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Upper Bounds on List Star Chromatic Index of Sparse Graphs

Jia Ao LI
School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071*, P. R. China*
E-mail: *lijiaao@nankai.edu.cn*

Katie HORACEK
Department of Mathematics, Frostburg State University, Frostburg, MD 21532*, USA*
E-mail: *kmhoracek@frostburg.edu*

Rong LUO
Department of Mathematics, West Virginia University, Morgantown, WV 26506*, USA*
E-mail: *rluo@math.wvu.edu*

Zheng Ke MIAO[1)]
School of Mathematics and Statistics, Jiangsu Normal University,
Xuzhou 221116*, P. R. China*
E-mail: *zkmiao@jsnu.edu.cn*

Abstract A *star k-edge-coloring* is a proper k-edge-coloring such that every connected bicolored subgraph is a path of length at most 3. The star chromatic index $\chi'_{st}(G)$ of a graph G is the smallest integer k such that G has a star k-edge-coloring. The list star chromatic index $ch'_{st}(G)$ is defined analogously. The star edge coloring problem is known to be NP-complete, and it is even hard to obtain tight upper bound as it is unknown whether the star chromatic index for complete graph is linear or super linear. In this paper, we study, in contrast, the best linear upper bound for sparse graph classes. We show that for every $\varepsilon > 0$ there exists a constant $c(\varepsilon)$ such that if $\mathrm{mad}(G) < \frac{8}{3} - \varepsilon$, then $ch'_{st}(G) \leq \frac{3\Delta}{2} + c(\varepsilon)$ and the coefficient $\frac{3}{2}$ of Δ is the best possible. The proof applies a newly developed coloring extension method by assigning color sets with different sizes.

Keywords Star edge coloring, list edge coloring, maximum average degree

MR(2010) Subject Classification 05C05, 05C15

1 Introduction

Throughout this paper we consider finite simple graphs. The star coloring of a graph, introduced by Grünbaum [5], is a proper vertex coloring such that the union of any two color classes induces a star forest. This notion has attracted much attention since a 2001 paper by Fertin, Raspaud and Reed [4]. A *star edge coloring* of a graph G is obtained from a star coloring of its line

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1) Corresponding author

graph $L(G)$, and it is a proper edge coloring such that every connected bicolored subgraph is a path with at most 3 edges. The notion of the star edge coloring is intermediate between acyclic edge coloring (where every bicolored subgraph is acyclic) and strong edge coloring (where every bicolored connected subgraph has at most two edges).

The star chromatic index of G, denoted by $\chi'_{st}(G)$, is the smallest integer k such that G is star k-edge-colorable. In 2008, Liu and Deng [11] showed that $\chi'_{st}(G) \leq \lceil 16(\Delta-1)^{\frac{3}{2}} \rceil$ when $\Delta \geq 7$. Dvořák, Mohar and Šámal [3] presented the following upper and lower bounds for complete graphs:

$$2n(1+o(1)) \leq \chi'_{st}(K_n) \leq n\frac{2^{2\sqrt{2}(1+o(1))\sqrt{\log n}}}{(\log n)^{\frac{1}{4}}}.$$

And they applied it to show a near-linear upper bound that for any graph G with maximum degree Δ, $\chi'_{st}(G) \leq \Delta \cdot 2^{O(1)\sqrt{\log \Delta}}$. The star chromatic index is hard to compute, as Lei, Shi and Song [10] proved that it is NP-complete to determine whether $\chi'_{st}(G) \leq 3$ for a subcubic graph G.

Dvořák, Mohar and Šámal [3] also studied star edge coloring of subcubic graphs. They proved that $\chi'_{st}(G) \leq 7$ for any subcubic graph G and proposed the following conjecture.

Conjecture 1.1 ([3]) *If G is a subcubic graph, then $\chi'_{st}(G) \leq 6$.*

Star edge coloring is naturally generalized to the list version: Given a list assignment L which assigns to each edge e a finite set $L(e)$, a graph is said to be L-star-edge-colorable if G has a star edge coloring c such that $c(e) \in L(e)$ for each edge e. L is called an edge k-list if each $L(e)$ is a set of size k. A graph G is *star k-edge-choosable* if G is L-star-edge-colorable for any edge k-list L. The *list star chromatic index* of a graph G, denoted by $ch'_{st}(G)$, is the minimum k such that G is star k-edge-choosable.

Lužar, Mockovčiaková and Soták [12] proved the following result for subcubic graphs which solves a problem proposed by Dvořák, Mohar and Šámal in [3].

Theorem 1.2 ([12]) *If G is a subcubic graph, then $ch'_{st}(G) \leq 7$.*

Kerdjoudj, Kostochka and Raspaud [7], Kerdjoudj and Kostochka [8] and Kerdjoudj, Pradeep and Raspaud [9] studied the list star edge coloring of graphs with small maximum average degree where the maximum average degree of a graph G, denoted $\text{mad}(G)$, is $\max\{\frac{2|E(G[H])|}{|V(H)|} : H \subseteq G\}$.

Theorem 1.3 ([7, 8]) *Let G be a subcubic graph. Then each of the following holds.*

(i) *If* $\text{mad}(G) < \frac{7}{3}$, *then* $ch'_{st}(G) \leq 5$.

(ii) *If* $\text{mad}(G) < \frac{5}{2}$, *then* $ch'_{st}(G) \leq 6$.

Theorem 1.4 ([8, 9]) (i) *If* $\text{mad}(G) < \frac{7}{3}$, *then* $ch'_{st}(G) \leq 2\Delta(G) - 1$.

(ii) *If* $\text{mad}(G) < \frac{5}{2}$, *then* $ch'_{st}(G) \leq 2\Delta(G)$.

(iii) *If* $\text{mad}(G) < \frac{8}{3}$, *then* $ch'_{st}(G) \leq 2\Delta(G) + 1$.

(iv) *If* $\text{mad}(G) < \frac{14}{5}$, *then* $ch'_{st}(G) \leq 2\Delta(G) + 2$.

(v) *If* $\text{mad}(G) < 3$, *then* $ch'_{st}(G) \leq 2\Delta(G) + 3$.

Bezegová et al. [1] and Deng et al. [2] independently proved that for each tree T with maximum degree Δ, its star chromatic index $\chi'_{st}(T) \leq \lfloor \frac{3\Delta}{2} \rfloor$. This result was extended to list star chromatic index by Han et al. in [6].

Theorem 1.5 ([6]) *For any tree T with maximum degree Δ,*

$$\chi'_{st}(T) \le ch'_{st}(T) \le \left\lfloor \frac{3\Delta}{2} \right\rfloor.$$

Furthermore, this bound is sharp.

