bar{N}_G^2(v, w) \neq \emptyset$ .*

We only prove the first statement, the second one can be proved similarly. Suppose to the contrary that  $N_G^1(w) \cup \bar{N}_G^2(v, w) = \emptyset$ . Set  $V_0 = N_G(u) \cup \{u, v, w\}$  and  $G' = G - V_0$ . Then  $G'$  has no isolated vertex and so, by the minimality of  $G$ ,  $G'$  has an odd induced subgraph  $H'$  with  $|V(H')| \geq \frac{2}{5}|V(G')|$ . Let  $S$  be a maximum subset of  $N_G^1(u) \cup N_G^2(u, v) \cup N_G^2(u, w)$  so that  $s = |S|$  is odd. Then  $S_0 = S \cup \{u\}$  induces an odd subgraph  $K \cong K_{1,s}$  of  $G[V_0]$ , furthermore  $|S_0|/|V_0| \geq (s+1)/(t_1+t_2+t_3+3) \geq 2/5$ . Note that  $N_G(S_0) \cap V(H') = \emptyset$  and  $S_0 \cap N_G(V(H')) = \emptyset$ . Therefore,  $H = K \cup H'$  is an odd induced subgraph of  $G$  with  $|V(H)| \geq \frac{2}{5}|V_0| + \frac{2}{5}|V(G')| = \frac{2}{5}|V(G)|$ , a contradiction. The claim is true.

*Case 1*  $N_G^1(v) = \emptyset$ .

*Subcase 1.1*  $|N_G(w) \setminus (N_G^2(u, w) \cup \{u, v\})| \leq 1$ .

Note that  $|N_G(w) \setminus (N_G^2(u, w) \cup \{u, v\})| \leq 1$  implies that  $|N_G^1(w) \cup \bar{N}_G^2(v, w)| \leq 1$ .

By Claim 3,  $|N_G^1(w) \cup \bar{N}_G^2(v, w)| = 1$  and so

$N_G(w) \setminus (N_G^2(u, w) \cup \{u, v\}) = N_G^1(w) \cup \bar{N}_G^2(v, w)$ . Let  $N_G^1(w) \cup \bar{N}_G^2(v, w) = \{x\}$  and set  $V_0 = N_G(u) \cup \{u, v, w, x\}$  and  $G' = G - V_0$ . Then  $G'$  has no isolated vertex and so, by the minimality of  $G$ ,  $G'$  has an odd induced subgraph  $H'$  with  $|V(H')| \geq \frac{2}{5}|V(G')|$ . Let  $S$  be a maximum subset of  $N_G^1(u) \cup N_G^2(u, v) \cup N_G^2(u, w)$

so that  $s = |S|$  is odd. Then  $S_0 = S \cup \{u\}$  induces an odd subgraph  $K \cong K_{1,s}$  of  $G[V_0]$ , furthermore  $|S_0|/|V_0| \geq (s + 1)/(t_1 + t_2 + t_3 + 4) \geq 2/5$  unless  $t_1 + t_2 + t_3 = 2$ . Note that  $N_G(S_0) \cap V(H') = \emptyset$  and  $S_0 \cap N_G(V(H')) = \emptyset$ . Hence  $H = K \cup H'$  is an odd induced subgraph of  $G$  with  $|V(H)| \geq \frac{2}{5}|V_0| + \frac{2}{5}|V(G')| = \frac{2}{5}|V(G)|$  provided that  $t_1 + t_2 + t_3 \neq 2$ , a contradiction.

For  $t_1 + t_2 + t_3 = 2$ , notice that  $E_G(w, V(G')) = \emptyset$  because  $N_G(w) \setminus (N_G^2(u, w) \cup \{u, v\}) = N_G^1(w) \cup \bar{N}_G^2(v, w)$ . If  $E_G(v, V(G')) = \emptyset$  then  $G$  is a graph of order six, it can be easily checked that  $G$  cannot be a counterexample. If  $t_3 = 2$  then

$S_0 = N_G^2(u, w) \cup \{w, x\}$  induces an odd subgraph  $K \cong K_{1,3}$ , and therefore

