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Nowhere-zero 3-flow of graphs with small independence number

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ABSTRACT

Tutte's 3-flow conjecture states that every 4-edge-connected graph admits a nowhere-zero 3-flow. In this paper, we characterize all graphs with independence number at most 4 that admit a nowhere-zero 3-flow. The characterization of 3-flow verifies Tutte's 3-flow conjecture for graphs with independence number at most 4 and with order at least 21. In addition, we prove that every odd-5-edge-connected graph with independence number at most 3 admits a nowhere-zero 3-flow. To obtain these results, we introduce a new reduction method to handle odd wheels.

1. Introduction

Graphs in this paper are finite and loopless, but may contain parallel edges. We follow [2] for undefined terms and notation. For a graph G, let $\alpha(G)$, $\kappa'(G)$, and $\delta(G)$ denote the independence number, the edge-connectivity, and the minimum degree of G, respectively. For vertex subsets $U, W \subseteq V(G)$, let $[U, W]_G = \{uw \in E(G) | u \in U, w \in W\}$. When $U = \{u\}$ or $W = \{w\}$, we use $[u, W]_G$ or $[U, w]_G$ for $[U, W]_G$, respectively. We also use $\partial_G(S) = [S, V(G) - S]_G$ to denote an edge-cut of G. The subscript G may be omitted when G is understood from the context.

Let $D = D(G)$ be an orientation of G. For each $v \in V(G)$, let $E_D^+(v)$ ($E_D^-(v)$, respectively) be the set of all arcs directed out from (into, respectively) v. An *integer flow* (D, f) of G is an orientation D and a mapping $f : E(G) \mapsto \mathbb{Z}$ such that, for every vertex $v \in V(G)$,

$$\sum_{e\in E_D^+(v)} f(e) - \sum_{e\in E_D^-(v)} f(e) = 0.$$

An integer flow (D, f) is called a *nowhere-zero k-flow* if $1 \le |f(e)| \le k - 1$, for each edge $e \in E(G)$.

Let $d_D^+(v) = |E_D^+(v)|$ and $d_D^-(v) = |E_D^-(v)|$ denote the out-degree and the in-degree of v under the orientation D, respectively. A graph G admits a *modulo 3-orientation*, or a mod 3-orientation for short if it has an orientation D such that $d_D^+(v) - d_D^-(v) \equiv 0 \pmod 3$ for every vertex $v \in V(G)$. It is well-known that a graph admits a nowhere-zero 3-flow if and only if it admits a modulo 3-orientation (see [7,15,18]). Therefore, in this paper, we will study nowhere-zero 3-flow in terms

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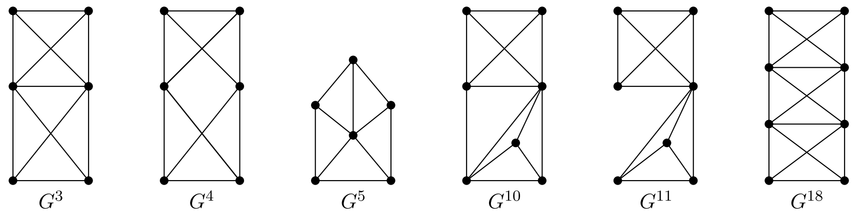

Fig. 1. Graphs in Theorems 1.3 and 1.4.

of modulo 3-orientation. The odd-edge-connectivity of a graph is defined as the minimum size of an edge-cut of odd size. A graph with low edge-connectivity may have high odd-edge-connectivity.

Tutte posed the following famous 3-Flow Conjecture, which appeared in 1970 s (see [2]).

Conjecture 1.1. *Every* 4*-edge-connected graph admits a nowhere-zero* 3*-flow.*

Thomassen [14] settled the weak version of 3-Flow Conjecture with edge-connectivity 8 replacing 4 and his result was further improved by Lovász, Thomassen, Wu and Zhang [11].

Theorem 1.2 (*Lovász et al. [11]*)**.** *Every odd-*7*-edge-connected graph admits a nowhere-zero* 3*-flow.*

Jaeger et al. [8] introduced the concept of group connectivity as generalizations of nowhere-zero flows. Let $Z(G, \mathbb{Z}_3) = \{b : V(G) \to \mathbb{Z}_3 \mid \sum_{v\in V(G)} b(v) \equiv 0 \pmod 3\}$. A graph G is $\mathbb{Z}_3$*-connected* if, for any $b \in Z(G, \mathbb{Z}_3)$, there is an orientation D such that $d_D^+(v) - d_D^-(v) \equiv b(v) \pmod 3$ for every vertex $v \in V(G)$. Let $\langle \mathbb{Z}_3 \rangle$ denote the family of all $\mathbb{Z}_3$-connected graphs. Jaeger et al. [8] pointed out that not every 4-edge-connected graph is $\mathbb{Z}_3$-connected, and they further conjectured that *every* 5*-edge-connected graph is* $\mathbb{Z}_3$*-connected.* This conjecture, if true, implies Tutte's 3-Flow Conjecture as Kochol [9] showed that the 3-Flow conjecture is equivalent to its restriction to 5-edge-connected graphs.

Luo et al. [12] characterized graphs with independence number two that admit a nowhere-zero 3-flow.

