A note on broom number and maximum edge-cut of graphs *

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Abstract: A model for cleaning a graph with brushes was first introduced by Messinger, Nowakowski, and Prałat in 2008. Later, they focused on the problem of determining the maximum number of brushes needed to clean a graph. This maximum number of brushes needed to clean a graph in the model is called the broom number of the graph. In this paper, we show that the broom number of a graph is equal to the size of a maximum edge-cut of the graph, and prove the $\mathcal{NP}$-completeness of the problem of determining the broom number of a graph. As an application, we determine the broom number exactly for the Cartesian product of two graphs.

Keywords: Broom number; Maximum edge-cut; Cleaning sequence; Cartesian product

1 Introduction

The cleaning model of decontaminating a graph with brushes was first proposed by Messinger, Nowakowski, and Prałat [4]. For any graph, suppose that all the vertices and edges are dirty. Each vertex has a number of brushes assigned to it and we try to clean the graph vertex by vertex. A vertex can be cleaned if it has at least as many brushes as dirty incident edges. When a vertex is cleaned, it sends exactly one brush along each edge

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incident to this vertex. Then the cleaned vertex and its incident dirty edges could be regarded as being deleted from the graph. We determine an initial configuration of brushes and a sequence of vertices which, being cleaned in that order, will result in every vertex and its incident edges being cleaned. Note that a dirty vertex may have no dirty incident edges but it still needs to be cleaned. Such a sequence of vertices is called a cleaning sequence. The focus of [1, 3, 6, 7] is, for a graph G, to determine the minimum number of brushes needed to clean the graph, called the brush number and denoted $b(G)$.

In this paper, we focus on the broom number, $B(G)$, for any graph G, that is, the maximum number of brushes that can be used to clean the graph G with the rule that every brush has to clean at least one edge. The broom number was introduced by Messinger, Nowakowski, and Prałat [5] along with general bounds and results for some classes of graphs, including Cartesian products; and later Prałat [7] considered the broom number for d-regular random graphs.

The following notations are used in the remainder of the paper. Let G be an undirected graph and $v \in V(G)$. The degree of v in G, denoted by $d_G(v)$, is the number of edges incident with v. For a subset $X \subset V(G)$, use $G[X]$ to denote the subgraph of G induced by X. For two disjoint subsets X and Y in $V(G)$, use $E_G[X,Y]$ to denote the set of edges between X and Y in G. Let D be a digraph and $x \in V(D)$. The out-degree (resp. in-degree) of x in D, denoted by $d_D^+(x)$ (resp. $d_D^-(x)$), is the number of out-going (resp. in-going) edges of x in D. For two disjoint subsets X and Y in $V(D)$, the notation $E_D(X,Y)$ denotes the set of directed edges of D whose tails are in X and heads are in Y, and $E_D[X,Y] = E_D(X,Y) \cup E_D(Y,X)$. Using these notations, we present another equivalent definition of the broom number.

Let $G = (V(G), E(G))$ be a finite simple undirected connected graph on n vertices and all the vertices and edges of G are initially dirty. We order the vertices of G as a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then the oriented graph G_α of G is given with respect to α in the following way:

$$(\alpha_i, \alpha_j) \in E(G_\alpha) \Leftrightarrow \alpha_i\alpha_j \in E(G) \text{ and } i < j \text{ for } i,j \in \{1,2,\ldots,n\}.$$

Define $w_\alpha : V(G) \to \mathbb{N}$ by

$$w_\alpha(v) = \begin{cases} d_{G_\alpha}^+(v) - d_{G_\alpha}^-(v), & \text{if } d_{G_\alpha}^+(v) > d_{G_\alpha}^-(v), \\ 0, & \text{otherwise,} \end{cases}$$

for each $v \in V(G)$. Clearly, if we put $w_\alpha(v)$ brushes on v for each $v \in V(G)$, then G can be cleaned along the sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. We call α a cleaning sequence of G. The brush number of the cleaning sequence α, denoted by $b_\alpha(G)$, is the sum of brushes put on $V(G)$, that is $b_\alpha(G) = \sum_{i=1}^{n} w_\alpha(\alpha_i)$.