The star edge coloring problem is very challenging and it is even unknown whether the star chromatic index for complete graph is linear or super linear. The list coloring is even much harder. In this paper, we study the list star edge coloring and present, in contrast, a best possible linear upper bound for the list star chromatic index for some graph classes. Specifically we show that $ch'_{st}(G) \le \frac{3\Delta}{2} + c$ for sparse graphs G with $\text{mad}(G) < \frac{8}{3}$ and this is the best possible up to constant c. To overcome some difficulties in the proof, we develop a new coloring extension method by requiring certain edges to be colored with different size of sets of colors. We believe that the idea is of independent interest and the method may be modified in the study of other list coloring problems.

Theorem 1.6 *Let $\varepsilon > 0$ be a real number and $d = 2\lceil \frac{8-3\varepsilon}{9\varepsilon} \rceil$. Let G be a graph with maximum average degree $\text{mad}(G) < \frac{8}{3} - \varepsilon$. Then*

$$ch'_{st}(G) \le \max\left\{\frac{3}{2}\Delta + \frac{d}{2} + 2, \Delta + 2d + 1\right\}.$$

For a planar graph G with girth g, it is well-known that the maximum average degree $\text{mad}(G) < \frac{2g}{g-2}$. Thus we have the following corollary.

Corollary 1.7 *Let G be a planar graph with maximum degree Δ and girth g.*

(a) *If $g \ge 9$, then $ch'_{st}(G) \le \max\{\frac{3\Delta}{2} + 11, \Delta + 37\}$.*

(b) *If $g \ge 16$, then $ch'_{st}(G) \le \frac{3\Delta}{2} + 4$.*

Note that we pay little effort to optimize the constant and girth condition in order to pursue a concrete and clear presentation. However in Section 4 we will present examples of planar graphs with maximum degree Δ and girth 4 and with star chromatic index at least $\frac{13\Delta}{8} - \frac{3}{4}$ for any $\Delta \ge 3$.

Before proceeding, we introduce some notation. For a vertex $v \in V(G)$, let $E_G(v) = \{vw : w \in N_G(v)\}$ and $d_G(v) = |E_G(v)|$. The subscript may be omitted if no confusion occurs. We denote $d_k(v) = |\{w \in N_G(v) : d(w) = k\}$, $d_{k^-}(v) = |\{w \in N_G(v) : d(w) \le k\}$, and $d_{k^+}(v) = |\{w \in N_G(v) : d(w) \ge k\}$, respectively. A *$t$-thread* in a graph G is a path $ux_1x_2 \cdots x_tv$ such that $d(x_1) = d(x_2) = \cdots = d(x_t) = 2$ and $\min\{d(u), d(v)\} \ge 3$. Given a proper edge coloring f of G, we use $c_f(v) = \{f(uv) : uv \in E_G(v)\}$ to denote the set of colors of the edges incident with the vertex v.

2 Proof of Theorem 1.6

In this section we will prove Theorem 1.6. For reduction purpose, we first prove a more technical and slightly stronger result on trees than Theorem 1.5 and then apply it to complete the proof of Theorem 1.6.

2.1 List Star Edge Coloring of Rooted Trees

Let G be a graph, $Z \subseteq E(G)$, and L be a list assignment of G. A *Z-star-sublist of L* is a list assignment L_0 satisfying the following two conditions:

(a) $L_0(e) \subseteq L(e)$ for each edge e and $|L_0(e)| \in \{1,3\}$ with $|L_0(e)| = 3$ if and only if $e \in Z$.

(b) Any edge coloring ϕ with $\phi(e) \in L_0(e)$ for each $e \in E(G)$ is a star edge coloring of G.

A vertex $v \in V(G)$ is called a *pre-pendent vertex* if $d(v) \geq 2$ and $d_1(v) \geq d(v) - 1$. In other words, the vertex v has at least $d(v) - 1$ neighbors of degree one. An edge $e = uv$ is called a *twig* of G if $d(u) = 1$ and v is a pre-pendent vertex.

Let T be a tree rooted at a vertex x. We use the terminology that a vertex v is in *Level* i if the distance between x and v is i. So x is in Level 0 and the neighbors of x are in Level 1. An edge is called in *Layer* i if its two endvertices are in Levels $i-1$ and i, respectively. Denote $\mathrm{Lay}(i)$ to be the set of all edges in Layer i.

Note that Theorem 1.5 follows from Theorem 2.1 below by taking $W = \emptyset$.

Theorem 2.1 *Let T be a tree with maximum degree Δ rooted at x and W be a set of twigs in T such that W is a matching and x is not a pre-pendent vertex of a twig in W. Then T has a W-star-sublist of L for any list assignment L satisfying the following*:

(a) $|L(e)| = 1$ *for any* $e \in \mathrm{Lay}(1)$ *and* $L(e) \neq L(e')$ *for any two distinct edges* $e, e' \in \mathrm{Lay}(1)$;

(b) *for each* $e \in \mathrm{Lay}(2)$, $|L(e)| \geq \Delta + \lfloor \frac{d(x)}{2} \rfloor$ *if* $e \notin W$ *and* $|L(e)| \geq \Delta + \lfloor \frac{d(x)}{2} \rfloor + 2$ *otherwise*;

(c) *for each* $e \in E(G) - \mathrm{Lay}(1) - \mathrm{Lay}(2)$, $|L(e)| \geq \Delta + \lfloor \frac{\Delta}{2} \rfloor$ *if* $e \notin W$ *and* $|L(e)| \geq \Delta + \lfloor \frac{\Delta}{2} \rfloor + 2$ *otherwise.*

Proof If there are two twigs whose pre-pendant vertices are adjacent, then there is an edge $uv \in E(T) - W$ such that each of u and v is a pre-pendant vertex of some twig in W. Thus T is a bistar and the result follows easily. Now we assume that W is an induced matching.

Note that for each edge e in Level 1, $L_0(e) = L(e)$. We proceed algorithmically from lower to higher levels to find L_0 as follows.

From Level 1 **to Level** 2 (construct $L_0(e)$ for each $e \in \mathrm{Lay}(2)$):

Denote $l = \Delta + \lfloor \frac{d(x)}{2} \rfloor - d(x)$ and $N(x) = \{y_1, \ldots, y_{d(x)}\}$. Suppose that the sublist for each $e \in \mathrm{Lay}(2)$ incident with y_i is selected for $i \leq t-1$. Now we select the sublists for edges incident with y_t. Denote $N(y_t) = \{u_0, u_1, \ldots, u_{d(y_t)-1}\}$ where $x = u_0$. In case when y_t is a pre-pendant vertex of a twig in W, $y_t u_{d(y_t)-1}$ is the twig in W.