$H = K \cup H'$  is an odd induced subgraph of  $G$  with

$$|V(H)| \geq 4 + \frac{2}{5}|V(G')| > \frac{2}{5}|V(G)|,$$

a contradiction. Hence suppose

$E_G(v, V(G')) \neq \emptyset$  and  $t_3 < 2$ . Reset  $V_0 = (N_G(u) \cup \{u, w, x\}) \setminus \{v\}$  and  $G' = G - V_0$ . Then  $G'$  has no isolated vertex and so, by the minimality of  $G$ ,  $G'$  has an odd induced subgraph  $L'$  with  $|V(L')| \geq \frac{2}{5}|V(G')|$ . If  $v \notin V(L')$  or  $vw, vx \notin E(G)$ , then  $\{w, x\} \cup V(L')$  induces an odd subgraph  $H$  of  $G$  with  $|V(H)| \geq 2 + \frac{2}{5}|V(G')| \geq \frac{2}{5}|V(G)|$ , a contradiction. So suppose  $v \in V(L')$  and  $vw \in E(G)$  or  $vx \in E(G)$ . If  $N_G(v) \cap V_0$  has two nonadjacent vertices, say  $\{a, b\}$ , then  $\{a, b\} \cup V(L')$  induces an odd subgraph of  $G$  with order at least  $\frac{2}{5}|V(G)|$ , a contradiction. This implies that  $N_G^2(u, v) = \emptyset$  (i.e.  $t_2 = 0$ ),  $vx \notin E(G)$  (i.e.  $x \in N_G^1(w)$ ) and  $vw, uw \in E(G)$  (otherwise, it is easy to choose two nonadjacent vertices from  $N_G^2(u, v) \cup \{u, w, x\}$ ). As  $t_1 + t_2 + t_3 = 2$ ,  $t_2 = 0$ , and  $t_3 < 2$ , we have  $t_1 > 0$ . Choose  $a \in N_G^1(u)$ , then  $\{a, u, w, x\} \cup V(L')$  induces an odd subgraph  $H$  of  $G$  with  $|V(H)| \geq 4 + \frac{2}{5}|V(G')| > \frac{2}{5}|V(G)|$ , a contradiction.

*Subcase 1.2*  $|N_G(w) \setminus (N_G^2(u, w) \cup \{u, v\})| \geq 2$ .

Choose  $x \in N_G^1(w) \cup \bar{N}_G^2(v, w)$  (this can be done because  $N_G^1(w) \cup \bar{N}_G^2(v, w) \neq \emptyset$  by Claim 3) and set  $V_0 = (N_G(u) \cup \{u, v, x\}) \setminus \{w\}$  and  $G' = G - V_0$ . Then  $G'$  has no isolated vertex and so, by the minimality of  $G$ ,  $G'$  has an odd induced subgraph  $H'$  with  $|V(H')| \geq \frac{2}{5}|V(G')|$ .

**Claim 4**  $w \in V(H')$ .

If  $w \notin V(H')$ , choose a maximum subset  $S$  of  $N_G(u) \setminus \{v, w\}$  so that  $s = |S|$  is odd, then  $S_0 = S \cup \{u\}$  induces an odd subgraph  $K \cong K_{1,s}$  of  $G[V_0]$  such that  $|S_0|/|V_0| = (s + 1)/(t_1 + t_2 + t_3 + 3) \geq 2/5$ . Clearly,  $N_G(S_0) \cap V(H_0) = \emptyset$  and  $S_0 \cap N_G(V(H_0)) = \emptyset$ . Hence  $H = K \cup H'$  is an odd induced subgraph of  $G$  with  $|V(H)| \geq \frac{2}{5}|V(G)|$ , a contradiction. The claim holds.