Definition 1.1 ([5]) The broom number of a given graph G, denoted by $B(G)$, is

$$B(G) = \max\{b_\alpha(G) : \alpha \text{ is a cleaning sequence of } G\}.$$

Note that, in the model, a cleaning sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ corresponds to an oriented graph G_α of G. Let $S = \{v \in V(G) : d_{G_\alpha}^+(v) > d_{G_\alpha}^-(v)\}$. Since $\sum_{v \in V(G)} d_{G_\alpha}^+(v) = \sum_{v \in V(G)} d_{G_\alpha}^-(v)$, it is easy to find that

$$\sum_{v \in S} (d_{G_\alpha}^+(v) - d_{G_\alpha}^-(v)) = \sum_{v \in V(G) \setminus S} (d_{G_\alpha}^-(v) - d_{G_\alpha}^+(v)),$$

which implies an enumerative formula of $b_\alpha(G)$ as follows.

$$b_\alpha(G) = \sum_{v \in S} (d_{G_\alpha}^+(v) - d_{G_\alpha}^-(v)) = \frac{1}{2} \sum_{v \in V(G)} |d_{G_\alpha}^+(v) - d_{G_\alpha}^-(v)|. \quad (1)$$

Equality (1) was indicated in [4] (see Theorem 2.3) and [3] (see Theorem 4.1).

In section 2, we obtain the main result of the paper, that is, the broom number of a graph is equal to the size of a maximum edge-cut of the graph, and further show that the problem to determine the broom number of a graph is $\mathcal{NP}$-complete. In Section 3, we give some applications based on the main results mentioned in Section 2, including the broom number for the Cartesian product of two graphs, and exact values of the broom numbers for the Cartesian products of paths, cycles, and cliques.

2 Main results

An edge-cut of a graph G is the set of edges between a subset S of $V(G)$ and its complement $\overline{S}$. A maximum edge-cut of G is an edge-cut with the maximum number of edges. Let $\text{maxcut}(G)$ denote the cardinality of a maximum edge-cut of G. To get the main result, we first give a lemma that articulates the relationship between the edge-cut and the broom number of cleaning sequence, which plays an important role in coping with broom number problems.

Lemma 2.1 For any graph $G = (V, E)$ and any positive integer k, there is a set $S \subset V$ such that $|E_G[S, \overline{S}]| \geq k$ if and only if there is a cleaning sequence α of G such that $b_\alpha(G) \geq k$.

Proof. First, suppose that α is a cleaning sequence of G with $b_\alpha(G) \geq k$. Denoted by G_α the oriented graph of G with respect to α. Let $S = \{v \in$

$V : d^+_{G_\alpha}(v) > d^-_{G_\alpha}(v)\}$ and $\overline{S} = V \setminus S$. By equality (1), we have

$$\begin{aligned}
b_\alpha(G) &= \sum_{v\in S} \left(d^+_{G_\alpha}(v) - d^-_{G_\alpha}(v)\right) \\
&= \sum_{v\in S} \left(d^+_{G_\alpha[S]}(v) - d^-_{G_\alpha[S]}(v)\right) + |E_{G_\alpha}(S,\overline{S})| - |E_{G_\alpha}(\overline{S},S)| \\
&= |E_{G_\alpha}(S,\overline{S})| - |E_{G_\alpha}(\overline{S},S)| \\
&\le |E_{G_\alpha}(S,\overline{S})| \\
&\le |E_G[S,\overline{S}]|,
\end{aligned}$$

that is $|E_G[S,\overline{S}]| \ge b_\alpha(G) \ge k$.