Step 1 Select $L_0(y_t u_i)$ for $y_t u_i$ for $1 \leq i \leq l$.

Note that $E(x)$ and W are disjoint. Thus $|L_0(e)| = 1$ for each $e \in E(x)$ and for each $1 \leq i \leq l$,

$$\left| L(y_t u_i) - \bigcup_{e \in E(x)} L_0(e) \right| \geq |L(y_t u_i)| - d(x) \geq \Delta + \left\lfloor \frac{d(x)}{2} \right\rfloor - d(x) = l.$$

Thus for edges $y_t u_1, \ldots, y_t u_l$, one can choose mutually disjoint $L_0(y_t u_1), \ldots, L_0(y_t u_l)$ such that $L_0(y_t u_i) \subseteq L(y_t u_i) - \bigcup_{e \in E(x)} L_0(e)$ for each $i = 1, \ldots, l$ with size 1 or 3 depending on whether $l = d(y_t) - 1$ and whether $y_t u_{d(y_t)-1} \in W$ or not.

Step 2 Select $L_0(y_t u_i)$ for $l + 1 \leq i \leq d(y_t) - 1$.

We select $L_0(y_t u_i)$ one by one for $l + 1 \leq i \leq d(y_t) - 1$. Note $y_t x = y_t u_0$ and for each $1 \leq j \leq d(y_t) - 2$, $y_t u_j \notin W$.

We forbid the colors in $L_0(y_t u_j)$ for each $j = 1, \ldots, i-1$ and in $L_0(x y_s)$ for each $s = t, \ldots, t + \lfloor \frac{d(x)}{2} \rfloor$ where the subindex s is taken mod $d(x)$.

Define

$$A_i = L(y_t u_i) - \bigcup_{j=1}^{i-1} L_0(y_t u_j) - \bigcup_{s=t}^{t+\lfloor \frac{d(x)}{2} \rfloor} L_0(x y_s).$$

Since

$$\begin{aligned}|A_i| &\geq |L(y_t u_i)| - (i-1) - \left\lfloor \frac{d(x)}{2} \right\rfloor - 1 \\ &= |L(y_t u_i)| - i - \left\lfloor \frac{d(x)}{2} \right\rfloor \\ &\geq 1 \quad (\text{or} \geq 3 \text{ if } y_t u_i \in W),\end{aligned}$$

one can choose $L_0(y_t u_i) \subset A_i$ with size 1 or 3 depending on whether $y_t u_i \in W$.

Assume that $L_0(e)$ is selected for all edges e in Layers up to Layer i. We are going to select $L_0(e)$ for edges e in Layer $i+1$ using the same strategy by modifying l and replacing $\lfloor \frac{d(x)}{2} \rfloor$ with $\lfloor \frac{\Delta}{2} \rfloor$.

From level i to level $i+1$ ($i \geq 2$): Let u be a vertex in level $i-1$ with a neighbor in level i of degree at least 2. Denote $N(u) = \{w_0, w_1, \ldots, w_{d(u)-1}\}$ where w_0 is the parent of u. Denote T_u to be the subtree rooted at u with $E(T_u) = \{uw_0\} \cup [\bigcup_{t=1}^{d(u)-1} E(w_t)]$. Then T_u is a rooted tree with two layers, and $E(u) \cap W = \emptyset$. Denote $W_u = W \cap E(T_u)$.

Suppose that the sublist for each edge $e \in \text{Lay}(i+1)$ incident with w_j is selected for each $j \leq t-1$. Now we select the sublist for edges incident with w_t. Denote $N(w_t) = \{u_0, u_1, \ldots, u_{d(w_t)-1}\}$ where $u = u_0$. In case when w_t is a pre-pendant vertex of a twig in W, $w_t u_{d(w_t)-1}$ is the twig in W. Let $l = \lfloor \frac{3\Delta}{2} \rfloor - d(u)$. Then we can follow Steps 1 and 2 in the above to pick $L_0(w_t u_i)$ for $1 \leq i \leq d(w_t) - 1$ (replacing $\lfloor \frac{d(x)}{2} \rfloor$ with $\lfloor \frac{\Delta}{2} \rfloor$).

Note that for each $l+1 \leq i \leq d(w_t) - 1$,

$$i + \left\lfloor \frac{\Delta}{2} \right\rfloor \geq l + 1 + \left\lfloor \frac{\Delta}{2} \right\rfloor = \left\lfloor \frac{3\Delta}{2} \right\rfloor - d(u) + 1 + \left\lfloor \frac{\Delta}{2} \right\rfloor \geq \Delta.$$

Therefore by the algorithm, we have the following observation.

Observation: *$L_0(w_t u_i) \cap L_0(uw_k) = \emptyset$ for each $i \in \{1, \ldots, d(u)-1\}$ and $k \in \{t, t+1, \ldots, t+\lfloor \frac{d(u)}{2} \rfloor\}$ (mod $d(u)$). Furthermore, for each $k \in \{1, 2, \ldots, \lfloor \frac{3\Delta}{2} \rfloor - d(w_0)\}$, $L_0(w_0 z) \cap L_0(uw_k) = \emptyset$ for each $z \in N_T(w_0)$ by the coloring process of $E_T(u)$.*

Let c be a coloring of $E(T)$ with $c(e) \in L_0(e)$ for each edge e. It remains to verify that c is a star edge coloring of T . Clearly, c is a proper edge coloring. Suppose to the contrary that $vwuyz$ is a path of length 4 in T such that $c(vw) = c(uy)$ and $c(wu) = c(yz)$. Denote $N(u) = \{w_0, w_1, \ldots, w_{d(w)-1}\}$ as above. Then $w = w_i$ and $y = w_j$ for some i, j. Without loss of generality, assume $i < j$.

We first assume $i \geq 1$. Since $c(vw_i) = c(uw_j)$, by Observation, we have $j - i \notin \{0, 1, \ldots, \lfloor \frac{d(u)}{2} \rfloor\}$ (mod $d(u)$). Similarly, we also have $i - j \notin \{0, 1, \ldots, \lfloor \frac{d(u)}{2} \rfloor\}$ (mod $d(u)$) as $c(w_i u) = c(w_j z)$. This implies $j - i \notin \{0, 1, \ldots, d(u) - 1\}$ (mod $d(u)$), a contradiction.

