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Flow extensions and group connectivity with applications☆

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ABSTRACT

We study the flow extension of graphs, i.e., pre-assigning a partial flow on the edges incident to a given vertex and aiming to extend to the entire graph. This is closely related to Tutte's 3-flow conjecture(1972) that every 4-edge-connected graph admits a nowhere-zero 3-flow and a $\mathbb{Z}_3$-group connectivity conjecture(3GCC) of Jaeger, Linial, Payan, and Tarsi(1992) that every 5-edge-connected graph G is $\mathbb{Z}_3$-connected. Our main results show that these conjectures are equivalent to their natural flow extension versions and present some applications. The 3-flow case gives an alternative proof of Kochol's result(2001) that Tutte's 3-flow conjecture is equivalent to its restriction on 5-edge-connected graphs and is implied by the 3GCC. It also shows a new fact that Grötzsch's theorem (that triangle-free planar graphs are 3-colorable) is equivalent to its seemly weaker girth five case that planar graphs of girth 5 are 3-colorable. Our methods allow to verify 3GCC for graphs with crossing number one, which is in fact reduced to the planar case proved by Richter, Thomassen and Younger(2017). Other equivalent versions of 3GCC and related partial results are obtained as well.

1. Introduction

We consider finite graphs without loops, but permitting parallel edges. A vertex of degree k is called a *k-vertex*. An edge-cut of size k is called a *k-cut* for convenience, and basically no vertex-cut

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would be involved in this paper. A graph is *essentially k-edge-connected* if for any $t < k$, every t-cut isolates a vertex. In a graph G, a function $\beta : V(G) \to \mathbb{Z}_3$ is called a *boundary function* of G if $\sum_{x\in V(G)} \beta(x) = 0$ in $\mathbb{Z}_3$. Let $Z(G, \mathbb{Z}_3)$ be the set of all boundary functions of G. We call an orientation D of G a *β-orientation* if it holds that $d^+_D(v)-d^-_D(v) = \beta(v)$ in $\mathbb{Z}_3$ for every vertex $v \in V(G)$. The special case of β-orientation with $\beta(x) = 0$ in $\mathbb{Z}_3$ for every vertex $x \in V(G)$ is known as a *mod 3-orientation* of G. It is well-known (cf. [9,19,20]) that searching for mod 3-orientations is equivalent to finding nowhere-zero 3-flows in graphs. Tutte's 3-Flow Conjecture (abbreviated as 3FC) in 1972(see [1]) is as follows.

3-**Flow Conjecture (3FC):** *Every 4-edge-connected graph admits a nowhere-zero 3-flow.*

The 3FC restricted to planar graphs is the dual of Grötzsch's 3-Coloring Theorem (3CT) that every triangle-free planar graph is 3-colorable. Applying the famous coloring extension techniques, Thomassen [14–16] presented short proofs of Grötzsch's 3CT and extended to its list version, as well as obtained his elegant 5-list-coloring theorem [13]. Even before Thomassen's coloring extension proofs, Steinberg and Younger [12] employed a flow extension method to confirm 3FC for planar and projective planar graphs, that is to pre-assign certain flow value to edges incident a given vertex and then to extend it to the entire graph. Motivated by the results of Steinberg and Younger, we say that a graph G is *$\mathcal{M}_3$-extendable at $z \in V(G)$* if for any pre-orientation D_0 of $\partial_G(z)$ with $d^+_{D_0}(z) \equiv d^-_{D_0}(z)$ (mod 3), D_0 can be extended to a mod 3-orientation D of G.

Kochol [6] obtained some interesting equivalent versions of the 3FC.

Theorem 1 (*Kochol [6]*)**.** *The following are equivalent.*
(i) (3FC) *Every 4-edge-connected graph admits a nowhere-zero 3-flow.*
(ii) Every 5-edge-connected graph admits a nowhere-zero 3-flow.
(iii) Every 5-edge-connected graph is $\mathcal{M}_3$-extendable at every 5-vertex.

A graph G is called *$\mathbb{Z}_3$-connected* if, for every $\beta \in Z(G, \mathbb{Z}_3)$, there exists a β-orientation in G (i.e., an orientation D such that $d^+_D(x) - d^-_D(x) \equiv \beta(x)$ (mod 3), $\forall x \in V(G)$). This group connectivity concept was introduced by Jaeger, Linial, Payan, and Tarsi [5] as a nonhomogeneous generalization of Tutte's nowhere-zero flow theory [18]. Jaeger et al. [5] posed the following $\mathbb{Z}_3$-Group Connectivity Conjecture, abbreviated as 3GCC.