Theorem 1.3 (*Luo et al. [12]*)**.** *Let G be a bridgeless graph with independence number $\alpha(G) \le 2$. Then G admits a nowhere-zero 3-flow if and only if G cannot be contracted to K_4 or G^3, and G is not one of three exceptional graphs, G^3, G^5, G^{18} (see Fig. 1).*

Yang et al. [17] further refined this result to characterize 3-edge-connected $\mathbb{Z}_3$-connected graphs with independence number two. To state their theorem, we need to introduce the concept of $\langle \mathbb{Z}_3 \rangle$-reduction first. Note that a K_1 is $\mathbb{Z}_3$-connected, which is called a trivial $\mathbb{Z}_3$-connected graph, and thus for any graph G, every vertex lies in a maximal $\mathbb{Z}_3$-connected subgraph of G. Let $H_1, H_2, \dots, H_c$ denote the collection of all maximal $\mathbb{Z}_3$-connected subgraph of G. We call $G' = G/(\cup_{i=1}^{c} E(H_i))$ the $\langle \mathbb{Z}_3 \rangle$*-reduction* of G, and we say that G is $\langle \mathbb{Z}_3 \rangle$*-reduced to* G'. A graph G is $\langle \mathbb{Z}_3 \rangle$*-reduced* if G does not have a nontrivial $\mathbb{Z}_3$-connected subgraph. By definition, the $\langle \mathbb{Z}_3 \rangle$-reduction of a graph is always $\langle \mathbb{Z}_3 \rangle$-reduced. It is shown in [10] that a graph G admits a nowhere-zero 3-flow (is $\mathbb{Z}_3$-connected, respectively) if and only if its $\langle \mathbb{Z}_3 \rangle$-reduction admits a nowhere-zero 3-flow (is $\mathbb{Z}_3$-connected, respectively). Moreover, the potential minimal counterexamples to Conjecture 1.1 must be $\langle \mathbb{Z}_3 \rangle$-reduced graphs. Therefore in order to describe nowhere-zero 3-flow and $\mathbb{Z}_3$-connectedness properties of certain family of graphs, it is sufficient to characterize all $\langle \mathbb{Z}_3 \rangle$-reductions of this family.

Theorem 1.4 (*Yang et al. [17]*)**.** *Let G be a* 3*-edge-connected graph with $\alpha(G) \le 2$. If G is not one of the* 18 *graphs of order at most* 8*, then G is $\mathbb{Z}_3$-connected if and only if G cannot be $\langle \mathbb{Z}_3 \rangle$-reduced to one of the graphs in $\{K_4, G^3, G^4, G^{10}, G^{11}\}$ (see Fig. 1).*

The purpose of this paper is to further extend Theorem 1.3 to graphs with independence number at most 4, and thus resolve the 3-Flow Conjecture for this family of graphs.

Denote $\mathcal{F}_1 = \{H \mid H$ is $\langle \mathbb{Z}_3 \rangle$-reduced without mod 3-orientation, $2 \le |V(H)| \le 15, \alpha(H) \le 4$ and $\kappa'(H) \le 3\}$, and let $\mathcal{F}_2 = \{H \mid H$ has no mod 3-orientation and $14 \le |V(H)| \le 20\}$.

Theorem 1.5. *Let G be a graph with $\alpha(G) \le 4$. Then G admits a nowhere-zero* 3*-flow if and only if $G \notin \mathcal{F}_2$ and the $\langle \mathbb{Z}_3 \rangle$-reduction of G is not in $\mathcal{F}_1$.*

Since each graph in $\mathcal{F}_1$ is of edge-connectivity at most 3, Theorem 1.5 immediately leads the following, which verifies the 3-Flow Conjecture for graphs with at least 21 vertices and independence number at most 4.

Theorem 1.6. *Every* 4*-edge-connected graph G with $|V(G)| \ge 21$ and $\alpha(G) \le 4$ admits a nowhere-zero* 3*-flow.*

In Section 3, we will show that Theorem 1.5 is equivalent to Theorem 1.6 (Lemma 3.3).

For graphs with independence number at most 3, we can eliminate the order requirement in Theorem 1.6 and prove the following theorem.

Theorem 1.7. *Every 4-edge-connected graph G with $\alpha(G) \leq 3$ admits a nowhere-zero 3-flow.*

In fact, in Section 3, we will prove slightly stronger results than Theorems 1.6 and 1.7 by replacing 4-edge-connectivity with odd-5-edge-connectivity.

Theorem 1.8. *Every odd-5-edge-connected graph G with $|V(G)| \geq 21$ and $\alpha(G) \leq 4$ admits a nowhere-zero 3-flow.*

Theorem 1.9. *Every odd-5-edge-connected graph G with $\alpha(G) \leq 3$ admits a nowhere-zero 3-flow.*

Remark. There are quite a few graphs in the family $\mathcal{F}_1$ that are far from being described by hand. In particular, the 18 special graphs of order at most 8 demonstrated by Yang et al. [17] can be modified to construct graphs in $\mathcal{F}_1$ by replacing a vertex of K_4 with one of those graphs. Also, many graphs obtained from 2-sum of two small non-3-flow admissible graphs are in $\mathcal{F}_1$.

While some splitting technique cannot be applied for $\mathbb{Z}_3$-connectedness, it seems very complicated to obtain analogous results for $\mathbb{Z}_3$-connectedness of graphs with small independence number via modifying the method of this paper and much more involved discussion on small graphs are needed. However, such characterization for $\mathbb{Z}_3$-connectedness is interesting. Note that Jaeger et al. [8] constructed a 4-edge-connected graph G of order 12 with $\alpha(G) = 3$, which is not $\mathbb{Z}_3$-connected.

The organization of the rest of the paper is as follows: Tools and preliminaries will be given in Section 2 and the proofs of the main results will be presented in Section 3.

2. Preliminaries

In this section, we display and develop some tools needed in the proofs of the main results.

2.1. Tools

Lemma 2.1 is a summary of certain basic properties from [8,10].