Now we assume $i = 0$. Since $c(uw_j) = c(w_0 v)$ and by Observation, $j \geq \lfloor \frac{3\Delta}{2} \rfloor - d(w_0) + 1$. Since $c(w_j z) = c(uw_0)$, $0 \notin \{j, j+1, \ldots, j + \lfloor \frac{d(u)}{2} \rfloor\}$ (mod $d(u)$), meaning $j + \lfloor \frac{d(u)}{2} \rfloor \leq d(u) - 1$.

Since $d(w_0) \le \Delta$ and $d(u) \le \Delta$, we have

$$d(u) \le \left\lfloor \frac{d(u)}{2} \right\rfloor + 1 + \left\lfloor \frac{d(u)}{2} \right\rfloor \le \left\lfloor \frac{3\Delta}{2} \right\rfloor - d(w_0) + 1 + \left\lfloor \frac{d(u)}{2} \right\rfloor \le j + \left\lfloor \frac{d(u)}{2} \right\rfloor \le d(u) - 1.$$

This is a contradiction again and thus completes the proof of the theorem. □

2.2 Completing the Proof of Theorem 1.6

The following structural result of graphs with small maximum average degree is needed in the proof of Theorem 1.6 and its proof will be completed in the next section.

Lemma 2.2 *Let $\varepsilon > 0$ be a real number and $a = 2\lceil \frac{8-3\varepsilon}{9\varepsilon} \rceil$. Let G be a graph with* $\mathrm{mad}(G) < \frac{8}{3} - \varepsilon$ *and minimum degree $\delta(G) \ge 2$. Then G contains one of the following t-threads $ux_1x_2\cdots x_tv$.*

(C1) $t \ge 4$;
(C2) $t = 3$ *and* $d(u) \le a$;
(C3) $t = 2$, $d(u) = 3$, *and* $d_{(a+1)^+}(u) = 0$;
(C4) $t = 2$, $d(u) \le a$, $d_2(u) = d(u)$, *and* $d(v) \le a$;
(C5) $t = 2$, $d(u) \le \frac{a}{2}$, $d_2(u) \ge d(u) - 1$, $d_{(a+1)^+}(u) = 0$, *and* $d(v) \le a$;
(C6) $t = 1$, $d(u) = 3$, $d_{(a+1)^+}(u) = 0$, $d(v) \le a$, *and* $d_2(v) = d(v)$;
(C7) $t = 1$, $d(u) = 3$, $d_{(a+1)^+}(u) = 0$, $d(v) \le \frac{a}{2}$, $d_2(v) \ge d(v) - 1$, *and* $d_{(a+1)^+}(v) = 0$.

Theorem 1.6 is a corollary of the following slightly stronger result.

Theorem 2.3 *Let $\varepsilon > 0$ be a real number and $d = 2\lceil \frac{8-3\varepsilon}{9\varepsilon} \rceil$. Let G be a graph with* $\mathrm{mad}(G) < \frac{8}{3} - \varepsilon$ *and maximum degree Δ, and let W be a set of twigs in G which is a matching. Let $k = \max\{\frac{3}{2}\Delta + \frac{d}{2} + 2, \Delta + 2d + 1\}$. Then for any k-list assignment L of $E(G)$, there exists a W-star-sublist of L.*

Proof Let the pair (G, W) be a counterexample to Theorem 2.3 with $|E(G)|$ minimum where W is a set of twigs. Thus G is connected and by Theorem 2.1 G is not a tree. Hence we have $\frac{8}{3} - \varepsilon > \mathrm{mad}(G) \ge 2$, which gives $\varepsilon < \frac{2}{3}$, and so $d \ge 4$.

Let G' be the subgraph of G obtained by recursively deleting vertices of degree one (i.e., G' is the maximal subgraph of G with minimum degree at least 2). Then no pre-pendent vertex of G is in $V(G')$ by definition and $G - E(G')$ is a forrest.

For each $x \in V(G')$, denote T_x to be the tree component of $G - E(G')$ rooted at x. Thus $V(T_x) \cap V(G') = \{x\}$. If $d_{G'}(x) = d_G(x)$, we call T_x *trivial* (i.e. it is a singleton).

Claim 1 If $d_{G'}(x) \le d$, then T_x is a star rooted at x.

Proof Let $E_0 = E(T_x) - E_G(x)$ be the set of edges in T_x not incident to x. Then the claim is equivalent to $E_0 = \emptyset$. Suppose to the contrary $E_0 \ne \emptyset$. Denote $G_1 = G - E_0$ and $W_1 = W \cap E(G_1)$. By the minimality of G, for each $e \in E(G_1)$, one can have a desired W_1-star-sublist $L_0^1(e)$ of $L(e)$ in G_1. Since $x \in V(G')$, x is not a pre-pendent vertex in G_1, so $|L_0^1(e)| = 1$ for each $e \in E_{G_1}(x)$. For each edge e in Lay(2) of T_x, let

$$L'(e) = L(e) - \bigcup_{e' \in E(x) - E(T_x)} L_0^1(e').$$

Since $\Delta \ge d_G(x) = d_{G'}(x) + d_{T_x}(x)$ and $d_{G'}(x) \le d$, we have

$$|L'(e)| \ge \frac{3}{2}\Delta + \frac{d}{2} + 2 - d_{G'}(x) \ge \Delta + \frac{\Delta - d_{G'}(x)}{2} + 2 \ge \Delta + \frac{d_{T_x}(x)}{2} + 2.$$

By Theorem 2.1, one can obtain a desired sublist L_0^2 of T_x with $L_0^1(e) = L_0^2(e)$ for each $e \in E_{T_x}(x)$. Then we combine L_0^1 and L_0^2 to obtain a W-star-sublist of L of G, a contradiction. □

Claim 2 Assume that $x \in V(G)$ such that $d_{G'}(x) \le d$, $d_{G'}(u) \le d$ for every $u \in N_{G'}(x)$, and $\sum_{u \in N_{G'}(x)} d_{G'}(u) \le 2d + 2$. Then $d_G(x) = d_{G'}(x)$. That is, T_x is trivial.