$\mathbb{Z}_3$**-Group Connectivity Conjecture (3GCC):** *Every 5-edge-connected graph is $\mathbb{Z}_3$-connected.*

The main purpose of this paper is to study some natural flow extension versions of 3FC and 3GCC, with some additional applications. In particular, using a unified approach, we provide a new proof of Theorem 1 (different from Kochol's 2-sum method [6]), and prove that some seemly stronger versions of 3GCC are actually equivalent to the original version, as shown in Theorem 2 below. Furthermore, as a byproduct of the new proof of Kochol's Theorem 1, it also indicates that those statements are equivalent within planar graphs, which implies that, by duality, Grötzsch's 3CT is exactly equivalent to its restriction on girth 5 case. This interesting fact seems not known before (since Kochol's arguments [6] need to construct nonplanar graphs).

Similar as the $\mathcal{M}_3$-extendability on mod 3-orientations, there is an analogous pre-orientation extension concept for $\mathbb{Z}_3$-group connectivity. This technique is notably one of the key ideas in the proof of Weak 3-Flow Conjecture by Thomassen [17], and subsequently improvement by Lovász, Thomassen, Wu and Zhang [9]. A graph is called *$\mathbb{Z}_3$-extendable at x*, if for any $\beta \in Z(G, \mathbb{Z}_3)$ and any pre-orientation D_x of $\partial_G(x)$ with $d^+_{D_x}(x) - d^-_{D_x}(x) \equiv \beta(x)$ (mod 3), D_x can be extended to a β-orientation D of G. A graph is *$\mathbb{Z}_3$-reduced* if it contains no $\mathbb{Z}_3$-connected subgraph of order at least two. We show the following statements are all equivalent to 3GCC, some of which have been appeared in [3] and shown to imply the 3GCC.

Theorem 2. *The following are equivalent.*
(a) (3GCC) *Every 5-edge-connected graph is $\mathbb{Z}_3$-connected.*
(b-i) Every 5-edge-connected graph is $\mathbb{Z}_3$-extendable at every 5-vertex.

(b-ii) Every 5-edge-connected essentially 6-edge-connected graph is $\mathbb{Z}_3$-extendable at every 5-vertex.
(c) Every $\mathbb{Z}_3$-reduced graph has minimum degree at most 4.
(d) Every 4-edge-connected graph with at most five 4-cuts is $\mathbb{Z}_3$-connected.

In particular, Theorem 2, using equivalent statement (c), provides another alternative proof (different from Theorem 1) of the fact that the validity of 3GCC implies 3FC. To see this, notice that the minimal counterexample G of 3FC is 5-regular by Mader's splitting lemma [10] (Lemma 3 below). Observe also that, if H is a $\mathbb{Z}_3$-connected subgraph of G, then a mod 3-orientation of G/H can be easily extended to G (cf. [3,5,9,20]), and so the minimal counterexample G must be $\mathbb{Z}_3$-reduced. Thus G is a 5-regular $\mathbb{Z}_3$-reduced graph, a contradiction to Theorem 2(c).

Restricted to planar graphs, applying the powerful flow extension techniques, a recent result of Richter, Thomassen and Younger [11] shows 3GCC and its flow extension version(Theorem 2 (b-i)) hold for planar graphs. The techniques in proving Theorems 1 and 2 allow us to obtain more equivalent statements of the Richter–Thomassen–Younger result, and to extend it to graphs with crossing number one.

Theorem 3. *Each of the following holds.*
(i) [7,11] Every 5-edge-connected planar graph is $\mathbb{Z}_3$-connected.
(ii) [11] Every 5-edge-connected planar graph is $\mathbb{Z}_3$-extendable at every 5-vertex.
(iii) Every $\mathbb{Z}_3$-reduced planar graph has minimum degree at most 4.
(iv) Every 5-edge-connected graph with crossing number at most one is $\mathbb{Z}_3$-connected.

For general graphs, we summarize some previous approach on each of the above statements of Theorem 2 from [3,9], and also provide new partial results for Theorem 2(d).

Theorem 4. *Each of the following holds.*
(a) [9] Every 6-edge-connected graph is $\mathbb{Z}_3$-connected.
(b-i) [9] Every 6-edge-connected graph is $\mathbb{Z}_3$-extendable at every vertex of degree at most 7.
(b-ii) [3] Every 5-edge-connected essentially 23-edge-connected graph is $\mathbb{Z}_3$-extendable at every 5-vertex.
(c) [3] Every $\mathbb{Z}_3$-reduced graph has minimum degree at most 5.
(d-i) Every 4-edge-connected graph with at most five 4-cuts and without 5-cuts is $\mathbb{Z}_3$-connected.
(d-ii) Every 5-edge-connected graph with at most seven 5-cuts is $\mathbb{Z}_3$-connected.