Lemma 2.1. *Let G be a graph. Each of the following holds:*
(i) *If $G \in \langle \mathbb{Z}_3 \rangle$ and $e \in E(G)$, then $G/e \in \langle \mathbb{Z}_3 \rangle$.*
(ii) *If $H \subseteq G$, and if both $H \in \langle \mathbb{Z}_3 \rangle$ and $G/H \in \langle \mathbb{Z}_3 \rangle$, then $G \in \langle \mathbb{Z}_3 \rangle$.*
(iii) *G admits a mod 3-orientation if and only if its $\langle \mathbb{Z}_3 \rangle$-reduction does.*
(iv) *$G \in \langle \mathbb{Z}_3 \rangle$ if and only if its $\langle \mathbb{Z}_3 \rangle$-reduction is K_1.*
(v) *A cycle C_n is $\mathbb{Z}_3$-connected if and only if $n = 2$.*
(vi) *The complete graph K_n is $\mathbb{Z}_3$-connected if and only if $n = 1$ or $n \geq 5$.*

Lemma 2.2 (*Han et al. [6]*)**.** *Every $\langle \mathbb{Z}_3 \rangle$-reduced graph has minimum degree at most 5.*

It has been extensively studied on the graphs admitting nowhere-zero 3-flows or being $\mathbb{Z}_3$-connected under degree conditions. For example, Barat and Thomassen [1] presented some degree conditions to ensure a simple graph to be $\mathbb{Z}_3$-connected. Fan and Zhou [4] and Luo et al. [13] characterized graphs admitting nowhere-zero 3-flow and all $\mathbb{Z}_3$-connected graphs under Ore-condition, respectively, where a simple graph G satisfies **Ore-condition**, if for every pair of nonadjacent vertices u and v in G, $d_G(u) + d_G(v) \geq |V(G)|$. Their results will be needed in our proofs to handle small graphs.

Theorem 2.3 (*Fan and Zhou [4]*)**.** *Let G be a simple graph on $n \geq 3$ vertices satisfying the Ore-condition. Then G admits a nowhere-zero 3-flow except for six specified small graphs (see Fig. 2(1)–(6)).*

Theorem 2.4 (*Luo et al. [13]*)**.** *Let G be a simple graph on $n \geq 3$ vertices satisfying the Ore-condition. Then G is $\mathbb{Z}_3$-connected except for 12 specified small graphs (see Fig. 2(1)–(12)).*

Let u_1v and u_2v be two distinct edges in G. Denote $G_{[v,u_1u_2]}$ to be the graph obtained from G by deleting the edges u_1v, u_2v and adding a new edge u_1u_2, which is called the *lifting operation* (see [11,14]). The following splitting lemma of Zhang [19] shows that the odd-edge-connectivity is preserved under certain lifting operation.

Lemma 2.5 (*Zhang [19]*)**.** *Let G be a graph with odd-edge-connectivity k. Assume there is a vertex $v \in V(G)$ with $d(v) \neq k$ and $d(v) \neq 2$. Then there exists a pair of edges u_1v, u_2v in $\partial_G(v)$ such that $G_{[v,u_1u_2]}$ preserves odd-edge-connectivity k.*

Remark. Lemma 2.5 does not apply to a vertex v of degree two as v is an isolated vertex in $G_{[v,u_1u_2]}$. While in most of the flow problems (include the proofs in this paper), a degree two vertex v does not appear in the minimal counterexamples since we could apply induction on the graph obtained from $G_{[v,u_1u_2]}$ by deleting the isolated vertex v. For this reason, we frequently ignore the discussion of degree two vertices when we apply Lemma 2.5.

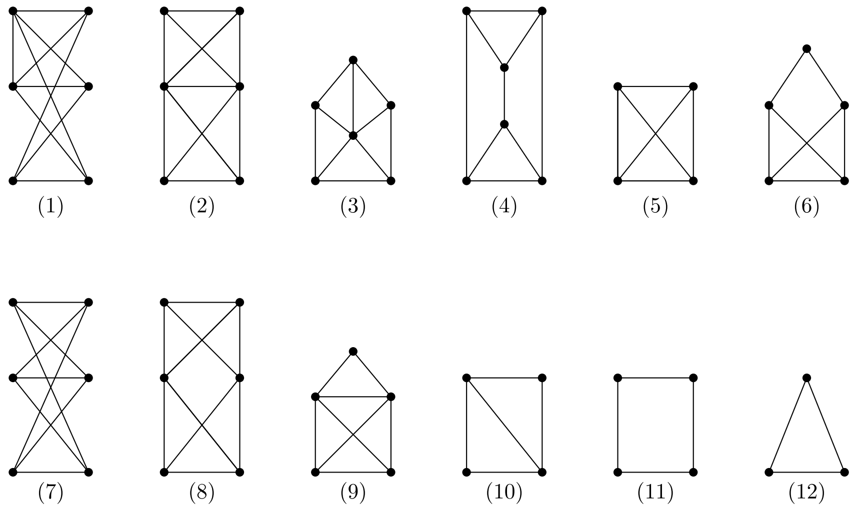

Fig. 2. Graphs in Theorems 2.3 and 2.4.

2.2. A new contraction method to handle odd wheels

A wheel W_n is the graph obtained from an n-cycle by adding a new vertex, called the center of the wheel, which is joined to every vertex of the n-cycle. W_n is odd (or even, respectively) if n is odd (or even, respectively). The complete graph K_4 can be viewed as a W_3.

Lemma 2.6. *Let k be a positive integer.*

(i) *(DeVos et al. [3]) Every even wheel W_{2k} is $\mathbb{Z}_3$-connected.*

(ii) *(Xu [16]) Let $b \in Z(W_{2k+1}, \mathbb{Z}_3)$. If there exists $b(v) \neq 0$ for some $v \in V(W_{2k+1})$, then there is an orientation D of W_{2k+1} such that $d^+_D(x) - d^-_D(x) \equiv b(x) \pmod 3$ for any $x \in V(W_{2k+1})$.*

Lemma 2.6(ii) tells that an odd wheel is almost $\mathbb{Z}_3$-connected except when the boundary $b \in Z(W_{2k+1}, \mathbb{Z}_3)$ is a constant zero function. Thus if a graph contains an odd wheel and if the resulting graph admits a nowhere-zero 3-flow (is $\mathbb{Z}_3$-connected, respectively) after contracting an odd wheel into a K_2, then so does (so is, respectively) the original graph. Therefore we have the following lemma.