Proof Suppose to the contrary that T_x is nontrivial. Since $d_{G'}(x) \le d$, $d_{G'}(u) \le d$ for every $u \in N_{G'}(x)$, by Claim 1, T_x and T_u are stars rooted at x and u respectively. By the minimality of G, $H = G - E(T_x)$ has a desired W-star-sublist L_0 of L. Denote

$$A(x) = \bigcup_{u \in N_{G'}(x),\ e \in E_{G'}(u)} L_0(e).$$

Then $|A(x)| \le 2d + 2$ since $|L_0(e)| = 1$ for each $e \in E_{G'}(u)$ with $u \in N_{G'}(x)$. For each $u \in N_{G'}(x)$, there is no path with four edges containing both one edge in T_u and one edge in T_x, since T_x and T_u are stars. Thus, to find sublists for edges in T_x, it suffices to exclude the colors in $A(x)$. Additionally, since x is not a pre-pendant vertex, we only need one available color for each edge in $E(T_x)$. For any edge $e \in E(T_x)$, $|L(e) \setminus A(x)| \ge \Delta + 2d + 1 - (2d+2) = \Delta - 1 \ge d_{T_x}(x)$. Thus, L_0 can be extended to be a desired W-star-sublist of L in G, a contradiction. □

Now we are ready to present main reductions by utilizing W-star-sublist argument.

Claim 3 There is no path xyz in G' such that $d_{G'}(x) \le 3$, $d_{G'}(u) \le d$ for every $u \in N_{G'}(x)$, and $d_{G'}(y) = d_{G'}(z) = 2$. Therefore G' doesn't contain Configurations (C1), (C2), or (C3).

Proof Suppose to the contrary that there is such a path xyz in G'. By Claim 1, T_z is a star and T_u is also a star for each $u \in N_{G'}(x)$.

Let x_1 (and x_2) be the other neighbor(s) of x. Since $\sum_{u \in N_{G'}(x)} d_{G'}(u) \le 2d + 2$ and $d_{G'}(x) + d_{G'}(z) \le 5 < 2d + 2$, by Claim 2, both T_x and T_y are trivial.

Let $H = G - xy$. Note $xy, yz \notin W$ and z is a pre-pendant vertex in H (but not in G). Let $W' = W \cup \{yz\}$. By the minimality of G, H has a W'-star-sublist L_0' of L. Since $yz \in W'$, $|L_0'(yz)| = 3$. We first pick a sublist L_0 of L for $G - xy$.

(1) $L_0(e) = L_0'(e)$ for each $e \in E(H) \setminus \{yz\}$.

(2) Pick any color

$$\alpha \in L_0'(yz) - L_0'(xx_1) \cup L_0'(xx_2) \text{ if } x_2 \text{ exists; and } \alpha \in L_0'(yz) - L_0'(xx_1) \text{ otherwise.}$$

Set $L_0(yz) = \{\alpha\}$. Note that this is possible since $xx_1 \notin W'$ (and $xx_2 \notin W'$).

For the edge xy, one can further pick a color

$$\beta \in A = L(xy) - \bigcup_{e \in E_{G'}(u), u \in \{z, x_1, x_2\}} L_0(e)$$

(or $\beta \in A = L(xy) - \bigcup_{e \in E_{G'}(u), u \in \{z, x_1\}} L_0(e)$ if x_2 does not exist), and let $L_0(xy) = \{\beta\}$, since we have $|A| \ge \Delta + 2d + 1 - (2d + 2) = \Delta - 1 \ge 1$.

It is easy to see that L_0 is a W-star-sublist of L in G. This contradicts the fact that G is a counterexample.

The second part of the claim follows from the fact that Configurations (C1), (C2) and (C3) all satisfy the conditions of Claim 3. □

Claim 4 There is no path $uxyz$ in G' such that $d_{G'}(u) \le d$, $d_{G'}(w) \le d$ for each $w \in N_{G'}(u)$, $\sum_{w\in N_{G'}(u)} d_{G'}(w) \le 2d+2$, $d_{G'}(x) = d_{G'}(y) = 2$, and $d_{G'}(z) \le d$. Therefore G' does not contain Configurations (C4) or (C5).

Proof By Claim 1, T_z is a star rooted at z. By Claim 2, T_u, T_x, and T_y are trivial. Note $xy \notin W$.

Let $H = G - xy$. By the minimality of G, H has a W-star-sublist L_0 of L. Since T_u is trivial, we have $d_G(u) + d_G(z) \le d + \Delta$, and so

$$|L(xy) - \bigcup_{e\in E_G(u)\cup E_G(z)} L_0(e)| \ge \Delta + 2d + 1 - (\Delta + d) = d + 1 > 1.$$

Pick a color $\alpha \in L(xy) - \bigcup_{e\in E_G(u)\cup E_G(z)} L_0(e)$ and let $L_0(xy) = \{\alpha\}$. Therefore L_0 is extended to be a W-star-sublist of L in G, a contradiction. □

Claim 5 There is no path uxy in G' such that $d_{G'}(y) \le 3$, $d_{G'}(v) \le d$ for each $v \in N_{G'}(y)$, $d_{G'}(x) = 2$, $d_{G'}(u) \le d$, $d_{G'}(w) \le d$ for each $w \in N_{G'}(u)$, and $\sum_{w\in N_{G'}(u)} d_{G'}(w) \le 2d+2$. Therefore G' does not contain Configurations (C6) or (C7).

Proof Suppose to the contrary that there is such a path. We first show $d_{G'}(y) = 3$. Otherwise if $d_{G'}(y) = 2$, let z be the other neighbor distinct from x. Then $uxyz$ is a path forbidden in Claim 4. This contradiction implies $d_{G'}(y) = 3$. Denote $N_{G'}(y) = \{x, y_1, y_2\}$.

By Claim 2, T_y, T_x, and T_u are all trivial. By Claim 1 both T_{y_1} and T_{y_2} are stars.

Let $H = G - xy$. By minimality of G, H has a W-star-sublist L_0 of L. We consider two cases.

Case 1 $L_0(xu) \subset L_0(yy_1) \cup L_0(yy_2)$. Without loss of generality, assume $L_0(xu) = L_0(yy_1)$.

Then $L_0(xu) \cap L_0(yy_2) = \emptyset$. Note that in this case it is allowed to have $L_0(xy) = L_0(wy_2)$ for a leaf edge wy_2 in G. Let

$$\alpha \in A = L(xy) - \bigcup_{w\in N_G(y_1)} L_0(y_1w) - \bigcup_{w\in N_{G'}(y_2)} L_0(y_2w) - \bigcup_{w\in N_{G'}(u)} L_0(uw)$$

and set $L_0(xy) = \{\alpha\}$. This is possible since $|A| \ge \Delta + 2d + 1 - (\Delta + d + d) = 1$.

Case 2 $L_0(xu) \cap [L_0(yy_1) \cup L_0(yy_2)] = \emptyset$.