**Claim 5**  $t_3 \leq t_1 + t_2$ .

If  $t_3 \geq t_1 + t_2 + 1$ , choose a maximum subset  $S_0$  of  $N_G^2(u, w) \cup \{x\}$  so that  $|S_0|$  is even, then  $|S_0|/|V_0| = |S_0|/(t_1 + t_2 + t_3 + 3) \geq |S_0|/(2t_3 + 2) \geq 2/5$  unless  $t_3 = 2$  and  $t_1 + t_2 = 1$ . Therefore,  $S_0 \cup V(H')$  induces an odd subgraph  $H$  of  $G$  with  $|V(H)| \geq \frac{2}{5}|V(G)|$  unless  $t_3 = 2$  and  $t_1 + t_2 = 1$ . For  $t_3 = 2$  and  $t_1 + t_2 = 1$ , reset  $V_0 = (N_G(u) \cup \{u, v\}) \setminus \{w\}$  and  $G' = G - V_0$ , then, again by the minimality of  $G$ ,  $G'$  has an odd induced subgraph  $L'$  with  $|V(L')| \geq \frac{2}{5}|V(G')|$ . If  $w \in V(L')$ , set

$S_0 = N_G^2(u, w)$ , and if  $w \notin V(L')$ , set  $S_0 = \{u, y\}$ , where  $y$  is a vertex in  $N_G^2(u, w)$ . In both cases,  $S_0 \cup V(L')$  induces an odd subgraph  $H$  of  $G$  with  $|V(H)| \geq \frac{2}{5}|V(G)|$ . Therefore, we always obtain a contradiction and so the claim follows.

Now let  $S$  be a maximum subset of  $N_G^1(u) \cup N_G^2(u, v)$  so that  $s = |S|$  is even if  $uw \in E(G)$ , and  $s = |S|$  is odd if  $uw \notin E(G)$ . Set  $S_0 = S \cup \{u, x\}$  if  $uw \in E(G)$  and  $S_0 = S \cup \{u\}$  if  $uw \notin E(G)$ . Clearly,  $S_0 \cup V(L')$  induces an odd subgraph  $H$  of  $G$  and furthermore, for  $uw \in E(G)$ ,  $|S_0|/|V_0| \geq (s+2)/(t_1+t_2+t_3+3) \geq (t_1+t_2+1)/(2t_1+2t_2+3) \geq 2/5$ ; and for  $uw \notin E(G)$ ,  $|S_0|/|V_0| = (s+1)/(t_1+t_2+t_3+3) \geq 2/5$  unless  $t_1+t_2 = 2, t_3 = 1$  or  $t_1+t_2 = 2, t_3 = 2$  or  $t_1+t_2 = t_3 = 4$ . Therefore, but some exceptions,  $H$  is an odd induced subgraph with  $|V(H)| \geq \frac{2}{5}|V(G)|$ , a contradiction. Note that all the exceptions occur under the assumption  $uw \notin E(G)$ . In the following of the case, we show that each of the three exceptions cannot occur in the minimal counterexample  $G$  as well.

For  $t_1 + t_2 = 2$  and  $t_3 = 1$ , reset  $V_0 = N_G(u) \cup \{u, v\}$  and let  $G' = G - V_0$ , then, by the minimality of  $G$ ,  $G'$  has an odd induced subgraph  $L'$  with  $|V(L')| \geq \frac{2}{5}|V(G')|$ . Choose a vertex  $a \in N_G^1(u) \cup N_G^2(u, v)$ , then  $S_0 = \{u, a\}$  induces an odd subgraph  $K \cong K_{1,1}$  of  $G[V_0]$ . As  $N_G(S_0) \cap V(L') = \emptyset$  and  $S_0 \cap N_G(V(L')) = \emptyset$ ,  $H = K \cup L'$  is an odd induced subgraph of  $G$  with  $|V(H)| \geq \frac{2}{5}|V(G)|$ , a contradiction.