Conversely, suppose that S is a subset of V with $|E_G[S,\overline{S}]| \ge k$. Define a cleaning sequence $\alpha = (v_1,\ldots,v_s,v_{s+1},\ldots,v_{|V|})$ of G, with $(v_1,\ldots,v_s)$ and $(v_{s+1},\ldots,v_{|V|})$ being arbitrary orderings of vertices in S and $\overline{S}$, respectively. By equality (1), we have

$$\begin{aligned}
b_\alpha(G) &= \tfrac{1}{2}\sum_{v\in V} |d^+_{G_\alpha}(v) - d^-_{G_\alpha}(v)| \\
&= \tfrac{1}{2}\sum_{v\in S} |d^+_{G_\alpha}(v) - d^-_{G_\alpha}(v)| + \tfrac{1}{2}\sum_{v\in \overline{S}} |d^+_{G_\alpha}(v) - d^-_{G_\alpha}(v)| \\
&\ge \tfrac{1}{2}\left|\sum_{v\in S} \left(d^+_{G_\alpha}(v) - d^-_{G_\alpha}(v)\right)\right| + \tfrac{1}{2}\left|\sum_{v\in \overline{S}} \left(d^+_{G_\alpha}(v) - d^-_{G_\alpha}(v)\right)\right| \\
&= \tfrac{1}{2}\left|\sum_{v\in S} \left(d^+_{G_\alpha[S]}(v) - d^-_{G_\alpha[S]}(v)\right) + |E_{G_\alpha}(S,\overline{S})|\right| \\
&\quad + \tfrac{1}{2}\left|\sum_{v\in \overline{S}} \left(d^+_{G_\alpha[\overline{S}]}(v) - d^-_{G_\alpha[\overline{S}]}(v)\right) - |E_{G_\alpha}(S,\overline{S})|\right| \\
&= \tfrac{1}{2}|E_{G_\alpha}(S,\overline{S})| + \tfrac{1}{2}|E_{G_\alpha}(S,\overline{S})| \\
&= |E_{G_\alpha}(S,\overline{S})| = |E_G[S,\overline{S}]|,
\end{aligned}$$

that is $b_\alpha(G) \ge |E_G[S,\overline{S}]| \ge k$. ∎

In [5], it was observed that $B(G) \ge \text{maxcut}(G)$. According to Lemma 2.1, Definition 1.1, and $\text{maxcut}(G) = \max\{|E_G[S,\overline{S}]| : S \subset V(G)\}$, it is clear to see that $B(G) = \text{maxcut}(G)$. Thus, to determine the broom number of a graph, it suffices to find the maximum edge-cut of the graph.

Theorem 2.2 For any graph G, the broom number of G equals to the size of a maximum edge-cut of G, that is

$$B(G) = \text{maxcut}(G).$$

Moreover, any cleaning sequence α with $b_\alpha(G) = B(G)$ induces a maximum edge-cut of G, and any maximum edge-cut of G gives a cleaning sequence α with $b_\alpha(G) = B(G)$.

Next, we will show the problem of determining the broom number of a graph is $\mathcal{NP}$-complete. In [2], the following problem, which is called SIMPLE MAX CUT, was shown to be $\mathcal{NP}$-complete for arbitrary graphs by Garey, Johnson, and Stockmeyer.

SIMPLE MAX CUT (SMC)
Instance: Graph $G = (V,E)$, positive integer K.
Question: Is there a set $S \subset V$ such that $|E_G[S,\overline{S}]| \ge K$?

In the following, we present the decision problem BROOM CLEANING and transform the $\mathcal{NP}$-complete problem SIMPLE MAX CUT to it, which implies the difficulty in determining the broom number of a graph.

BROOM CLEANING (BC)
Instance: Graph $G = (V,E)$, positive integer J.
Question: Is there a cleaning sequence α of G such that $b_\alpha(G) \ge J$?

Theorem 2.3 BROOM CLEANING is $\mathcal{NP}$-complete.

Proof. It is easy to see that **BC** $\in \mathcal{NP}$, because a nondeterministic algorithm need only guess a cleaning sequence and check in polynomial time whether the broom number of the cleaning sequence is no less than J.