In this case it is allowed to have $L_0(xy) = L_0(wy_i)$ for a leaf edge wy_i for each $i = 1, 2$. Let

$$\beta \in B = L(xy) - \bigcup_{w\in N_{G'}(y_1)} L_0(y_1w) - \bigcup_{w\in N_{G'}(y_2)} L_0(y_2w) - \bigcup_{w\in N_{G'}(u)} L_0(uw)$$

and set $L_0(xy) = \{\beta\}$, since $|B| \ge \Delta + 2d + 1 - (d + d + d) \ge 1$.

In either case, we can extend L_0 from H to G, a contradiction. □

By Claims 3, 4, and 5, G' does not contain configurations (C1)–(C7), which contradicts Lemma 2.2. This contradiction completes the proof of the theorem. □

3 Proof of Lemma 2.2

In this section, we will prove Lemma 2.2. For convenience, we copy the lemma in the following.

Lemma 3.1 *Let $\varepsilon > 0$ be a real number and $a = 2\lceil \frac{8-3\varepsilon}{9\varepsilon} \rceil$. Let G be a graph with* $\mathrm{mad}(G) < \frac{8}{3} - \varepsilon$ *and minimum degree $\delta(G) \ge 2$. Then, G contains one of the following t-threads $ux_1x_2\cdots x_tv$.*

(C1) $t \ge 4$;

(C2) $t = 3$ *and* $d(u) \le a$;
(C3) $t = 2$, $d(u) = 3$, *and* $d_{(a+1)^+}(u) = 0$;
(C4) $t = 2$, $d(u) \le a$, $d_2(u) = d(u)$, *and* $d(v) \le a$;
(C5) $t = 2$, $d(u) \le \frac{a}{2}$, $d_2(u) \ge d(u) - 1$, $d_{(a+1)^+}(u) = 0$, *and* $d(v) \le a$;
(C6) $t = 1$, $d(u) = 3$, $d_{(a+1)^+}(u) = 0$, $d(v) \le a$, *and* $d_2(v) = d(v)$;
(C7) $t = 1$, $d(u) = 3$, $d_{(a+1)^+}(u) = 0$, $d(v) \le \frac{a}{2}$, $d_2(v) \ge d(v) - 1$, *and* $d_{(a+1)^+}(v) = 0$.

Proof If $\varepsilon \ge \frac{2}{3}$, then $\mathrm{mad}(G) < 2$ and so G is acyclic, contradicting $\delta(G) \ge 2$. Thus $0 < \varepsilon < \frac{2}{3}$.

We prove by contradiction and proceed by the discharging method. Suppose to the contrary that there is no t-thread as in (C1)–(C7). For each $x \in V(G)$, define the initial charge $M(x) = d(x) - (\frac{8}{3} - \varepsilon)$. Note that $M(x) = 2 - \frac{8}{3} + \varepsilon = -\frac{2}{3} + \varepsilon$ for each 2-vertex x. For each 3^+-vertex x, $M(x) \ge \frac{1}{3} + \varepsilon > 0$.

Obtain a second charge $M'(x)$ by the following rule:

R1: *Each* $(a+1)^+$*-vertex* y *sends* $\frac{d(y)-8/3+\varepsilon}{d_{a^-}(y)}$ *to each* a^-*-neighbor, if* $d_{a^-}(y) \ne 0$.

Obtain a third charge $M''(x)$ by the following rule:

R2: *Each* 3^+*-vertex* y *sends* $\frac{M'(y)}{d_2(y)}$ *to each* 2*-neighbor if* $d_2(y) \ne 0$.

It is of interest to consider the amount sent from a 3^+-vertex y to a 2-neighbor, given properties of y. Firstly, we compute charges of several types in the following. Note that the function $\frac{c-8/3+\varepsilon}{c}$ is increasing with respect to c given $\varepsilon < 2/3$.

(A) By **R1**, if y is a $(a+1)^+$-vertex, then y sends each a^--neighbor x at least

$$\frac{a+1-8/3+\varepsilon}{a+1} = 1 - \frac{8-3\varepsilon}{6\lceil (8-3\varepsilon)/(9\varepsilon)\rceil + 3} > 1 - \frac{8-3\varepsilon}{6(8-3\varepsilon)/(9\varepsilon)} = 1 - \frac{3\varepsilon}{2}.$$

Thus y sends x at least $1 - \frac{3\varepsilon}{2}$.

(B) Assume that y is a 3-vertex with a 2-neighbor x.

• If $d_2(y) = 1$, then y sends x exactly $\frac{1}{3} + \varepsilon$.

• Assume that y has a $(a+1)^+$-neighbor z. Then, $d_2(y) \le 2$ and y receives at least $1 - \frac{3\varepsilon}{2}$ from z by (A). Thus y sends x at least $\frac{1}{2}[(\frac{1}{3} + \varepsilon) + (1 - \frac{3\varepsilon}{2})] \ge \frac{2}{3} - \frac{\varepsilon}{4}$.

(C) Assume that $4 \le d(y) < \frac{a}{2} + 1$. Let x be a 2-neighbor of y.

• If $d_2(y) = d(y)$, then y sends x at least $\frac{4-8/3+\varepsilon}{4} = \frac{1}{3} + \frac{\varepsilon}{4}$.

• If $d_2(y) \le d(y) - 1$, then y sends x at least $\frac{4-8/3+\varepsilon}{3} \ge \frac{4}{9}$.

• If $d_2(y) \le d(y) - 2$, then y sends x at least $\frac{4-8/3+\varepsilon}{2} \ge \frac{2}{3}$.

• Assume that y has a $(a+1)^+$-neighbor z. Then, $d_2(y) \le d(y) - 1$ and y receives at least $1 - \frac{3\varepsilon}{2}$ from z by (A). Thus, y sends x at least $\frac{1}{3}[4 - \frac{8}{3} + \varepsilon + (1 - \frac{3\varepsilon}{2})] \ge \frac{7}{9} - \frac{\varepsilon}{6} \ge \frac{2}{3}$.

(D) Assume $\frac{a}{2} + 1 \le d(y) \le a$.