For  $t_1 + t_2 = t_3 = 2$ . If  $|N_G^1(w) \cup \bar{N}_G^2(v, w)| \leq 2$ , reset  $V_0 = N_G(u) \cup N_G^1(w) \cup \bar{N}_G^2(v, w) \cup \{u, v, w, x\}$ , then  $G' = G - V_0$  has no isolated vertex and so, by the minimality of  $G$ ,  $G'$  has an odd induced subgraph  $L'$  with  $|V(L')| \geq \frac{2}{5}|V(G')|$ . Let  $S$  be a subset of  $N_G(u) \setminus \{v\}$  with  $s = |S| = 3$  (this can be done because  $|N_G(u)| \geq t_1 + t_2 + t_3 = 4$ ). Then  $S_0 = S \cup \{u\}$  induces an odd subgraph  $K \cong K_{1,3}$  of  $G[V_0]$  and therefore  $H = K \cup L'$  is an odd induced subgraph of  $G$  with  $|V(H)| \geq \frac{2}{5}|V(G)|$ , a contradiction. Now suppose  $|N_G^1(w) \cup \bar{N}_G^2(v, w)| \geq 3$ . Choose a vertex  $y \in N_G^1(w) \cup \bar{N}_G^2(v, w)$  with  $y \neq x$ . Reset  $V_0 = N_G(u) \cup \{u, v, x, y\}$  and  $G' = G - V_0$ , then, by the minimality of  $G$ ,  $G'$  has an odd induced subgraph  $L'$  with  $|V(L')| \geq \frac{2}{5}|V(G')|$ . Let  $S_0 = N_G^2(u, w) \cup \{x, y\}$  if  $w \in V(L')$ , and let  $S_0 = S \cup \{u\}$  if  $w \notin V(L')$ , where  $S$  is a maximum subset of  $N_G(u) \setminus \{v\}$  with  $s = |S| = 3$ . Clearly,  $S_0 \cup V(L')$  induces an odd subgraph  $H$  of  $G$  with  $|V(H)| > \frac{2}{5}|V(G)|$ , a contradiction.

For  $t_1 + t_2 = t_3 = 4$ , reset  $V_0 = N_G(u) \cup \{u, v\}$  and  $G' = G - V_0$ , then  $G'$  has no isolated vertices and so, by the minimality of  $G$ ,  $G'$  has an odd induced subgraph  $L'$  with  $|V(L')| \geq \frac{2}{5}|V(G')|$ . Let  $S_0 = N_G^2(u, w)$  if  $w \in V(L')$ , or let  $S_0 = N_G^1(u) \cup N_G^2(u, v) \cup \{u\}$  if  $w \notin V(L')$ . Then  $|S_0|/|V_0| \geq 2/5$  and  $S_0 \cup V(L')$  induces an odd subgraph  $H$  of  $G$  with  $|V(H)| \geq \frac{2}{5}|V(G)|$ , a contradiction.

This proves Case 1. By symmetry, we may also assume  $N_G^1(w) \neq \emptyset$  to verify the following remaining case.

*Case 2*  $N_G^1(v) \neq \emptyset$ .

Choose  $x \in N_G^1(v)$  and  $y \in N_G^1(w)$ , set  $V_0 = (N_G(u) \cup \{u, x, y\}) \setminus \{v, w\}$  and  $G' = G - V_0$ .

**Claim 6**  $G'$  has no isolated vertex.

Suppose to the contrary that  $G'$  has isolated vertices. Then  $v$  or  $w$  must be an isolated vertex of  $G'$ . Without loss of generality, assume  $v$  is an isolated vertex of  $G'$ . Then  $D_G(v) = 1$ . But  $N_G^1(v) \neq \emptyset$ , this is a contradiction to Lemma 4.

Hence  $G'$  has no isolated vertex and so, by the minimality of  $G$ ,  $G'$  has an odd induced subgraph  $H'$  with  $|V(H')| \geq \frac{2}{5}|V(G')|$ .

**Claim 7**  $H'$  contains at least one of  $\{v, w\}$ .

Suppose to the contrary that  $H'$  contains none of  $\{v, w\}$ . Let  $S$  be a maximum subset of  $N_G(u) \setminus \{v, w\}$  so that  $s = |S|$  is odd. Then  $S_0 = S \cup \{u\}$  induces an odd subgraph  $K \cong K_{1,s}$  with  $|S_0|/|V_0| = (s + 1)/(t_1 + t_2 + t_3 + 3) \geq 2/5$ . Note that  $N_G(S_0) \cap V(H') = \emptyset$  and  $S_0 \cap N_G(V(H')) = \emptyset$ . Thus  $H = K \cup H'$  is an odd induced subgraph of  $G$  with  $|V(H)| \geq \frac{2}{5}|V(G)|$ , a contradiction.

**Claim 8** If  $w \in V(H')$  then  $t_3 \leq t_1 + t_2$ . Symmetrically, if  $v \in V(H')$  then  $t_2 \leq t_1 + t_3$ .