Note that Jaeger et al. [5] constructed a 4-edge-connected non-$\mathbb{Z}_3$-connected graph with fifteen 4-cuts and without 5-cuts. This indicates that Theorem 4(d-i) is almost tight.

In the next section, we first present some preliminaries, and then prove Theorems 1–3. The proof of Theorem 4(d-i)(d-ii) will be completed in Section 3.

2. Flow extensions

2.1. Preliminaries

Before proceeding we introduce a few more notation. For a vertex subset $A \subset V(G)$, we use $\partial_G(A)$ to denote the set of edges with one end in A and the other in A^c, where $A^c = V(G) \setminus A$ is the complement of A. Let $d_G(A) = |\partial_G(A)|$ be the number of edges between A and A^c. When $A = \{x\}$, we shall use $\partial_G(x)$ for $\partial_G(\{x\})$ and $d_G(x)$ for $d_G(\{x\})$, respectively. Sometimes the subscripts may be omitted for convenience if the graph G is understood from context.

In a graph G, a k-cut $\partial(A)$ is called *a k-critical-cut* with respect to A if $d(A) \le k$ and for any $B \subsetneq A$, $d(B) > k$; we also say that A is a *k-critical-set*. The following observation follows easily from the definition.

Observation 1. *Let G be a k-edge-connected graph with exactly q k-cuts. Denote $A_1, A_2, \ldots, A_t$ to be all distinct k-critical-set A such that $\partial(A)$ is a k-critical-cut. Then each of the following holds.*
(i) $A_i \cap A_j = \emptyset$ for any $i \ne j$.
(ii) If $q = 1$, then $t = 2$ and $A_2 = V(G) \setminus A_1$.

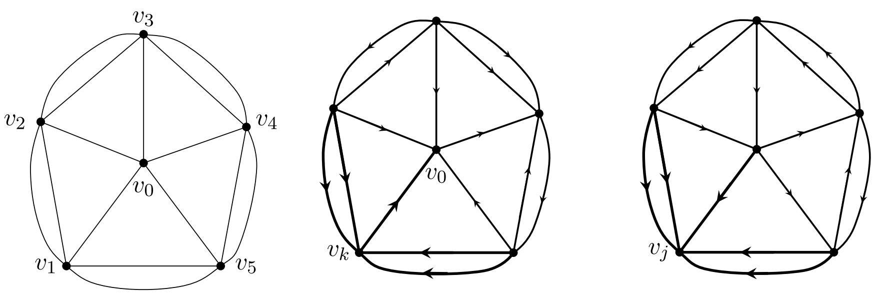

Fig. 1. The graph W and its orientations.

(iii) If $q \geq 2$, then $\partial(A_i) \neq \partial(A_j)$ for any $i \neq j$. Hence $t \leq q$.
(iv) Construct a graph G' from G by adding a new vertex x and connecting x and A_i with a new edge for each $i = 1, \ldots, t$. Then all the edge-cuts other than $\partial_{G'}(x)$ in G' have size at least $k+1$.

Let G be a graph with a 5-vertex $x \in V(G)$. In a mod 3-orientation D of G, the edges in $\partial(x)$ at x is oriented either as 4 ingoing and 1 outgoing, or as 1 ingoing and 4 outgoing. So we call such an edge in $\partial(x)$ a *minor-edge* at x if its orientation is different from other edges in $\partial(x)$.

A major step of our arguments relies on the following property of flows on the graph W depicted in Fig. 1. Formally, W denotes the graph with vertex set $V(W) = \{v_0, v_1, \ldots, v_5\}$ and edge multiset

$$E(W) = \{v_5v_1, v_5v_1\} \bigcup_{i=1}^{4} \{v_iv_{i+1}, v_iv_{i+1}\} \bigcup_{i=1}^{5} \{v_0v_i\}.$$

Lemma 1. *(i) For any mod 3-orientation D of W, there exists a vertex v_k with $1 \leq k \leq 5$ such that v_0v_k is the minor-edge at v_k.*
(ii) Let $\beta \in Z(W, \mathbb{Z}_3)$ be a boundary function such that $\beta(v_i) = 1$ in $\mathbb{Z}_3$ for each $i = 0, 1, \ldots, 5$. Then for any β-orientation D of W, there exists a vertex $v_j \in V(W)$ such that $d_D^+(v_j) = 0$ and $d_D^-(v_j) = 5$.

