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The wide-diameter of regular graphs

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Abstract: The diameter with width m of a graph G is defined as the minimum integer d for which between any two distinct vertices in G there exist at least m internally disjoint paths of length of at most d. It was shown that the tight upper bound on m-diameter of w-regular w-connected graph with order n is $\left\lfloor \frac{(n-2)(w-2)}{(w-m+1)(3m-w-4)} \right\rfloor + 1$ for any integer m with $\left\lceil \frac{2w+5}{3} \right\rceil \leqslant m \leqslant w$. Some known results can be deduced or improved from the obtained result.

Key words: graphs; connectivity; diameter; wide-diameter; regular graphs; networks; fault tolerance

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正则图的宽直径

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摘要:宽度为 m 的图 G 的直径是最小整数 d,使得 G 中任何两顶点之间至少存在 m 条其长度都不超过 d 的内点不交的路. 对于任何满足 $\left\lceil \frac{2w+5}{3} \right\rceil \leqslant m \leqslant w$ 的整数 m,给出了 n 阶 w 正则 w 连通图的 m 宽直径的上界为 $\left\lfloor \frac{(n-2)(w-2)}{(w-m+1)(3m-w-4)} \right\rfloor + 1$. 它能导出和改进某些已知结果.

关键词:图论;连通度;直径;宽直径;正则图;网络;容错性

0 Introduction

We follow Ref. [1] for graph-theoretical terminology and notation not defined here. A graph $G=(V, E)$ always means a simple graph, where $V=V(G)$ is the vertex-set and $E=E(G)$ is the edge-set of G.

It is well known that interconnection networks play an important role in parallel computing/communication systems. An interconnection

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network can be modeled by a graph $G=(V, E)$, where V is the set of processors and E is the set of communication links in the network. Fault tolerance and transmission delay of the networks are often measured by connectivity and diameter of the corresponding graph, respectively. To consider fault tolerance and efficiency, all these systems demand the existence of a large number of internally disjoint paths connecting any two vertices, each of which is of short length. In the circumstances, any exclusive consideration of connectivity or diameter is not comprehensive. This has motivated us to consider the following concepts by combining connectivity with diameter rather naturally.

Let G be a connected graph, and x, y be two distinct vertices of G. The distance with width m from x to y, m-distance for short and denoted by $d_m(G; x, y)$, is the minimum integer d for which there are at least m internally disjoint (x, y)-paths of length at most d in G. The diameter with width m of G, m-diameter for short and denoted by $d_m(G)$, is defined as the maximum m-distance $d_m(G; x, y)$ over all ordered pairs (x, y) of vertices of G.

The concept of the wide-diameter was first proposed by Hsu et al.[2-4], Flandrin and Li[5], independently. From the definition of wide-diameter, it is clear that $d_1(G)$ is the diameter $d(G)$ and

$$d(G) = d_1(G) \leqslant d_2(G) \leqslant \cdots \leqslant d_{m-1}(G) \leqslant d_m(G).$$

This implies that the wide-diameter is a generalization of the diameter. If G is w-connected, i. e. the connectivity $\kappa(G) \geqslant w$, then $d_w(G)$ certainly exists by the well-known Menger's theorem (see Ref. [1, Theorem 4.3]). The maximum value of k for which $d_w(G)$ is well defined is the connectivity $\kappa(G)$. Thus the concept of wide diameter is a combination of connectivity and diameter. It is not only an important parameter to measure fault tolerance and efficiency of parallel processing computer networks, but also an attractive research topic in graph theory. However, Hsu[2] proved that the problem determining $d_w(G)$ is NP-complete for general w-connected graphs. As far the main research attention is to determine the wide-diameters for some special networks, a list on these results is referred to Ref. [6, Section 4.4]. The tight upper bound on the wide-diameter of a w-regular w-connected graph of order n is $d_w(G) \leqslant \left\lfloor \frac{n}{2} \right\rfloor$ for $w \geqslant 3$ obtained by Hsu and Luczak[4]. In this paper, we present more general upper bounds stated as follows. For a w-regular w-connected graph G of order n and any integer m with $\left\lceil \frac{2w+5}{3} \right\rceil \leqslant m \leqslant w$, the m-diameter

$$d_m(G) \leqslant \left\lfloor \frac{(n-2)(w-2)}{(w-m+1)(3m-w-4)} \right\rfloor + 1.$$

The proof is in Section 2. In Section 1, we give some preliminaries used in our proof.