We show that  $t_3 \leq t_1 + t_2$  when  $w \in V(H')$ . Suppose to the contrary that  $t_3 \geq t_1 + t_2 + 1$ . Let  $S_0$  be a maximum subset of  $N_G^2(u, w) \cup \{y\}$  such that  $|S_0|$  is even. Then  $S_0 \cup V(H')$  induces an odd subgraph  $H$  of  $G$  with  $|V(H)| = |S_0| + |V(H')| \geq \frac{2}{5}|V_0| + \frac{2}{5}|V(G')| = \frac{2}{5}|V(G)|$  unless  $t_3 = 2$  and  $t_1 + t_2 = 1$ .

For  $t_3 = 2$  and  $t_1 + t_2 = 1$ , reset  $V_0 = (N_G(u) \cup \{u, x\}) \setminus \{v, w\}$  and  $G' = G - V_0$ , then  $G'$  has no isolated vertex and so  $G'$  has an odd induced subgraph  $L'$  with  $|V(L')| \geq \frac{2}{5}|V(G')|$  by the minimality of  $G$ . If  $w \in V(L')$ , let  $S_0 = N_G^2(u, w)$ , then  $S_0 \cup V(L')$  induces an odd subgraph  $H$  with  $|V(H)| \geq \frac{2}{5}|V(G)|$ . Now suppose  $w \notin V(L')$ . If  $v \notin V(L')$ , choose a vertex  $z$  from  $N_G^1(u) \cup N_G^2(u, v)$ , then  $\{u, z\} \cup V(L')$  induces an odd subgraph  $H$  of  $G$  with  $|V(H)| \geq \frac{2}{5}|V(G)|$ . Hence  $v \in V(L')$ , choose a vertex  $z \in N_G^2(u, v) \cup \{u\}$  which is adjacent to  $v$ , then  $\{x, z\} \cup V(L')$  induces an odd subgraph  $H$  of  $G$  with  $|V(H)| \geq \frac{2}{5}|V(G)|$ . In all cases we get contradictions and so the claim follows.

We shall show that certain special cases cannot occur in the minimal counterexample  $G$ , which would be helpful to eliminate exception values in later discussion.

**Claim 9** If  $uw \notin E(G)$  then none of the following occurs in the minimal counterexample  $G$ .

- (a)  $t_1 + t_2 = 2$  and  $t_3 = 1$ ;
- (b)  $t_1 + t_2 = t_3 = p$ ,  $p = 2$  or  $4$ .

For  $t_1 + t_2 = 2$  and  $t_3 = 1$ , reset  $V_0 = (N_G(u) \cup \{u, x\}) \setminus \{w, v\}$  and  $G' = G - V_0$ , then  $G'$  has no isolated vertex and so, by the minimality of  $G$ ,  $G'$  has an odd induced subgraph  $L'$  with  $|V(L')| \geq \frac{2}{5}|V(G')|$ . If  $v \in V(L')$ , choose a vertex  $z \in N_G^2(u, v) \cup \{u\}$  which is adjacent to  $v$ , note that  $uw \notin E(G)$ , then  $\{x, z\} \cup V(L')$  induces an odd subgraph  $H$  of  $G$  with  $|V(H)| \geq \frac{2}{5}|V(G)|$ , a contradiction. Hence  $v \notin V(L')$ , choose a vertex  $z$  from  $N_G^1(u) \cup N_G^2(u, v)$ , then  $\{u, z\} \cup V(L')$  induces an odd subgraph  $H$  of  $G$  with  $|V(H)| \geq \frac{2}{5}|V(G)|$ , a contradiction.