Proof. (i) Suppose to the contrary that, in a mod 3-orientation D of G each edge v_0v_k is not the minor-edge at v_k for $k = 1, 2, \ldots, 5$. We count the *deficiency* $d_D^+(v) - d_D^-(v)$ at each vertex $v \in V(W)$. By symmetry, we may assume that under orientation D the edges in $\partial(v_0)$ at vertex v_0 are oriented as 4 ingoing and 1 outgoing (with deficiency -3). As each v_0v_k is not the minor-edge at v_k for $k = 1, 2, \ldots, 5$, it holds that four of $\{v_1, v_2, \ldots, v_5\}$ are received orientations as 1 ingoing and 4 outgoing (with deficiency 3), and the other one is opposite as 4 ingoing and 1 outgoing (with deficiency -3). So the deficiency at all the vertices are four 3's and two -3's. This is a contradiction to the fact that $\sum_{v \in V(W)} (d_D^+(v) - d_D^-(v)) = 0$.
(ii) The proof is similar to (i) by counting deficiency at each vertex. Let D be a β-orientation of W. Then for each vertex $v \in V(W)$, $d_D^+(v) - d_D^-(v) \equiv \beta(v) \equiv 1 \pmod 3$, and so the deficiency $d_D^+(v) - d_D^-(v) \in \{1, -5\}$. Since $\sum_{i=0}^{5} (d_D^+(v_i) - d_D^-(v_i)) = 0$, there exists a vertex v_j with $0 \leq j \leq 5$ such that $d_D^+(v) - d_D^-(v) = -5$ as desired. □

We also need the following lemma about $\mathbb{Z}_3$-extendability in [3].

Lemma 2 ([3]). *Let G be a graph with $x \in V(G)$. Then G is $\mathbb{Z}_3$-extendable at x if and only if $G - x$ is $\mathbb{Z}_3$-connected.*

For a graph G with $uz, vz \in E(G)$, a *splitting* at z is an operation to delete edges uz, vz and add a new edge uv. If z is an even vertex of G, a *complete splitting* at z is to apply splitting operations on all the edges of $\partial_G(z)$ in pairs and then delete the isolated vertex z to obtain the resulting graph. The following Mader's splitting lemma shows that it is possible to preserve the edge connectivity after splitting operations.

Lemma 3 (*Mader [10]*)**.** *Let G be a k-edge-connected graph with a t-vertex $z \in V(G)$. If $t \geq k+2$, then there exists a splitting at z such that the resulting graph is k-edge-connected. If t is even, then there exists a complete splitting at z such that the resulting graph is k-edge-connected.*

2.2. Proofs of Theorems 1–3

In this subsection, we present the proofs of Theorems 1–3 using a unified construction method through properties given in Lemma 1.

An Alternative Proof of Theorem 1. Clearly, "(i)⇒(ii)" holds and a standard argument could show that "(iii)⇒(i)". We provide a proof of "(iii)⇒(i)" here for completeness, which is similar to Kochol's proof in [6]. Specifically, let G be a counterexample of 3FC (statement (i)) with $|E(G)| + |V(G)|$ minimized. Then G is 5-regular by Lemma 3. And G must contain nontrivial 4-cuts; otherwise G is 5-edge-connected, and so (i) follows by (iii). Among all nontrivial 4-cuts of G, we select a 4-cut $\partial(A)$ with $|A|$ as small as possible. Then $|V(G)| - 1 > |A| \geq 2$ and we have

$$d_G(A') = |\partial_G(A')| \geq 5 \text{ for any } A' \subsetneq A. \tag{1}$$

Contract A to obtain a new graph $G_1 = G/A$. Thus G_1 is 4-edge-connected, and so admits a mod 3-orientation D_1 by the minimality of G. Then we contract A^c to obtain another new graph $G_2 = G/A^c$, where A^c is contracted to become a new vertex x. Pre-orient the edges in $\partial_{G_2}(x)$ the same as $\partial_{D_1}(A^c)$. Hence the edges in $\partial_{G_2}(x)$ are oriented as two ingoing and two outgoing. Obtain a new graph G_3 from G_2 by replacing an ingoing edge at x with two outgoing edges. Hence x is a 5-vertex now, and G_3 is 5-edge-connected by (1). Moreover, the pre-orientation at x is still balanced mod 3. By (iii), this pre-orientation can be extended to a mod 3-orientation D_3 of G_3. Then, after deleting the edges of $\partial_{G_3}(x)$, the combination of D_1 and the rest of D_3 gives a mod 3-orientation of G. Hence (iii) implies (i).