We transform **SMC** to **BC**. Let $G = (V,E)$ and positive integer K constitute any instance of **SMC**. The corresponding instance of **BC** is provided by the graph G and the integer $J = K$. By Lemma 2.1, there is a set $S \subset V$ such that $|E_G[S,\overline{S}]| \ge K$ if and only if there is a cleaning sequence α of G such that $b_\alpha(G) \ge K$. The result follows from the $\mathcal{NP}$-completeness of **SMC**. ∎

3 Applications

A subgraph H of a graph G is called a maximum bipartite subgraph of G, if H has a maximum number of edges among all bipartite subgraphs of G. Obviously, the subgraph induced by a maximum edge-cut of G is a maximum bipartite subgraph of G, and the edge-set of a maximum bipartite subgraph of G is also a maximum edge-cut of G. Therefore, the problem of finding a maximum bipartite subgraph is essentially the same as the problem of finding a maximum edge-cut.

The Cartesian product of graphs G_1 and G_2, written as $G_1 \Box G_2$, is the graph with vertex set $V(G_1)\times V(G_2)$ where (u_1,u_2) is adjacent to (v_1,v_2) if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. For Cartesian product graph, its broom number can be expressed in terms of the broom numbers of factor graphs.

Theorem 3.1 Let G_1 and G_2 be two graphs. Then

$$B(G_1 \square G_2) = |V(G_2)|B(G_1) + |V(G_1)|B(G_2).$$

Proof. Let H_i be a maximum bipartite subgraph of G_i for each $i \in \{1,2\}$. Then H_i is a spanning subgraph of G_i and $E(H_i)$ is a maximum edge cut of G_i for each $i \in \{1,2\}$. Since the Cartesian product of two graphs is bipartite if and only if both of them are bipartite, it is easy to find that $H_1 \square H_2$ is a maximum bipartite subgraph of $G_1 \square G_2$ with $|E(H_1 \square H_2)| =$ maxcut$(G_1 \square G_2)$. Therefore, by Theorem 2.2, we have

$$\begin{aligned} B(G_1 \square G_2) &= \mathrm{maxcut}(G_1 \square G_2) = |E(H_1 \square H_2)| \\ &= |V(H_2)||E(H_1)| + |V(H_1)||E(H_2)| \\ &= |V(G_2)|\mathrm{maxcut}(G_1) + |V(G_1)|\mathrm{maxcut}(G_2) \\ &= |V(G_2)|B(G_1) + |V(G_1)|B(G_2). \end{aligned}$$

By Theorem 2.2, it is easy to get $B(P_n) = \mathrm{maxcut}(P_n) = n-1$, $B(C_{2k}) = \mathrm{maxcut}(C_{2k}) = 2k$, $B(C_{2k+1}) = \mathrm{maxcut}(C_{2k+1}) = 2k$, and $B(K_n) = \mathrm{maxcut}(K_n) = \lfloor \frac{n^2}{4} \rfloor$ (also see Theorem 3.6 in [5]). Thus, by Theorem 3.1, we have the following result.

Corollary 3.2 For any positive integers m and n, we have

$$B(P_m \square P_n) = 2mn - (m+n),$$

$$B(P_m \square C_n) = \begin{cases} 2mn - n, & \text{if } n \text{ is even}, \\ 2mn - n - m, & \text{if } n \text{ is odd}, \end{cases}$$

$$B(C_m \square C_n) = \begin{cases} 2mn, & \text{both of } m, n \text{ are even}, \\ 2mn - m, & n \text{ is odd and } m \text{ is even}, \\ 2mn - m - n, & \text{both of } m, n \text{ are odd}, \end{cases}$$

$$B(C_m \square K_n) = \begin{cases} nm + m\lfloor \frac{n^2}{4} \rfloor, & \text{if } m \text{ is even}, \\ n(m-1) + m\lfloor \frac{n^2}{4} \rfloor, & \text{if } m \text{ is odd}, \end{cases}$$

$$B(K_m \square K_n) = \begin{cases} \frac{mn}{4}(m+n), & \text{both of } m, n \text{ are even}, \\ n\frac{m^2}{4} + m\lfloor \frac{n^2}{4} \rfloor, & n \text{ is odd and } m \text{ is even}, \\ n\lfloor \frac{m^2}{4} \rfloor + m\lfloor \frac{n^2}{4} \rfloor, & \text{both of } m, n \text{ are odd}. \end{cases}$$

Corollary 3.2 gives the exact value of $B(K_m \square K_n)$, which also improves the result of Theorem 5.2 in [5].

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