Lemma 2.7. *Let G be a connected graph that contains a W_{2k+1} as a proper subgraph of G. Let X, Y be a partition of $V(W_{2k+1})$, and let $G_{[X,Y]}$ be the graph obtained from G by deleting the edges of $E(W_{2k+1})$, contracting X and Y into x and y, respectively, and adding a new edge xy (see Fig. 3).*

(i) *If $G_{[X,Y]}$ has a mod 3-orientation, then so does G.*

(ii) *If $G_{[X,Y]}$ is $\mathbb{Z}_3$-connected, then G is $\mathbb{Z}_3$-connected.*

An edge cut $\partial_G(S) = [S, V(G) - S]$ in a connected graph G is *essential* if at least two components of $G - \partial_G(S)$ are nontrivial, where a component is called *nontrivial* if it contains at least one edge. A graph is *essentially k-edge-connected* if it does not have an essential edge cut with fewer than k edges. Observe that, in a highly essentially edge-connected graph, if we contract an odd wheel into a single edge as described in Lemma 2.7, then the edge connectivity of the resulting graph cannot drop too much. To formulate this for later application, we define the following special contraction of odd wheels.

Definition 2.8. Let G be a connected graph and W_{2k+1} be a proper subgraph of G. $G_1 = G_{[X,Y]}$ is a W-**contraction** of G if X, Y form a partition of $V(W_{2k+1})$ and one of X, Y consists of two adjacent vertices in the $(2k+1)$-cycle of W_{2k+1} (see Fig. 3).

Note that, in a W-contraction of G, the original 4 edges in $[X, Y]_G$ are replaced by a single edge $K_2 = xy$. Hence an essential edge-cut of size k in G results in an edge-cut of size at least $k - 3$ in the W-contraction. It is also obvious that any W-contraction of G has minimal degree at least 5 provided that G is 5-edge-connected. Therefore, we obtain the following proposition.

Proposition 2.9. *Let G be a 5-edge-connected essentially 8-edge-connected graph. If G contains an odd wheel as a proper subgraph, then every W-contraction of G remains 5-edge-connected.*

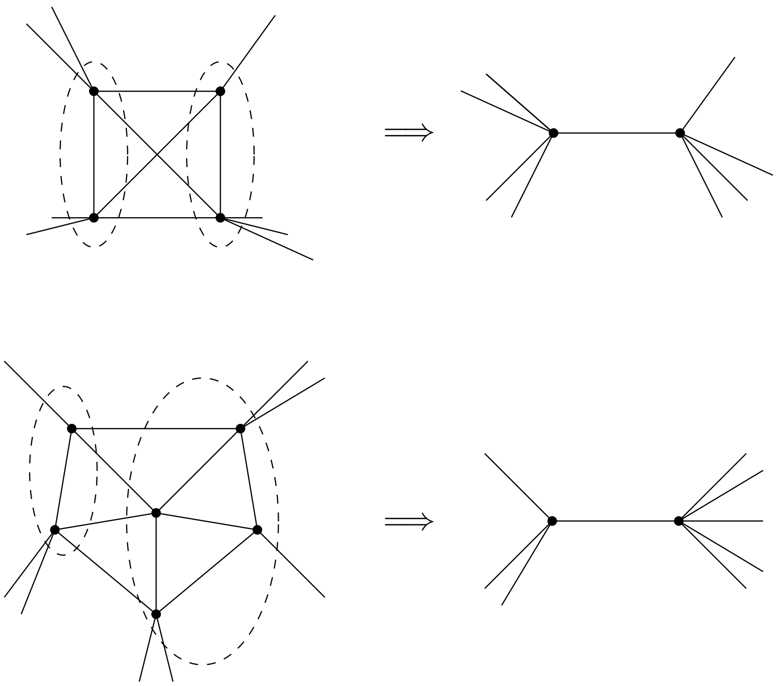

Fig. 3. Example of contraction in Lemma 2.7.

2.3. Small graphs

We shall discuss certain graphs of small order to serve for the induction basis of the proofs.

Denote $r(n, \mathbb{Z}_3) = \max\{|E(G)| \mid |V(G)| = n \text{ and } G \text{ is } \langle\mathbb{Z}_3\rangle\text{-reduced}\}$. We determine $r(n, \mathbb{Z}_3)$ when n is small in the following, which is needed in later proof.

Lemma 2.10. $r(1, \mathbb{Z}_3) = 0$, $r(2, \mathbb{Z}_3) = 1$, $r(3, \mathbb{Z}_3) = 3$, $r(4, \mathbb{Z}_3) = 6$, $r(5, \mathbb{Z}_3) = 8$, $r(6, \mathbb{Z}_3) = 11$, $r(7, \mathbb{Z}_3) = 13$.

Proof. Since a $\langle\mathbb{Z}_3\rangle$-reduced graph is simple by Lemma 2.1(v), it is routine to compute $r(n, \mathbb{Z}_3)$ when $n \leq 4$. For $n = 5$, $K_5 - e$ is not $\langle\mathbb{Z}_3\rangle$-reduced for any edge e in K_5 because it contains a $\mathbb{Z}_3$-connected subgraph, namely the wheel W_4 (by Lemma 2.6(i)). However, it is straightforward to show that K_5 deleting two incident edges is $\langle\mathbb{Z}_3\rangle$-reduced (see Fig. 2 (9)). Therefore $r(5, \mathbb{Z}_3) = 8$. We are to show $r(6, \mathbb{Z}_3) = 11$ and $r(7, \mathbb{Z}_3) = 13$ below.