The major task remaining is to show that "(ii)⇒(iii)". The method below is principally different from Kochol's proof in [6]. We hope this new method may shed some light on attacking 3FC and 3GCC.

Assume that statement (ii) holds that every 5-edge-connected graph has a mod 3-orientation. Suppose to the contrary that there is a 5-edge-connected graph G and a 5-vertex $x \in V(G)$ with pre-orientation D_x that is not $\mathcal{M}_3$-extendable to a mod 3-orientation of G. Recall that W denotes the graph depicted in Fig. 1. We construct a new graph H by replacing each vertex of W with a copy of $G - x$, where each edge v_0v_k ($1 \leq k \leq 5$) is corresponded to the minor-edge at x of D_x in that copy. More precisely, denote $\partial_G(x) = \{xx_0, xx_1, \ldots, xx_4\}$, where xx_0 is the minor-edge in pre-orientation D_x. (Notice that we allow $x_i = x_j$ for $i \neq j$, when $\partial_G(x)$ contains parallel edges.) The construction of the new graph H is as follows. Attach six copies of $G - x$, say $G^0, G^1, \ldots, G^5$, whose vertices corresponding to $x_0, \ldots, x_5$ are $x_0^i, \ldots, x_5^i$ for $i = 0, \ldots, 5$. First, replace the vertex v_0 of W with G^0 by putting the end v_0 of edge v_0v_i in the position of x_{i-1}^0 for each $i = 0, \ldots, 4$. Then, for each $j = 1, \ldots, 5$, replace the vertex v_j of W with G^j by putting the end v_j of edge v_jv_0 in the position of x_0^j, and putting the end v_j of other edges in $\partial_W(v_j)$ matching to $x_1^j, x_2^j, x_3^j, x_4^j$, respectively. The constructed new graph H is depicted in Fig. 2.

It is routine to check that H is 5-edge-connected by the 5-edge-connectivity of W and copies of G. Since statement (ii) holds, H admits a mod 3-orientation D. Contract all copies of $G - x$ to obtain a graph W and consider the orientation D restricted to W. By Lemma 1(i), there exists a vertex v_k of W, corresponding to the contraction of G^k (for some $k \in \{1, \ldots, 5\}$), such that v_0v_k is the minor-edge at v_k. Now in H contract all the vertices in $V(H) \setminus V(G^k)$ to become a new vertex x. Then this results a copy of G, consisting of a vertex x and $G^k = G - x$. The orientation D restricted to it provides a mod 3-orientation D_k. Moreover, the edge xx_0^k is a minor-edge at x under D_k. If D_k agrees with D_x at x, then D_k is a mod 3-orientation extended from D_x, a contradiction. Otherwise, we reverse the orientation of all edges from D_k to obtain another mod 3-orientation D_k^*. Now D_k^* agrees with D_x at x since xx_0^k is still the minor-edge at x under D_k^*. This is a contradiction again, completing the proof of Theorem 1. ■

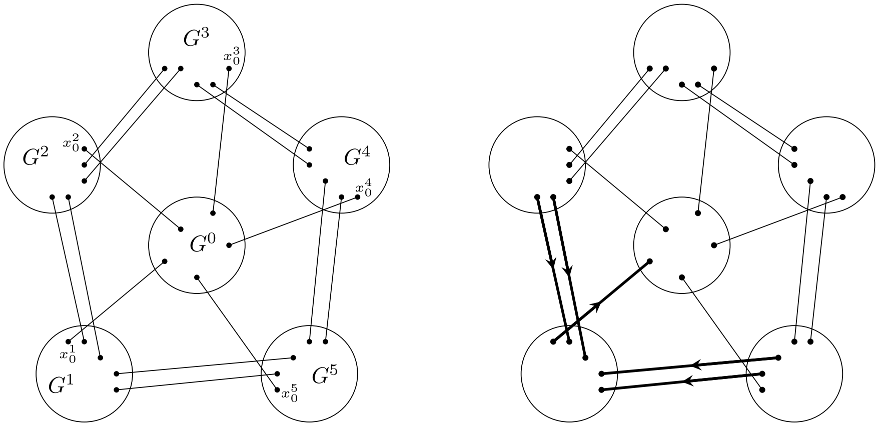

Fig. 2. The constructed graph H and its orientation for proving Theorem 1.