• If $d_2(y) = d(y)$, then y sends x at least

$$\frac{a/2+1-8/3+\varepsilon}{a/2+1} = 1 - \frac{8/3-\varepsilon}{\lceil (8-3\varepsilon)/(9\varepsilon)\rceil + 1} \ge 1 - \frac{3\varepsilon(8-3\varepsilon)}{8+6\varepsilon}.$$

• If $d_2(y) \le d(y) - 1$, then y sends x at least

$$\frac{a/2+1-8/3+\varepsilon}{a/2} = 1 - \frac{5/3-\varepsilon}{\lceil (8-3\varepsilon)/(9\varepsilon)\rceil} \ge 1 - \frac{5/3-\varepsilon}{(8-3\varepsilon)/(9\varepsilon)} = 1 - \frac{3\varepsilon(5-3\varepsilon)}{8-3\varepsilon}.$$

Clearly, $\sum_{x\in V(G)} M''(x) = \sum_{x\in V(G)} M'(x) = \sum_{x\in V(G)} M(x) < 0$ since $\mathrm{mad}(G) < \frac{8}{3} - \varepsilon$. As $M''(y) \ge 0$ for each 3^+-vertex y, we have $\sum_{x\in V_2(G)} M''(x) < 0$. Here $V_2(G)$ denotes the set of 2-vertices of G. In the following, we shall show that each of the t-threads not forbidden in G receives nonnegative charge to yield a contradiction.

• Let $ux_1x_2x_3v$ be a 3-thread. Since (C2) is forbidden, we have $\min\{d(u), d(v)\} \geq a+1$. By (A), u sends x_1 at least $1-\frac{3\varepsilon}{2}$, and v sends x_3 at least $1-\frac{3\varepsilon}{2}$ as well. Thus, $M''(x_1)+M''(x_2)+M''(x_3) \geq 3(\varepsilon-\frac{2}{3})+2(1-\frac{3\varepsilon}{2})=0$.

• Let ux_1x_2v be a 2-thread. We further divide our discussion according to the value of $d(u)$:

(1) Assume $d(u)=3$. Since (C3) is forbidden, we conclude $d_{(a+1)^+}(u) \geq 1$. Then, u sends x_1 at least $\frac{2}{3}-\frac{\varepsilon}{4}$ by (B).

(a) Assume $d(v) \geq \frac{a}{2}+1$. Then, v sends x_2 at least $1-\frac{3\varepsilon(8-3\varepsilon)}{8+6\varepsilon}$ by (D). Since $\varepsilon < \frac{2}{3}$, we have $M''(x_1)+M''(x_2) \geq 2(\varepsilon-\frac{2}{3})+(\frac{2}{3}-\frac{\varepsilon}{4})+(1-\frac{3\varepsilon(8-3\varepsilon)}{8+6\varepsilon})=\frac{117\varepsilon^2-48\varepsilon+16}{12(4+3\varepsilon)}>0$.

(b) Assume $3 \leq d(v) < \frac{a}{2}+1$. Since (C4) is forbidden and $d_2(u)<d(u)$, we have $d_2(v)<d(v)$.

Case 1 Assume $d_{(a+1)^+}(v) \geq 1$. Then, v sends x_2 at least $\frac{2}{3}-\frac{\varepsilon}{4}$ by (B) or (C). Hence $M''(x_1)+M''(x_2) \geq 2(\varepsilon-\frac{2}{3})+2(\frac{2}{3}-\frac{\varepsilon}{4})>0$.

Case 2 Assume $d_{(a+1)^+}(v)=0$. Since (C3) is forbidden, $d(v) \geq 4$, and since (C5) is forbidden, $d_2(v) \leq d(v)-2$. Then, v sends x_2 at least $\frac{2}{3}$ by (C) and so $M''(x_1)+M''(x_2) \geq 2(\varepsilon-\frac{2}{3})+(\frac{2}{3}-\frac{\varepsilon}{4})+\frac{2}{3}>0$.

(2) Assume $\min\{d(u), d(v)\} \geq 4$. Then, u sends x_1 at least $\frac{1}{3}+\frac{\varepsilon}{4}$ by (C).

(a) Assume $d(v) \geq a+1$. Then, v sends x_2 at least $1-\frac{3\varepsilon}{2}$ by (A), so $M''(x_1)+M''(x_2) \geq 2(\varepsilon-\frac{2}{3})+(\frac{1}{3}+\frac{\varepsilon}{4})+(1-\frac{3\varepsilon}{2})>0$.

(b) Assume $\max\{d(u), d(v)\} < \frac{a}{2}+1$. Since (C4) is forbidden, we have $d_2(u)<d(u)$ and $d_2(v)<d(v)$. Since (C5) is forbidden, for $w \in \{u, v\}$, either $d_{(a+1)^+}(w) \geq 1$ or $d_2(w) \leq d(w)-2$. If $d_{(a+1)^+}(w) \geq 1$, then w sends its 2-neighbors at least $\frac{2}{3}$ by (C). If $d_2(w) \leq d(w)-2$, then w sends its 2-neighbors at least $\frac{2}{3}$ by (C). Thus, $M''(x_1)+M''(x_2) \geq 2(\varepsilon-\frac{2}{3})+2\cdot\frac{2}{3}>0$.

(c) Assume $a \geq d(v) \geq \frac{a}{2}+1$. Since $d_2(v)<d(v)$, v sends x_2 at least $1-\frac{3\varepsilon(5-3\varepsilon)}{8-3\varepsilon}$ by (D).

Case 1 $d(u) \leq \frac{a}{2}$. Then, u sends x_1 at least $\frac{2}{3}$ by (C). Thus,

$$M''(x_1)+M''(x_2) \geq 2\left(\varepsilon-\frac{2}{3}\right)+\frac{2}{3}+\left(1-\frac{3\varepsilon(5-3\varepsilon)}{8-3\varepsilon}\right)=\frac{9\varepsilon^2+8}{3(8-3\varepsilon)}>0.$$

Case 2 $d(u) \geq \frac{a}{2}+1$. Then we have

$$M''(x_1)+M''(x_2) \geq 2\left(\varepsilon-\frac{2}{3}\right)+2\left(1-\frac{3\varepsilon(5-3\varepsilon)}{8-3\varepsilon}\right)=\frac{(6\varepsilon-4)^2}{3(8-3\varepsilon)}>0.$$

• Let uxv be a 1-thread.

(1) Assume $\min\{d(u), d(v)\} \geq 4$. Then by (C), $M''(x) \geq (\varepsilon-\frac{2}{3})+2(\frac{1}{3}+\frac{\varepsilon}{4})>0$.

(2) Assume $d(u)=3$ and $d(v) \geq a+1$. Then, u sends x at least $\frac{1}{3}(3-\frac{8}{3}+\varepsilon)=\frac{1}{9}+\frac{\varepsilon}{3}$. Then by (A), $M''(x) \geq (\varepsilon-\frac{2}{3})+(\frac{1}{9}+\frac{\varepsilon}{3})+(1-\frac{3\varepsilon}{2})=\frac{4}{9}-\frac{\varepsilon}{6}>0$.