Let G be a $\langle\mathbb{Z}_3\rangle$-reduced graph of order 6. Since every subgraph of a $\langle\mathbb{Z}_3\rangle$-reduced graph is also $\langle\mathbb{Z}_3\rangle$-reduced, we have $|E(G)| \leq \delta(G) + r(5, \mathbb{Z}_3) = \delta(G) + 8$. By Lemma 2.1(v), G is simple. If $\delta(G) \leq 2$, then $|E(G)| \leq 10$. If $\delta(G) \geq 3$, then G satisfies Ore-condition and thus by Theorem 2.4, we have $|E(G)| \leq 11$ with equality if and only if G is isomorphic to G^3 in Fig. 1. This proves $r(6, \mathbb{Z}_3) = 11$ and the only $\langle\mathbb{Z}_3\rangle$-reduced graph of order 6 with 11 edges is G^3.

Clearly, the graph obtained from G^3 by adding a new vertex with two nonparallel edges connecting to G^3 is $\langle\mathbb{Z}_3\rangle$-reduced. So $r(7, \mathbb{Z}_3) \geq 13$. Let G be a $\langle\mathbb{Z}_3\rangle$-reduced graph of order 7. Then, by Lemma 2.1(v), G is simple and by Theorem 2.4, $\delta(G) \leq 3$. If $\delta(G) \leq 2$, then $|E(G)| \leq 2 + r(6, \mathbb{Z}_3) \leq 13$. Assume $\delta(G) = 3$. Then $|E(G)| \leq 14$. If $|E(G)| = 14$, then $G - v = G^3$ for any degree 3 vertex v. Then G must be the graph obtained from G^3 by adding a new vertex v adjacent to both degree 5 vertices and one degree 3 vertex. Thus G contains a W_4. So G is not $\langle\mathbb{Z}_3\rangle$-reduced by Lemma 2.6(i), a contradiction. Hence $|E(G)| \leq 13$ and $r(7, \mathbb{Z}_3) = 13$. □

The proposition below follows directly from the definitions of $r(n, \mathbb{Z}_3)$ and $\langle\mathbb{Z}_3\rangle$-reduced graphs.

Proposition 2.11. *Let G be a $\langle\mathbb{Z}_3\rangle$-reduced graph and $S \subset V(G)$. Then*

$$|\partial_G(S)| \geq \delta(G)|S| - 2r(|S|, \mathbb{Z}_3).$$

By applying Lemma 2.10 and Proposition 2.11 with straightforward calculation, we have the following lemma immediately.

Lemma 2.12. *Let G be a $\langle\mathbb{Z}_3\rangle$-reduced graph with $|V(G)| \leq 15$. If $\delta(G) \geq 5$, then G is 5-edge-connected and is essentially 8-edge-connected. That is, for any $S \subset V(G)$ with $\min\{|S|, |\bar{S}|\} \geq 2$,*

$$|\partial_G(S)| \geq 8.$$

We also need the following orientation theorem of Hakimi [5] to handle small graphs.

Theorem 2.13 (*Hakimi [5]*). *Let G be a graph and $\ell : V(G) \mapsto \mathbb{Z}$ be a function such that $\sum_{v\in V(G)}\ell(v) = 0$ and $\ell(v) \equiv d_G(v) \pmod 2$, $\forall v \in V(G)$. Then the following are equivalent.*
(i) *G has an orientation D such that $d^+_D(v) - d^-_D(v) = \ell(v)$, $\forall v \in V(G)$.*
(ii) *$|\sum_{v\in S}\ell(v)| \le |\partial_G(S)|$, $\forall S \subset V(G)$.*

Lemma 2.14. *Every odd-5-edge-connected graph of order at most 13 admits a mod 3-orientation.*

Proof. Let G be a counterexample with $|V(G)| + |E(G)|$ minimized. Then G is a $\langle\mathbb{Z}_3\rangle$-reduced graph by Lemma 2.1(iii). By Lemma 2.5, G is 5-regular, which implies that $|V(G)|$ is even. If $|V(G)| \le 10$, then G has a mod 3-orientation by Theorem 2.3, a contradiction. Assume $|V(G)| = 12$ in the following.

Since every even wheel is $\mathbb{Z}_3$-connected by Lemma 2.6(i), G does not contain an even wheel. If G contains an odd wheel, then we apply W-contraction, and the resulting graph is still 5-edge-connected by Lemma 2.12 and Proposition 2.9. This yields a smaller counterexample by Lemma 2.7. Thus we obtain the following.

Fact A. *G does not contain a wheel as a subgraph. In particular,*
(i) *G contains no K_4, W_4, W_5;*
(ii) *for any vertex $v \in V(G)$, $G[N_G(v)]$ has no cycle, and therefore, $|E(G[N_G(v)])| \le 4$.*

Let [X,Y] be a maximum edge cut of G with $|X| \le |Y|$. Since G is 5-regular, we have

$$|[x, Y]_G| \ge 3, \text{ for any } x \in X \text{ and } |[y, X]_G| \ge 3, \text{ for any } y \in Y. \tag{1}$$

Hence

$$3|Y| \le |[X, Y]_G| \le 5|X| = 5(12 - |Y|),$$

which implies that $|Y| \le 7$ and thus $5 \le |X| \le |Y| \le 7$ since $|X| + |Y| = 12$.

If $|X| = 5$ and $|Y| = 7$, denote $Y_0 = \{y \in Y : |[y, X]_G| = 3\}$. It follows that $5|X| - 2|E(G[X])| = |[X, Y]_G| \ge 3|Y_0| + 4(7 - |Y_0|)$, which implies that

$$|Y_0| \ge 3 + 2|E(G[X])| \text{ and } |E(G[X])| \le 2. \tag{2}$$

By (2), there is a vertex $y_0 \in Y_0$ such that y_0 is adjacent to an isolated vertex in $G[X]$. Since y_0 has only three neighbors in X, we have the following.