For  $t_1 + t_2 = t_3 = p$ ,  $p = 2, 4$ , reset  $V_0 = (N_G(u) \cup \{u\}) \setminus \{v, w\}$  and  $G' = G - V_0$ , then  $G'$  has no isolated vertex and so, by the minimality of  $G$ ,  $G'$  has an odd induced subgraph  $L'$  with  $|V(L')| \geq \frac{2}{5}|V(G')|$ . If  $w \in V(L')$ , note that  $|N_G^2(u, w)| = t_3 = p$  is even,  $N_G^2(u, w) \cup V(L')$  induces an odd subgraph  $H$  of  $G$  with  $|V(H)| \geq \frac{2}{5}|V(G)|$ ,

a contradiction. So suppose  $w \notin V(L')$ . If  $uv \notin E(G)$  or  $v \notin V(L')$ , choose a subset  $S$  of  $N_G^2(u, w)$  so that  $|S| = p - 1$ , then  $S \cup \{u\} \cup V(L')$  induces an odd subgraph  $H$  of  $G$  with  $|V(H)| \geq \frac{2}{5}|V(G)|$ , a contradiction. Hence  $uv \in E(G)$  and  $v \in V(L')$ . If  $x \in V(L')$ , then  $N_G^2(u, w) \cup \{u\} \cup (V(L') \setminus \{x\})$  induces an odd subgraph  $H$  with  $|V(H)| = p + 1 + |V(L')| - 1 \geq \frac{2}{5}|V(G)|$ , a contradiction. Hence  $x \notin V(L')$ . Then  $N_G^2(u, w) \cup \{u\} \cup V(L') \cup \{x\}$  induces an odd subgraph  $H$  of  $G$  with  $|V(H)| \geq \frac{2}{5}|V(G)|$ , a contradiction. This proves the claim.

By Claim 7, we may assume, without loss of generality,  $w \in V(H')$ . Hence, by Claim 8,  $t_3 \leq t_1 + t_2$ . Now we divide the discussion into two subcases below.  
*Subcase 2.1*  $v \notin V(H')$ .

Let  $S$  be a maximum subset of  $N_G^1(u) \cup N_G^2(u, v)$  such that  $s = |S|$  is odd if  $uw \notin E(G)$  or  $s = |S|$  is even if  $uw \in E(G)$ . Set  $S_0 = S \cup \{u\}$  if  $uw \notin E(G)$  or  $S_0 = S \cup \{u, y\}$  if  $uw \in E(G)$ . Note that  $s = t_1 + t_2$  or  $t_1 + t_2 - 1$  depending on the parity of  $t_1 + t_2$  and  $|S_0| = s + 1$  or  $s + 2$  depending on  $uw \notin E(G)$  or  $uw \in E(G)$ . Notice that  $t_3 \leq t_1 + t_2$ , we have  $|S_0|/|V_0| = |S_0|/(t_1 + t_2 + t_3 + 3) \geq 2/5$  unless  $uw \notin E(G)$  and  $t_1 + t_2 = 2, t_3 = 1$ , or  $t_1 + t_2 = t_3 = 2$ , or  $t_1 + t_2 = t_3 = 4$ . Therefore  $S_0 \cup V(H')$  induces an odd subgraph  $H$  of  $G$  with  $|V(H)| \geq \frac{2}{5}|V(G)|$  but three exceptions. However, none of the exceptions occur in  $G$  by Claim 9. This yields a contradiction and verifies Subcase 2.1.

*Subcase 2.2*  $v \in V(H')$ .

By Claim 8, we have  $t_3 \leq t_1 + t_2$  and  $t_2 \leq t_1 + t_3$ . Furthermore, we have the following claim.

**Claim 10** *We have  $t_2 + t_3 \leq t_1$ .*

Suppose to the contrary that  $t_2 + t_3 \geq t_1 + 1$ . Let  $S_v$  be a maximum subset of  $N_G^2(u, v) \cup \{x\}$  such that  $|S_v|$  is even, let  $S_w$  be a maximum subset of  $N_G^2(u, w) \cup \{y\}$  such that  $|S_w|$  is even, and set  $S_0 = S_u \cup S_v$ . Then  $S_0 \cup V(H')$  induces an odd subgraph of  $G$ . By checking the parity of  $t_2$  and  $t_3$  with certain calculation, we have  $|S_0|/|V_0| \geq \frac{2}{5}$  unless  $t_1 = 1, t_2 + t_3 = 2$  and  $t_i, i = 2, 3$ , is even. But this exception cannot occur because  $t_3 \leq t_1 + t_2$  and  $t_2 \leq t_1 + t_3$ , a contradiction. Hence the claim holds.