(3) Assume $d(u)=3$ and $d_{(a+1)^+}(u) \geq 1$. Then, u sends x at least $\frac{2}{3}-\frac{\varepsilon}{4}$ by (B), so $M''(x) \geq (\varepsilon-\frac{2}{3})+(\frac{2}{3}-\frac{\varepsilon}{4})>0$.

(4) Assume $d(u)=3, d_{(a+1)^+}(u)=0$, and $d(v) \geq \frac{a}{2}+1$. Since (C6) is forbidden, $d_2(v) < d(v)$. Thus, v sends x at least $1-\frac{3\varepsilon(5-3\varepsilon)}{8-3\varepsilon}$ by (D), so

$$M''(x) \geq \left(\varepsilon-\frac{2}{3}\right)+\left(\frac{1}{9}+\frac{\varepsilon}{3}\right)+\left(1-\frac{15\varepsilon-9\varepsilon^2}{8-3\varepsilon}\right)=\frac{45\varepsilon^2-51\varepsilon+32}{9(8-3\varepsilon)}>0.$$

(5) Assume $d(u)=3, d_{(a+1)^+}(u)=0$, and $d(v)\le \frac{a}{2}$. Since (C7) is forbidden, $d_2(v)\le d(v)-2$ or $d_{(a+1)^+}(v)\ge 1$. Thus by (C),

$$M''(x)\ge \left(\varepsilon-\frac{2}{3}\right)+\left(\frac{1}{9}+\frac{\varepsilon}{3}\right)+\frac{2}{3}>0.$$

All the t-threads allowed in G are examined in the above arguments. This proves that every t-thread in G receives nonnegative charge, a contradiction to $\sum_{x\in V_2(G)} M''(x)<0$. □

4 Concluding Remarks

Wang et al. [13] obtain the following upper bounds on the non-list star edge coloring for planar graphs with large girth.

Theorem 4.1 ([13]) *Let G be a planar graph with maximum degree Δ and girth g.*

(i) *If G has no cycles of length 4, then $\chi'_{st}(G)\le \lfloor 1.5\Delta\rfloor+18$.*

(ii) *If $g\ge 5$, then $\chi'_{st}(G)\le \lfloor 1.5\Delta\rfloor+13$.*

(iii) *If $g\ge 8$, then $\chi'_{st}(G)\le \lfloor 1.5\Delta\rfloor+3$.*

Their proof in [13] applies a clever edge-partition technique and assigns certain specific colors to certain given part of edges. However their methods seem not to be easily extendable to the list version of the problem. We believe that the girth conditions in Corollary 1.7 are not tight. For planar graphs with girth 4, we have an infinite family of such graphs whose list star chromatic index can not be bounded by $\frac{3\Delta}{2}+c$.

Proposition 4.2 *For each integer $\Delta\ge 3$, there exists a planar graph K of girth 4 with maximum degree Δ such that*

$$ch'_{st}(K)\ge \chi'_{st}(K)\ge \frac{13\Delta}{8}-\frac{3}{4}.$$

Proof Let K be a graph obtained from the complete bipartite graph $K_{2,\Delta}$ as follows: Let v_1,v_2 be the Δ-vertices of $K_{2,\Delta}$, and let $u_1,\dots,u_\Delta$ be the 2-vertices of $K_{2,\Delta}$. Obtain K by adding $(\Delta-2)$ leaves to u_i $(1\le i\le \Delta)$, so that u_i is now a Δ-vertex. Let ϕ be a star $(\Delta+k)$-edge-coloring of K. We shall show that $k\ge \frac{5\Delta}{8}-\frac{3}{4}$ below.

We first claim $|c_\phi(v_1)\cap c_\phi(v_2)|\le \frac{\Delta}{2}$. By contradiction, we may assume, without loss of generality, that $\phi(v_1u_i)\in c_\phi(v_2)$ for each $1\le i\le \lfloor\frac{\Delta}{2}\rfloor+1$. Then it follows from the Pigon-Hole Principle that there exist $1\le i,j\le \lfloor\frac{\Delta}{2}\rfloor+1$ such that $\phi(v_2u_i)=\phi(v_1u_j)$. Since $\phi(v_1u_i)\in c_\phi(v_2)$, we denote $\phi(v_1u_i)=\phi(v_2u_\ell)$. Then $u_\ell v_2u_iv_1u_j$ is a bicolored path (or cycle) of length four, a contradiction.

Now we assume, wlog, that $\phi(v_1u_i)=i$ for each $i\in[\Delta]$ and $\phi(v_2u_j)=\Delta+j$ for each $j\in[t]$, for some t with $\frac{\Delta}{2}\le t\le k$. There are $(\Delta-2)\Delta$ leaves incident with $u_1,\dots,u_\Delta$. To determine the number of colors needed to color those leaves, we may view that each u_i is incident with $\Delta+k-2$ colored pseudo-leaves (that is, u_i sees all colors) and then delete certain colored pseudo-leaves to obtain a proper star-edge coloring of K. For each color pair (i,j) with $1\le i<j\le\Delta$, the pseudo-leaf with color i incident with u_j and the pseudo-leaf with color j incident with u_i, together with v_1u_i, v_1u_j form a bicolored path of length four, and so at least one of the pseudo-leaves should be deleted. There are $\binom{\Delta}{2}$ such colored pairs (i,j), which implies at least $\binom{\Delta}{2}$ pseudo-leaves (with colors in $\{1,2,\dots,\Delta\}$) are deleted. Similarly, for each color pair $(\Delta+i,\Delta+j)$ with $1\le i<j\le t$, one of the pseudo-leaves, either the one with color

$\Delta + i$ incident with u_j or the one with color $\Delta + j$ incident with u_i, should be deleted. Thus at least $\binom{t}{2}$ pseudo-leaves (with color in $\{\Delta+1, \ldots, \Delta+t\}$) are deleted. On the other hand, there are $(\Delta-2)\Delta$ remaining leaves, which implies there are $k\Delta$ pseudo-leaves are deleted. Thus, we have

$$k\Delta \geq \binom{\Delta}{2} + \binom{t}{2}.$$

Since $t \geq \frac{\Delta}{2}$, we conclude that $k \geq \frac{5\Delta}{8} - \frac{3}{4}$. This completes the proof of the proposition. □

In view of Theorem 4.1, Proposition 4.2 and Corollary 1.7, we conjecture that, in the list star edge coloring problem, tree-like upper bound holds for planar graphs with girth at least 5.

Conjecture 4.3 *There exists a constant $c > 0$ such that for any planar graph G of girth at least 5 with maximum degree Δ, we have*

$$ch'_{st}(G) \leq \frac{3\Delta}{2} + c.$$

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