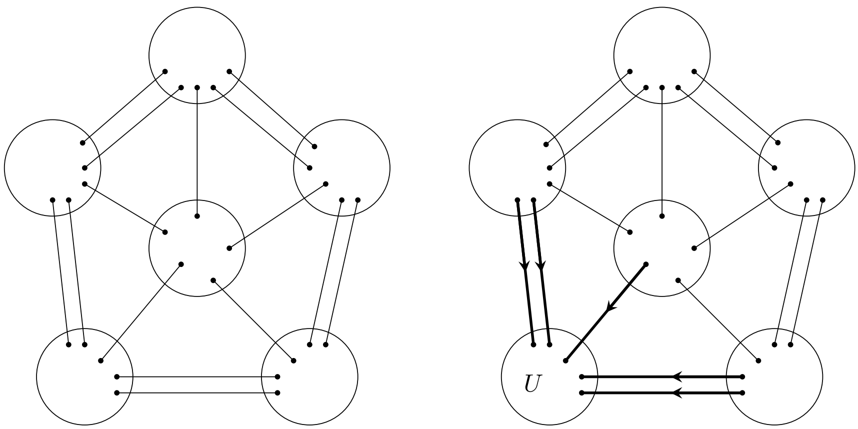

Fig. 3. The constructed graph for proving Theorems 2 and 3(ii)(iii).

With a little more thought, one can observe that in proving Theorem 1, if the graph G is planar, then the constructed graph H can be modified to planar as well, see similar construction in Fig. 3 below. (This is because the positions of $x_0, \ldots, x_4$ can be shifted cyclically in a planar embedding.) Thus we obtain the following corollary for planar graphs. It suggests that Grötzsch's 3CT is exactly equivalent to its restriction to girth 5 case, a fact maybe not known before.

Corollary 5. *The following are equivalent versions of Grötzsch's 3CT.*
(a) Every triangle-free planar graph is 3*-colorable.*
(b) Every planar graph of girth 5 *is* 3*-colorable.*

By applying arguments dual to the proof above (using dual graph of W and dual constructions), one may also show that Grötzsch's 3CT is also equivalent to the statement that any pre-coloring of a 5-cycle in a triangle-free planar graph can be extended to a 3-coloring of the entire graph, a useful strengthening theorem proved by Thomassen [14].

Now we prove Theorem 2 using similar constructions, but employing Lemma 1(ii) instead. The argument presented here is a slight modification of that in the author's Ph.D dissertation [8].

Proof of Theorem 2. The relations of some of those statements have been investigated in [3]. The proofs of "(b-i)⇔(b-ii)" and "(b-ii)⇒(c)⇒(a)" have been presented in [3,8]. Clearly, we also have "(d)⇒(a)". We shall complete the proof of Theorem 2 by showing "(b-i)⇒(d)" and "(a)⇒(b-i)" below.

Proof of "(b-i)⇒(d)": Let G be a 4-edge-connected graph with at most five 4-edge-cuts. Denote $A_1, A_2, \ldots, A_t$ to be all distinct 4-critical-sets A such that $\partial(A)$ is a 4-critical-cut. Then $t \leq 5$ by Observation 1. The conclusion is clear if $t = 0$. We may assume $1 \leq t \leq 5$. Construct a new graph G' from G by adding a new vertex z, connecting z and A_1 with $6 - t$ new edges, and connecting z and A_i with a new edge for each $i = 2, \ldots, t$. Then $d_{G'}(z) = 5$ and G' is 5-edge-connected by Observation 1(iv). By the validity of Theorem 2(b-i), G' is $\mathbb{Z}_3$-extendable at z. Then it follows from Lemma 2 that $G = G' - z$ is $\mathbb{Z}_3$-connected. This proves "(b-i)⇒(d)".

Proof of "(a)⇒(b-i)": Suppose to the contrary that G is a 5-edge-connected graph which is not $\mathbb{Z}_3$-extendable at a given 5-vertex z. By Lemma 2, $G - z$ is not $\mathbb{Z}_3$-connected, and thus $G - z$ has no β_1-orientation for some boundary function β_1 of $G - z$. Denote $\partial(z) = \{zu_1, zu_2, \ldots, zu_5\}$. (Note that u_i, u_j may represent the same vertex for distinct i and j when $\partial(z)$ contains parallel edges.) We orient the edge zu_i from z to u_i for each $i = 1, \ldots, 5$ to obtain a pre-orientation D_z. Let β be a boundary function of G such that $\beta(z) = 2$ and $\beta(x) = \beta_1(x) - \alpha(x)$ in $\mathbb{Z}_3$ for any $x \in V(G) \setminus \{z\}$, where $\alpha(x)$ is the number of directed edges from z to x. (In particular, $\beta(x) = \beta_1(x)$ in $\mathbb{Z}_3$ for any $x \in V(G) - \cup_{i=1}^{5}\{u_i\} \cup \{z\}$.) Clearly, $\beta \in Z(G, \mathbb{Z}_3)$ and