Now, we choose a set  $S_0$  according to the following rules:

- (i) If  $uv \in E(G), uw \in E(G)$ , let  $S_0 = S_u \cup \{u, x, y\}$ , where  $S_u$  is the maximum subset of  $N_G^1(u)$  with size odd;
- (ii) If  $uv \in E(G), uw \notin E(G)$ , let  $S_0 = S_u \cup \{u, x\}$ , where  $S_u$  is the maximum subset of  $N_G^1(u)$  with size even;
- (iii) If  $uv \notin E(G), uw \in E(G)$ , let  $S_0 = S_u \cup \{u, y\}$ , where  $S_u$  is the maximum subset of  $N_G^1(u)$  with size even;
- (iv) If  $uv \notin E(G), uw \notin E(G)$ , let  $S_0 = S_u \cup \{u\}$ , where  $S_u$  is the maximum subset of  $N_G^1(u)$  with size odd.

Then  $S_0 \cup V(H')$  induces an odd subgraph of  $G$  by definition. It remains to compute  $|S_0|/|V_0|$ .

If  $t_1$  is odd, we have  $|S_0|/|V_0| \geq (t_1 + 1)/(t_1 + t_2 + t_3 + 3) \geq 2/5$  by Claim 10 in each of the cases (i)–(iv). If  $t_1$  is even, it follows from Claim 10 that  $|S_0|/|V_0| \geq (t_1 + 2)/(t_1 + t_2 + t_3 + 3) \geq 2/5$  in each of the cases (i)–(iii), and in the case (iv),

$|S_0|/|V_0| = t_1/(t_1+t_2+t_3+3) \geq 2/5$  unless  $t_1 = 2, t_2+t_3 = 2$  or  $t_1 = 4, t_2+t_3 = 4$ . Therefore,  $S_0 \cup V(H')$  induces an odd subgraph  $H$  of  $G$  with  $|V(H)| \geq \frac{2}{5}|V(G)|$  unless  $t_1 = t_2 + t_3 = 2$  or  $t_1 = t_2 + t_3 = 4$ .

For  $t_1 = t_2 + t_3 = p, p = 2, 4$ , reset  $V_0 = N_G(u) \cup \{u\}$  and  $G' = G - V_0$ , then  $G'$  has no isolated vertex and so, by the minimality of  $G, G'$  has an odd induced subgraph  $L'$  with  $|V(L')| \geq \frac{2}{5}|V(G')|$ . Choose a subset  $S$  of  $N_G^1(u)$  so that  $|S| = p - 1$ , then  $S \cup \{u\} \cup V(L')$  induces an odd subgraph  $H$  of  $G$  with  $|V(H)| \geq \frac{2}{5}|V(G)|$ , a contradiction.

The proof of the lemma is completed. □

The following three structural properties of the minimum counterexample  $G$  are direct consequence of Lemmas 4 and 5.

**Corollary 6** *Let  $V_1$  be the set of all 1-vertices in  $G$  and let  $P = N_G(V_1)$ . Suppose  $G_1 = G - V_1$ , then  $d_{G_1}(x) \geq 3$  for any  $x \in P$ .*

*Proof* Suppose to the contrary that there is a vertex  $x \in P$  with  $d_{G_1}(x) \leq 2$ . If  $d_{G_1}(x) = 0$  then  $G$  is isomorphic to a star, which cannot be a counterexample. Hence  $0 < d_{G_1}(x) \leq 2$ . This implies that  $0 < D_G(x) \leq 2$ . But  $|N_G^1(x)| \geq 1$ , this is a contradiction to Lemmas 4 or 5. □

**Corollary 7**  *$G$  has no adjacent 2-vertices.*

*Proof* Suppose to the contrary that  $G$  has two adjacent 2-vertices  $u, v$ . Then  $D_G(u) \leq 2$ . Let  $v_1 = N_G(v) \setminus \{u\}$ . Then  $v \in N_G^2(u, v_1)$ , which is a contradiction to Lemmas 4 or 5. □