Fact B. *If $|X| = 5$ and $|Y| = 7$, then $N_G(y_0) \cap X$ induces a graph with at most one edge.*

We define a function ℓ as follows. If $|X| = |Y| = 6$, set $\ell(x) = 3$ for any $x \in X$ and $\ell(y) = -3$ for any $y \in Y$; if $|X| = 5$ and $|Y| = 7$, set $\ell(x) = 3$ for any $x \in X$, $\ell(y_0) = 3$ and $\ell(y) = -3$ for any $y \in Y \setminus \{y_0\}$.

As $\sum_{v\in V(G)}\ell(v) = 0$ and by Theorem 2.13, there exists an $S_0 \subset V(G)$ with $|S_0| \le 6$ such that

$$\left|\sum_{v\in S_0}\ell(v)\right| > |\partial_G(S_0)|. \tag{3}$$

Clearly, by (1), we have

$$S_0 \not\subseteq X \text{ and } S_0 \not\subseteq Y. \tag{4}$$

By (3) and Lemma 2.12, $|S_0| \ge 4$ and thus we have

$$\left|\sum_{v\in S_0}\ell(v)\right| > |\partial_G(S_0)| \ge 8. \tag{5}$$

We consider three cases according to $|S_0|$ in the following.

Case 1: $|S_0| = 4$.

Since $|\sum_{v\in S_0}\ell(v)| > |\partial_G(S_0)| \ge 8$, by (5), we have $|\sum_{v\in S_0}\ell(v)| = 12$. Thus, by (4), we have $|X| = 5$, $|Y| = 7$ and $S_0 \cap Y = \{y_0\}$. However, it follows from Fact B that

$$\partial_G(S_0)| \ge 5|S_0| - 2|E(G[S_0])| \ge 20 - 2 \times (1 + 3) = |\sum_{v\in S_0}\ell(v)|, \tag{6}$$

a contradiction to (5).

Case 2: $|S_0| = 5$.

With a similar calculation as in Case 1, by Proposition 2.11, $|\partial_G(S_0)| \geq \delta(G)|S_0| - 2r(5, \mathbb{Z}_3) \geq 9$, which implies $|\sum_{v\in S_0} \ell(v)| = 15$ by (3). Thus $|X| = 5$, $|Y| = 7$ and $S \cap Y = \{y_0\}$ by (4). When $|X| = 5$ and $|Y| = 7$, we have $|E(G[X])| \leq 2$ by (2). Since $|[y_0, X]_G| = 3$ by the choice of y_0, we have $|E(G[S_0])| \leq 5$. Therefore, $\partial_G(S_0)| \geq 5|S_0| - 2|E(G[S_0])| \geq 15$, which contradicts (3).

Case 3: $|S_0| = 6$.

In this case, when $|S_0 \cap X| = 2$ or 3, we have $|\sum_{v\in S_0} \ell(v)| \leq 6$, a contradiction to (5). Thus $|S_0 \cap X| = 1, 4$, or 5 by (4).

If $|S_0 \cap X| = 1$ or 5, then either $|S_0 \cap X| = 1$ or $|S_0 \cap Y| = 1$ and by (3), we have

$$12 \geq |\sum_{v\in S_0} \ell(v)| > |\partial_G(S_0)| = 5|S_0| - 2|E(G[S_0])| = 30 - 2|E(G[S_0])|,$$

which implies $|E(G[S_0])| \geq 10$.

Let w be the vertex in $G[S_0]$ such that $S_0 \cap X = \{w\}$ or $S_0 \cap Y = \{w\}$. Since $d_{G[S_0]}(w) \leq d_G(w) = 5$, we have

$$|E(G[N_G(w)])| \geq |E(G[S_0])| - d_{G[S_0]}(w) \geq 5,$$

contradicting Fact A(ii).

If $|S_0 \cap X| = 4$, then $|S_0 \cap Y| = 2$. Since $|\partial_G(S_0)| \geq 8$ by (5), we have $|\sum_{v\in S_0} \ell(v)| = 12$, implying that $|X| = 5$, $|Y| = 7$ and $y_0 \in S_0 \cap Y$. We claim that

$$|E(G[S_0])| \leq 9. \tag{7}$$

If $|E(G[X])| = 2$, we have $|Y_0| = 7$ by (2). Thus $|[S_0 \cap X, S_0 \cap Y]| \leq 6$. Therefore

$$|E(G[S_0])| \leq |E(G[X])| + |[S_0 \cap X, S_0 \cap Y]| + |E(G[S_0 \cap Y])| \leq 9.$$

Now assume $|E(G[X])| \leq 1$. Denote $(S_0 \cap Y) \setminus \{y_0\} = \{z\}$. Since $|[y_0, X]_G| = 3$ by the choice of y_0, we have $|[S_0 \cap X, S_0 \cap Y]| + |E(G[S_0 \cap Y])| \leq |[y_0, X]_G| + d_G(z) \leq 3 + 5$. Therefore, (7) holds as well by the same inequality above. This proves (7).

By (7), we have

$$12 = |\sum_{v\in S_0} \ell(v)| > |\partial_G(S_0)| = 5|S_0| - 2|E(G[S_0])| \geq 30 - 18,$$

a contradiction. This contradiction completes the proof of the lemma. □

Corollary 2.15. *Let G be a graph with $|V(G)| \leq 15$. If G is 5-edge-connected and contains a K_4, then G admits a nowhere-zero 3-flow.*

Proof. Let G be a counterexample with $|E(G)|$ minimum. Denote $\{v_1, v_2, v_3, v_4\}$ to be the vertex set of a K_4 in G. We first show that G is a $\langle \mathbb{Z}_3\rangle$-reduced graph. Suppose to the contrary that H is a maximal nontrivial $\mathbb{Z}_3$-connected subgraph of G. Since G/H admits no nowhere-zero 3-flow, G/H does not contain a K_4 by the minimality of G, and $|V(G/H)| \geq 14$ by Lemma 2.14. So $|V(H)| = 2$, meaning that H consists of some parallel edges. Moreover one edge of K_4, say v_1v_2, is included in H. Then $V(H) = \{v_1, v_2\}$ and H contains a digon v_1v_2. This implies that $G[\{v_1, v_2, v_3, v_4\}]$ is $\mathbb{Z}_3$-connected by Lemma 2.1(ii)(v), a contradiction to the maximality of H. This proves that G is a $\langle \mathbb{Z}_3\rangle$-reduced graph.