1 Preliminaries

Let G be a w-connected graph, x and y two distinct vertices of G. A set of w internally disjoint (x, y)-paths in G is called an (x, y)-container in G with width w, denoted by $\mathscr{P}_w(G; x, y)$. The length of $\mathscr{P}_w(G; x, y)$, denoted by $l(\mathscr{P}_w(G; x, y))$, is the largest length of paths in $\mathscr{P}_w(G; x, y)$. The w-distance between x and y of G is defined as

$$d_w(G; x, y) = \min\{ l(\mathscr{P}_w(G; x, y)) : \forall\, \mathscr{P}_w(G; x, y)\},$$

and the w-diameter of G is defined as

$$d_w(G) = \max\{ d_w(G; x, y) : \forall\, x, y \in V(G)\}.$$

When we write an (x, y)-container $\mathscr{P}_w(G; x, y) = \{P_1, P_2, \cdots, P_w\}$, we always assume $\varepsilon(P_1) \leqslant \varepsilon(P_2) \leqslant \cdots \leqslant \varepsilon(P_w)$. Thus $\varepsilon(P_w)$ is the length of $\mathscr{P}_w(G; x, y)$. Let

$$\left.\begin{array}{l} B_i = V(P_i)\setminus\{x, y\},\ b_i = |B_i|,\ i = 1, 2, \cdots, w, \\ B = B_1 \cup B_2 \cup \cdots \cup B_{w-1} \cup \{x, y\}, \\ A = V(G)\setminus(B \cup B_w) \end{array}\right\} \tag{1}$$

If the order of G is n, then

$$n = 2 + |A| + \sum_{i=1}^{w} b_i \tag{2}$$

By the definition of $d_w(G)$, there is an (x, y)-

container $\mathscr{P}_w(G; x, y)=\{P_1, P_2, \cdots, P_w\}$ such that

$$d_w(G) = d_w(G; x, y) = \varepsilon(P_w).$$

Since $b_w \geqslant b_{w-1} \geqslant \cdots \geqslant b_2 \geqslant 1$ and $b_1 \geqslant 0$, we have

$$n = 2 + |A| + \sum_{i=1}^{w} b_i \geqslant 2 + (w-m+1)b_m + (m-2) \quad (3)$$

for any integer m with $2 \leqslant m \leqslant w$. From (3), we immediately obtain a simple upper bound of $d_m(G)$:

$$d_m(G) \leqslant b_m + 1 \leqslant \left\lfloor \frac{n-m}{w-m+1} \right\rfloor + 1.$$

An (x, y)-container $\mathscr{P}_w(G; x, y)$ is called to be minimum if it is of the minimum sum of path-lengths over all (x, y)-containers with width w. For two disjoint subsets X and Y in V(G), we use $E_G(X, Y)$ to denote the set of edges between X and Y in G. Let $I_w = \{1, 2, \cdots, w\}$.

Lemma 1.1[6, Theorem 4.4.1] Let $\mathscr{P}_w(G; x, y) = \{P_1, P_2, \cdots, P_w\}$ be a minimum (x, y)-container in G, S_j an independent set in P_j. Then for any $i \in I_w$ with $i \neq j$ and $b_i > 0$, there are at most $b_i + |S_j| - 1$ edges between S and B_i, that is,

$$|E_G(S_j, B_i)| \leqslant b_i + |S_j| - 1.$$

Lemma 1.2 Let

$$\mathscr{P}_w(G; x, y) = \{P_1, P_2, \cdots, P_w\}$$

be a minimum (x, y)-container in G. If $b_i > 0$ and $b_j > 0$ for some $i, j \in I_w$ with $i \neq j$, then

$$|E_G(B_i, B_j)| \leqslant 2b_i + b_j - 2.$$

Proof If $b_j = 1$ then the conclusion is true clearly. Assume $b_j > 1$ below. Then B_j can be partitioned into two disjoint subsets S_j and $\overline{S}_j$ such that both S_j and $\overline{S}_j$ are independent sets in P_j. By Lemma 1.1,

$$|E_G(B_i, S_j)| \leqslant b_i + |S_j| - 1,$$
$$|E_G(B_i, \overline{S}_j)| \leqslant b_i + |\overline{S}_j| - 1.$$

It follows that

$$|E_G(B_i, B_j)| = |E_G(B_i, S_j)| + |E_G(B_i, \overline{S}_j)| \leqslant 2b_i + |S_j| + |\overline{S}_j| - 2 = 2b_i + b_j - 2.$$

The lemma follows. □

2 Main results

In this section, we will prove our main results in this paper, which is stated as follows.

Theorem 2.1 For a w-regular w-connected graph G of order n and any integer m with $\left\lceil \frac{2w+5}{3} \right\rceil \leqslant m \leqslant w$, the m-diameter

$$d_m(G) \leqslant \left\lfloor \frac{(n-2)(w-2)}{(w-m+1)(3m-w-4)} \right\rfloor + 1.$$

Proof Let x and y be two vertices in G such that $d_m(G) = d_m(G; x, y)$, and $\mathscr{P}_w(G; x, y) = \{P_1, P_2, \cdots, P_w\}$ a minimum (x, y)-container of G. Then

$$d_m(G) = d_m(G; x, y) \leqslant \varepsilon(P_m) = b_m + 1,$$

where b_m is defined in (1). To prove our theorem, we only need to prove that

$$b_m \leqslant \left\lfloor \frac{(n-2)(w-2)}{(w-m+1)(3m-w-4)} \right\rfloor \quad (4)$$