**Corollary 8**  *$G$  has no vertex  $u$  with  $d_G(u) \geq 3$  so that  $D_G(u) \leq 2$ .*

*Proof* Suppose to the contrary that  $G$  has a vertex  $u$  with  $d_G(u) \geq 3$  and  $D_G(u) \leq 2$ . By Lemmas 4 and 5,  $u$  has no neighbor of degree at most 2 since  $G$  cannot be isomorphic to a star. This implies  $D_G(u) \geq d_G(u) \geq 3$ , a contradiction. □

### 3 Proof of Theorem 3

Before giving the proof, we need some definition and structural properties of graphs with treewidth at most 2. A graph  $G$  contains a graph  $H$  as a *minor* if  $H$  can be obtained from a subgraph of  $G$  by contracting edges, and  $G$  is called  *$H$ -minor free* if  $G$  does not have  $H$  as a minor. It is well known that

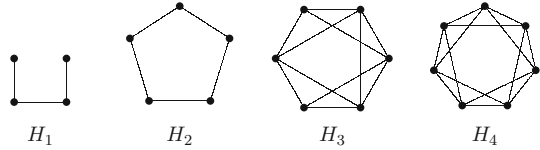
**Proposition 1** (Proposition 12.4.2 [3]) *A graph has treewidth at most 2 if and only if it is  $K_4$ -minor free.*

For  $K_4$ -minor free graphs, Lih et al. [4] gave a powerful structural property of them.

**Lemma 9** (Lemma 2 [4]) *If  $G$  is a  $K_4$ -minor free graph, then one of the following holds:*

- (a)  $\delta(G) \leq 1$ ;
- (b) *there exist two adjacent 2-vertices;*

**Fig. 1** Graphs  $H_k$  with treewidth  $k$  and  $\frac{f(H_k)}{|V(H_k)|} = \frac{2}{k+3}$  for  $k = 1, 2, 3, 4$



(c) *there exists a vertex  $u$  with  $d_G(u) \geq 3$  such that  $D_G(u) \leq 2$ .*

*Proof of Theorem 3* Let  $G$  be a minimum counterexample with respect to the order of  $G$ . By the minimality of  $G$ ,  $G$  must be connected. Let  $V_1$  be the set of all 1-vertices in  $G$  and  $P = N_G(V_1)$ . Let  $G_1 = G - V_1$ . By Corollaries 6 and 7,  $\delta(G_1) \geq 2$  and  $G_1$  has no adjacent 2-vertices. Clearly,  $tw(G_1) \leq 2$  and hence  $G_1$  is  $K_4$ -minor free. By Lemma 9,  $G_1$  has a vertex  $u$  with  $d_{G_1}(u) \geq 3$  and  $D_{G_1}(u) \leq 2$ . Clearly,  $d_G(u) = d_{G_1}(u) + |N_G^1(u)|$  and the adding of the vertices of  $N_G^1(u)$  to  $G_1$  does not increase the value of  $D_{G_1}(u)$ . So  $D_G(u) = D_{G_1}(u) \leq 2$ , this is a contradiction to Corollary 8. The proof of Theorem 3 is completed.  $\square$

### 4 Concluding Remarks

Let

$$\mathcal{G}_k = \{G : tw(G) \leq k \text{ and } G \text{ contains no isolated vertex}\},$$

and  $c_k = \min_{G \in \mathcal{G}_k} \frac{f(G)}{|V(G)|}$ . Since each graph in  $\mathcal{G}_k$  has chromatic number at most  $k + 1$ , Scott’s result [7] implies  $c_k \geq \frac{1}{2k+2}$ . The follow graphs  $H_k$  in Fig. 1 gives an upper bound  $c_k \leq \frac{2}{k+3}$  for  $k = 1, 2, 3, 4$ . Note that the graph  $H_4$  is found by Caro [2], which is the smallest known ratio of  $\frac{f(G)}{|V(G)|}$  for all graphs  $G$ . As we have known, Theorem 2 of Radcliffe and Scott [6] and the upper bound of  $c_k$  implies  $c_1 = 1/2$ , and in this paper, we show that  $c_2 = 2/5$  (Theorem 3). As a far more step, we want ask the question: what is the exact value  $c_k$  for graphs in  $\mathcal{G}_k$ . It is plausible that  $c_3 = \frac{1}{3}$  and  $c_4 = \frac{2}{7}$ .

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