$$D_z \text{ cannot be extended to a } \beta\text{-orientation of } G. \tag{2}$$

Now, we replace each vertex of the graph W (see Fig. 1) with a copy of $G - z$, where each u_i is connected with an edge of W (see Fig. 3). Let H be the resulting graph. Define a boundary function β^* of H such that β^* is consistent with β in each copy of $G - z$. Note that β^* is indeed a boundary function of H as $\sum_{v\in V(H)} \beta^*(v) = 6\sum_{v\in V(G-z_0)} \beta(v) \equiv 0 \pmod 3$. Since H is 5-edge-connected, we have a β^*-orientation D^* of H by the validity of Theorem 2(a). Under the orientation D^*, we consider the oriented graph W obtained from H by contracting all the copies of $G - z$. By Lemma 1(ii), there exists a vertex with indegree 5. We uncontract this vertex and denote its corresponding vertex set of H by U. Then H/U^c is isomorphic to G, where the contracted vertex y plays the same role as z. Furthermore, the orientation D^* restricted to H/U^c gives a β-orientation of H/U^c since all the edges incident with y are directed out of y. This contradicts to (2) that D_z cannot be extended to a β-orientation of G. The proof is completed. ■

Now we prove Theorem 3 using similar arguments as in the proof of Theorem 2.

Proof of Theorem 3. The proof of "(i)⇒(ii)" is the same as the proof of Theorem 2 "(a)⇒(b-i)" above. Notice that when G is planar, the new constructed graph H from W and copies of $G - x$ is also planar, and hence "(i)⇒(ii)" holds. The proof of "(ii)⇒(iii)" is also straightforward by employing Lemma 2, similar as proving Theorem 2 "(b-ii)⇒(c)" in [3]. If there exists a $\mathbb{Z}_3$-reduced graph with minimal degree at least 5, we choose a vertex set S such that $\partial(S)$ is a 4-critical-set. Then $|S| \geq 2$, and contract S^c to obtain a graph $G_1 = G/S^c$, where x is the vertex set S^c contracted into. Add $5 - |\partial_G(S)|$ edge between x and S in G_1 to result a new planar graph G_2. Hence G_2 is 5-edge-connected. By (ii), G_2 is $\mathbb{Z}_3$-extendable at x, which shows that $G[S] = G_2 - x$ is $\mathbb{Z}_3$-connected by Lemma 2, a contradiction to the fact that G is $\mathbb{Z}_3$-reduced.

Now we prove "(ii)⇒(iv)" with similar arguments. Let G be a 5-edge-connected graph embedded on the plane such that the only crossing is between x_1x_2 and y_1y_2. We delete edges x_1x_2, y_1y_2 and add a new vertex z with edges $zx_1, zx_2, zy_1, zy_2, zy_2$. Let G' be the resulting graph. Then G' is a 5-edge-connected planar graph with a 5-vertex z. By (ii), G' is $\mathbb{Z}_3$-extendable at z, and hence $G' - z = G - x_1x_2 - y_1y_2$ is $\mathbb{Z}_3$-connected by Lemma 2. Thus G is $\mathbb{Z}_3$-connected. This completes the proof of Theorem 3. ■

One may wonder whether the proof of Theorem 3 extends to the "doublecross graphs", graphs can be drawn in the plane with two crossings incident with the infinite region. We are unable to reduce it to planar case as in Theorem 3. Similar phenomenon happens for Four Color Theorem(4CT) of planar graphs. Jaeger [4] proved that every bridgeless cubic graph with at most one crossing has a nowhere-zero 4-flow (equivalently, is 3-edge-colorable), which is reduced to the planar case, an equivalent version of 4CT, that every bridgeless cubic planar graph has a nowhere-zero 4-flow. However, for doublecross cubic graphs, Edwards, Sanders, Seymour and Thomas [2] employed the whole arguments of 4CT proofs (and many more works) to accomplish their proof that every bridgeless doublecross cubic graph has a nowhere-zero 4-flow.

3. Graphs with Few Small Critical-cuts

We prove Theorem 4(d-i)(d-ii) in this section. Evidently, Theorem 4(d-ii) is easily derived by Theorem 4(b-i) and Observation 1. However, Theorem 4(d-i) seems not to be deduced from the current version of Theorem 4(b-i). We shall apply the full version of the flow extension theorem of Lovász et al. [9].