To the end, let E_1 be a set of edges with only one end-vertex in $\bigcup_{i=m}^{w} B_i$ and $E_1 \cap (\bigcup_{i=m}^{w} E(P_i)) = \emptyset$; E_2 be a set of edges with at least one end-vertex in $\bigcup_{i=1}^{m-1} B_i \cup A$, $E_2 \cap E(A) = \emptyset$ and $E_2 \cap (\bigcup_{i=1}^{m-1} E(P_i)) = \emptyset$; E_3 be a set of edges with one end-vertex in B_i and another in B_j for all i, j with $m \leqslant i \neq j \leqslant w$, that is,

$$E_1 = \{e = uv \in E(G) \mid u \text{ or } v \in \bigcup_{i=m}^{w} B_i \text{ and } \{u, v\} \nsubseteq \bigcup_{i=m}^{w} B_i \text{ and } e \notin \bigcup_{i=m}^{w} E(P_i)\};$$

$$E_2 = \{e = uv \in E(G) \mid u \text{ or } v \in \bigcup_{i=1}^{m-1} B_i \cup A \text{ and } \{u, v\} \nsubseteq A \text{ and } e \notin \bigcup_{i=1}^{m-1} E(P_i)\};$$

$$E_3 = \bigcup_{m \leqslant i < j \leqslant w} E_G(B_i, B_j).$$

Clearly, $E_1 \subseteq E_2$ and, since G is w-regular,

$$|E_1| + 2|E_3| = (w-2)\sum_{i=m}^{w} b_i \quad (5)$$

Since $\mathscr{P}_w(G; x, y)$ is a minimum (x, y)-container of G, every vertex in A has at most 3 neighbors in B_i for each $i \in I_w$, and so

$$|E_2| \leqslant (w-2)\sum_{i=1}^{m-1} b_i + 3(w-m+1)|A| = (w-2)\left(\sum_{i=1}^{m-1} b_i + |A|\right) + (2w-3m+5)|A|.$$

Noting that $2w-3m+5 \leqslant 0$ if $\left\lceil \frac{2w+5}{3} \right\rceil \leqslant m \leqslant w$, and $n = 2 + |A| + \sum_{i=1}^{w} b_i$, we have that

$$|E_2| \leqslant (w-2)(n-2-\sum_{i=m}^{w} b_i) \qquad (6)$$

By Lemma 1.2,

$$|E_G(B_i, B_j)| \leqslant 2b_i + b_j - 2 \leqslant \frac{3}{2}(b_i + b_j) - 2$$

if $b_i \leqslant b_j$. It follows that

$$|E_3| = \sum_{m \leqslant i < j \leqslant w} |E_G(B_i, B_j)| \leqslant \sum_{m \leqslant i < j \leqslant w} (2b_i + b_j - 2) \leqslant \sum_{m \leqslant i < j \leqslant w} \left(\frac{3}{2}(b_i + b_j) - 2\right) = (w-m)\sum_{i=m}^{w} \frac{3}{2} b_i - (w-m)(w-m+1),$$

that is,

$$|E_3| \leqslant (w-m)\sum_{i=m}^{w} \frac{3}{2} b_i - (w-m)(w-m+1) \qquad (7)$$

It follows from (5) and (7) that

$$|E_1| = (w-2)\sum_{i=m}^{w} b_i - 2|E_3| \geqslant (w-2)\sum_{i=m}^{w} b_i - 2(w-m)\sum_{i=m}^{w} \frac{3}{2} b_i + 2(w-m)(w-m+1) \geqslant (3m-2w-2)\sum_{i=m}^{w} b_i,$$

that is,

$$|E_1| \geqslant (3m-2w-2)\sum_{i=m}^{w} b_i \qquad (8)$$

By $|E_1| \geqslant |E_2|$, (6) and (8), we have that

$$(3m-2w-2)\sum_{i=m}^{w} b_i \leqslant (w-2)(n-2-\sum_{i=m}^{w} b_i) \qquad (9)$$

Since $b_m \leqslant \cdots \leqslant b_{w-1} \leqslant b_w$, by (9), we have that

$$(w-m+1)(3m-w-4)b_m \leqslant (3m-w-4)\sum_{i=m}^{w} b_i \leqslant (n-2)(w-2),$$

that is,

$$b_m \leqslant \left\lfloor \frac{(n-2)(w-2)}{(w-m+1)(3m-w-4)} \right\rfloor.$$

Thus, (4) holds, and so the theorem follows. □

Corollary 2.2 Let G be a w-regular w-connected graph of order n. If $w \geqslant 5$, then $d_w(G) \leqslant \left\lfloor \frac{n}{2} \right\rfloor$.

In Ref. [4], Hsu and Luczak gave some examples to show that the upper bound $\left\lfloor \frac{n}{2} \right\rfloor$ can be attained, and so this upper bound can not be improved for general w-regular w-connected graphs.

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