Since G is $\langle \mathbb{Z}_3\rangle$-reduced and $\delta(G) \geq 5$, G is essentially 8-edge-connected by Lemma 2.12. Applying W-contraction on G, by Proposition 2.9, the resulting graph G' remains 5-edge-connected and has order at most 13. By Lemma 2.14, G' admits a nowhere-zero 3-flow. Therefore, G admits a nowhere-zero 3-flow by Lemma 2.7(i), a contradiction to the choice of G. □

3. Proofs of Theorems 1.8 and 1.9

This section will devote proofs of Theorems 1.8 and 1.9. We start with some lemmas. For a vertex subset X of $V(G)$, denote the neighbor set of X to be $N(X) = \{y | y \notin X$ and there exists $x \in X$ such that $xy \in E(G)\}$.

Lemma 3.1. *Let G be a graph with $\alpha(G) \leq t$.*

(i) *For any nonempty $X \subset V(G)$, $\alpha(G - (X \cup N(X))) \leq t - 1$.*

(ii) *For any maximal $\mathbb{Z}_3$-connected subgraph H of G with $|V(H)| \geq t + 1$, $\alpha(G - V(H)) \leq t - 1$.*

Proof. (i) is obvious.

Now we prove (ii). Denote $J = G - V(H)$. Suppose to the contrary that $\alpha(J) = t$. Let $\{v_1, \ldots, v_t\}$ be an independent set of size t in J. By Lemma 2.1(ii)(v), we have $|[v_i, V(H)]| \leq 1$ for each $1 \leq i \leq t$. Since $|V(H)| \geq t + 1$, there exists a vertex $u \in V(H)$ such that $|[u, \{v_1, \ldots, v_t\}]_G| = 0$. Thus $\{v_1, \ldots, v_t, u\}$ is an independent set of size $t + 1$ in G, yielding a contradiction to $\alpha(G) \leq t$. □

Lemma 3.2. *Let G be a $\langle\mathbb{Z}_3\rangle$-reduced graph.*
(a) *If $\alpha(G) \le 2$, then $|V(G)| \le 8$. Moreover, if $|V(G)| = 8$, then G contains a K_4.*
(b) *If $\alpha(G) \le 3$, then $|V(G)| \le 14$. Moreover, if $|V(G)| = 14$, then G is 5-edge-connected and contains a K_4.*
(c) *If $\alpha(G) \le 4$, then $|V(G)| \le 20$.*

Proof. (a) Let G be a $\langle\mathbb{Z}_3\rangle$-reduced graph with $\alpha(G) \le 2$. By Lemma 2.1(vi), G does not contain a K_5. Thus if $\kappa'(G) \in \{0, 1\}$, $|V(G)| \le 8$ and if $|V(G)| = 8$, then G contains a K_4. (a) is also true by Theorem 1.4 if $\kappa'(G) \ge 3$.

Assume $\kappa'(G) = 2$. Let $[X_1, X_2]_G$ be a 2-edge-cut of G, where X_1, X_2 form a partition of $V(G)$. Then $|N(X_i)| \le 2$ for each $i \in \{1, 2\}$. By Lemmas 2.1(vi) and 3.1(i), for $i \in \{1, 2\}$, $G - (X_i \cup N(X_i))$ is a complete graph of size at most 4. Since $|N(X_i)| \le 2$, we have

$$|V(G)| \le 4 + |X_i| + |N(X_i)| \le 6 + |X_i|.$$

If $|X_i| \le 2$ for some $i \in \{1, 2\}$, then we have $|V(G)| \le 8$ and $|V(G)| = 8$ implies that G contains a K_4. If both $|X_1| \ge 3$ and $|X_2| \ge 3$, then there is a vertex $x_1 \in X_1$ such that $|[x_1, X_2]_G| = 0$. Since $X_2 \subseteq V(G) - (\{x_1\} \cup N(x_1))$, by Lemma 3.1(i), $G[X_2]$ is a complete graph. Hence $|X_2| \le 4$ since G does not contain a K_5. Similarly, we have $|X_1| \le 4$. Thus $|V(G)| \le 8$. If $|V(G)| = 8$, then $G[X_1]$ is a K_4.

(b) By Lemma 2.2, $\delta(G) \le 5$. Thus by (a) and Lemma 3.1(i) we have $|V(G)| \le 1 + \delta(G) + 8 \le 14$ and if $|V(G)| = 14$, then G contains a K_4. When $|V(G)| = 14$, the above inequality is equality, implying every vertex in G is of degree at least 5. By Lemma 2.12, G is 5-edge-connected.

(c) By Lemma 2.2, $\delta(G) \le 5$. Thus by (b) and Lemma 3.1(i), we have $|V(G)| \le 1 + \delta(G) + 14 \le 20$. □

Now we are ready to prove Theorem 1.9.