Let G be a graph and β a boundary function. For a vertex set $A \subset V(G)$, denote its boundary $\beta(A) \equiv \sum_{x\in A} \beta(x) \pmod 3$. Define an integer valued mapping $\tau : 2^{V(G)} \mapsto \{0, \pm 1, \pm 2, \pm 3\}$ such that, for each vertex set $A \subset V(G)$, $\tau(A) \equiv d(A) \pmod 2$ and $\tau(A) \equiv \beta(A) \pmod 3$.

Theorem 6 (*Lovász et al. [9]*). *Let G be a graph, $\beta \in Z(G, \mathbb{Z}_3)$ and $z \in V(G)$. Let D_z be a pre-orientation of $\partial_G(z)$. Assume that*
(i) $|V(G)| \geq 3$,
(ii) $d(z) \leq 4 + |\tau(z)|$ and $d^+_{D_z}(z) - d^-_{D_z}(z) \equiv \beta(z) \pmod 3$, and
(iii) $d(A) \geq 4 + |\tau(A)|$ for each nonempty $A \subseteq V(G) - \{z_0\}$ with $|V(G) - A| \geq 2$.

Then D_z can be extended to a β-orientation of the entire graph G.

Now we are ready to prove Theorem 4(d-i)(d-ii) using Theorem 6.

Proof of Theorem 4. *Proof of (d-i):* Let G be a 4-edge-connected graph with at most five 4-cuts and without 5-cuts. Let $\beta \in Z(G, \mathbb{Z}_3)$ be a boundary function of G. We are going to show that G has a β-orientation. Similar to the previous section, we denote $A_1, A_2, \ldots, A_t$ to be all distinct 4-critical-sets of G. Note that $t \leq 5$ by Observation 1. Construct a new graph G' from G by adding a new vertex z, and for each $i = 1, \ldots, t$, adding a new edge between z and A_i, say zv_i (where $v_i \in A_i$). We pre-orient the edges in $\partial_{G'}(z)$ and modify the boundary appropriately to become a new boundary β' of G' such that $d_{G'}(A_i) = 4 + |\tau'(A_i)|$ for each $i = 1, \ldots, t$, where τ' denotes the τ-function corresponding to boundary β' in G'. Specifically, we orient the edge zv_i from z to v_i if $\tau(A_i) = 0$ or 2, and orient zv_i from v_i to z otherwise(i.e. $\tau(A_i) = -2$). Define the boundary β' of G' as follows. For any $x \in V(G') \setminus \{v_1, \ldots, v_t\}$, define $\beta'(x) = \beta(x)$; for each $i = 1, \ldots, t$, define $\beta'(v_i) = \beta(v_i) + 1$ if zv_i is oriented from v_i to z, and $\beta'(v_i) = \beta(v_i) - 1$ otherwise. Now, it is easy to see that $d_{G'}(A_i) = 4 + |\tau'(A_i)|$ for each $i = 1, \ldots, t$, and that Theorem 6 is applied for G' by checking conditions (i)(ii)(iii). That is, we have $d_{G'}(z) \leq 4 + |\tau'(z)|$ since $d_{G'}(z) \leq 5$ and by parity, and this verifies condition (ii) of Theorem 6. Let A be a nonempty subset of $V(G') - \{z\}$ with $|V(G') - A| \geq 2$. If $d_{G'}(A) \geq 6$, then we have $d_{G'}(A) \geq 4 + |\tau'(A)|$ by parity. Otherwise, we have $A = A_i$ for some i, and so $d_{G'}(A) = 4 + |\tau'(A)|$. Hence condition (iii) of Theorem 6 holds. By Theorem 6, the pre-orientation can be extended to a β'-orientation D' of G'. Notice that D' restricted to G provides a β-orientation of G. This proves (d-i).

Proof of (d-ii): The proof of (d-ii) is analogous to the proof of Theorem 2 "(b-i)⇒(d)". We add a new vertex z to connect each 5-critical-set to obtain a new graph G' such that $d_{G'}(z) = 7$. Then $G = G' - z$ is $\mathbb{Z}_3$-connected by Theorem 4(b-i) and Lemma 2. This completes the proof. ■

Note that, by Observation 1 the proof above is still valid for graphs with many 5-cuts but only at most seven 5-critical-cuts, with essentially the same proof.

Corollary 7. *Every 5-edge-connected graph with at most seven 5-critical-cuts is $\mathbb{Z}_3$-connected.*

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