Theorem 1.9. *Every odd-5-edge-connected graph G with $\alpha(G) \le 3$ admits a mod 3-orientation.*

Proof of Theorem 1.9. Let G be a counterexample with $|E(G)|$ minimum. By Lemma 2.5, the degree of each vertex is odd; otherwise we lift all the edges incident with vertices of even degrees by applying Lemma 2.5, and then delete all isolated vertices to obtain a smaller counterexample. Thus $\delta(G) \ge 5$ and G is $\langle\mathbb{Z}_3\rangle$-reduced. By Lemma 2.14, $|V(G)| \ge 14$. Moreover, $|V(G)| \le 14$ by Lemma 3.2(b). Therefore $|V(G)| = 14$ and G contains a K_4. By Lemma 2.12, G is 5-edge-connected. By Corollary 2.15, G admits a mod 3-orientation, a contradiction. □

Lemma 3.3. *Every graph G with $|V(G)| \ge 21$ and $\alpha(G) \le 4$ is $\langle\mathbb{Z}_3\rangle$-reduced to a graph of order at most 15.*

Proof. Let G_1 be the underlying simple graph of G, which is obtained from G by replacing parallel edges $[u, v]_G$ with a single edge uv for each $|[u, v]_G| \ge 2$ in G. Since $|V(G_1)| \ge 21$, G_1 contains a nontrivial $\mathbb{Z}_3$-connected subgraph by Lemma 3.2(c), say H_1. Then $G[V(H_1)]$ is $\mathbb{Z}_3$-connected and $|V(H_1)| \ge 5$ by Lemma 2.1(vi). Let H be a maximal $\mathbb{Z}_3$-connected subgraph of G containing $G[V(H_1)]$. Then we have $|V(H)| \ge |V(H_1)| \ge 5$. Let $J = G - V(H)$ and J' be its $\langle\mathbb{Z}_3\rangle$-reduction. Since $|V(H)| \ge 5$, we have $\alpha(J) \le 3$ by Lemma 3.1(ii). Thus J', the $\langle\mathbb{Z}_3\rangle$-reduction of J, is of order at most 14 by Lemma 3.2(b). Since $|V(G')| = |V(J')| + 1$, where G' is the $\langle\mathbb{Z}_3\rangle$-reduction of G, we have $|V(G')| \le 15$. □

Equivalence of Theorems 1.5 and 1.6: In Theorem 1.5, if $|V(G)| \ge 21$, then the reduction of G is of order at most 15 by Lemma 3.3. So Theorem 1.6 is in fact equivalent to Theorem 1.5 by Lemma 2.14.

Now we are ready to prove Theorem 1.8.

Theorem 1.8. *Every odd-5-edge-connected graph G of order at least 21 with $\alpha(G) \le 4$ admits a mod 3-orientation.*

Proof of Theorem 1.8. Let G be a counterexample and G' be its $\langle\mathbb{Z}_3\rangle$-reduction. We shall show that G' has a mod 3-orientation, which yields to a contradiction by Lemma 2.1(iii).

By Lemma 3.3, $|V(G')| \le 15$. Since G' is odd-5-edge-connected with no mod 3-orientation and by Lemma 2.14, $|V(G')| \ge 14$. Therefore, $14 \le |V(G')| \le 15$.

Let H be a maximal nontrivial $\mathbb{Z}_3$-connected subgraph of G as in Lemma 3.3. Recall that $|V(H)| \ge 5$. Denote v_1 to be the contraction of H in G', and let $J' = G' - v_1$. Notice that J' is the $\langle\mathbb{Z}_3\rangle$-reduction of $J = G - V(H)$. Hence $\alpha(J') \le \alpha(J) \le 3$ by Lemma 3.1(ii).

We show the following to lead a contradiction.

(I) $|V(G')| = 14$.

Suppose to the contrary $|V(G')| = 15$. Then $|V(J')| = 14$. So J' is 5-edge-connected and contains a K_4 by Lemma 3.2(b).

If $d_{G'}(v_1) < 5$, then $d_{G'}(v_1)$ is even since G' is odd-5-edge-connected. Applying Lemma 2.5 to lift all edges incident with v_1, the resulting graph is 5-edge-connected and of order 14. By Corollary 2.15, the resulting graph admits a mod 3-orientation and so does G', a contradiction.

If $d_{G'}(v_1) \ge 5$, then G' is 5-edge-connected as J' is 5-edge-connected. Since G' contains a K_4, it admits a 3-orientation by Corollary 2.15, a contradiction again. This proves (I).

(II) G' is 5-regular and thus by Lemma 2.12, G' is 5-edge-connected.

Let x be a vertex in G'. If $d(x)$ is even, applying Lemma 2.5 to lift all the edges incident with x, the resulting graph remains odd-5-edge-connected with 13 vertices. Thus it has a mod 3-orientation by Lemma 2.14, so does G' by Lemma 2.1(iii)(v), a contradiction. Thus $\delta(G') \geq 5$. By Lemma 2.12, G' is 5-edge-connected essentially 8-edge-connected.

Now assume $d_{G'}(x) \geq 7$. Since $\alpha(G') \leq 4$, let u, v be two adjacent vertices in $N_{G'}(x)$. Let $G'' = G'_{[x,uv]}$ be the graph obtained from G' by deleting the edges xu, xv and adding a new edge uv. Since G' is essentially 8-edge-connected and $\delta(G'') \geq 5$, G'' remains 5-edge-connected. Note that G'' contains a digon uv. Then G''/uv has 13 vertices and remains 5-edge-connected. Thus it has a mod 3-orientation by Lemma 2.14, so does G' by Lemma 2.1(iii)(v), a contradiction.

The final step: By (II), $\delta(J') \leq 4$. Let $z \in V(J')$ with $d_{J'}(z) \leq 4$. Since $\alpha(J') \leq 3$, by Lemma 3.1(i), $\alpha(J' - (\{z\} \cup N_{J'}(z))) \leq 2$. Note that $J' - (\{z\} \cup N_{J'}(z))$ is a $\langle \mathbb{Z}_3 \rangle$-reduced graph of order at least 8. Thus by Lemma 3.2(a), $J' - (\{z\} \cup N_{J'}(z))$ has exactly 8 vertices and contains a K_4 and so does G'. By (I), (II) and Corollary 2.15, G' admits a mod 3-orientation, a contradiction. This completes the proof of Theorem 1.8. □

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