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On 3-flow-critical graphs

Jiaao Li [a], Yulai Ma [b], Yongtang Shi [b], Weifan Wang [c], Yezhou Wu [d]

[a] *School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, China*
[b] *Center for Combinatorics and LPMC, Nankai University, Tianjin 300071, China*
[c] *Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang 321004, China*
[d] *Ocean College, Zhejiang University, Zhoushan, Zhejiang 316021, China*

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ABSTRACT

A bridgeless graph G is called *3-flow-critical* if it does not admit a nowhere-zero 3-flow, but G/e has one for any $e \in E(G)$. Tutte's 3-flow conjecture can be equivalently stated as that every 3-flow-critical graph contains a vertex of degree three. In this paper, we study the structure and extreme size of 3-flow-critical graphs. We apply structural properties to obtain lower and upper bounds on the size of 3-flow-critical graphs, that is, for any 3-flow-critical graph G on n vertices,

$$\frac{8n-2}{5} \leq |E(G)| \leq 4n - 10,$$

where each equality holds if and only if G is K_4. We conjecture that every 3-flow-critical graph on $n \geq 7$ vertices has at most $3n-8$ edges, which would be tight if true. For planar graphs, the best possible upper bound for the size of 3-flow-critical graphs on n vertices is $\frac{5n-8}{2}$, known from a result of Kostochka and Yancey (2014) on vertex coloring 4-critical graphs by duality.

1. Introduction

Graphs in this paper are finite and may contain parallel edges but no loops. We follow [1,14] for undefined notation and terminology. A vertex of degree k in a graph G is called a k-vertex. Denote by $V_k(G)$ ($V_{\leq k}(G)$ and $V_{\geq k}(G)$, respectively) the set of all vertices of degree k (at most k and at least

E-mail addresses: lijiaao@nankai.edu.cn (J. Li), ylma92@163.com (Y. Ma), shi@nankai.edu.cn (Y. Shi), wwf@zjnu.cn (W. Wang), yezhouwu@zju.edu.cn (Y. Wu).

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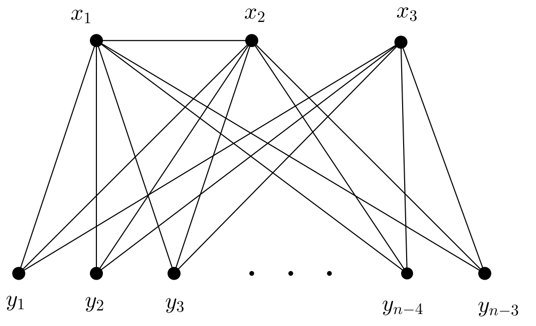

Fig. 1. The graph $K^+_{3,n-3}$.

k, respectively) in G. Let $n_k(G) = |V_k(G)|$, $n_{\le k}(G) = |V_{\le k}(G)|$, and $n_{\ge k}(G) = |V_{\ge k}(G)|$. If the graph G is understood from context, we may use n_k, $n_{\le k}$, and $n_{\ge k}$ for short, respectively.

Let $D = D(G)$ be an orientation of a graph G. For a vertex pair (u, v), denote $u \to v$ if there is an arc leaving u and entering v. For each $v \in V(G)$, we use $E^+_D(v)$ and $E^-_D(v)$ to denote the set of all arcs directed out of v and directed into v, respectively. An ordered pair (D, f) is called an *integer flow* of G if D is an orientation and f is a mapping from $E(G)$ to the integers such that every vertex $v \in V(G)$ is balanced, that is $\sum_{e\in E^+_D(v)} f(e) - \sum_{e\in E^-_D(v)} f(e) = 0$. An integer flow (D, f) is called a *nowhere-zero k-flow* if $1 \le |f(e)| \le k - 1$, $\forall e \in E(G)$.

As observed by Tutte [12], flow and coloring are dual concepts: a plane graph G admits a nowhere-zero k-flow if and only if the dual graph G^* is k-colorable. A graph G is called vertex coloring 4-*critical* if G is not 3-colorable but deleting any edge in G results in a 3-colorable graph. Motivated by this, we define a bridgeless graph G to be 3-*flow-critical* if G admits no nowhere-zero 3-flow but G/e has a nowhere-zero 3-flow for each edge $e \in E(G)$. Note that K_2 contains a bridge and thus is not considered as a 3-flow-critical graph.

The study of vertex coloring 4-critical graphs can be traced back to Dirac, Gallai and Ore in 1950s and 1960s (see [6]). It follows from Turán's Theorem that every 4-critical graph on $n \ge 5$ vertices has at most $\frac{1}{3}n^2$ edges, since any such graphs contain no K_4 as a subgraph. In [11], Toft constructed 4-critical graphs with more than $\frac{1}{16}n^2$ edges, while the optimal value remains unknown as of today. For the lower bound, resolving conjectures of Gallai and Ore on the density of 4-critical graphs, Kostochka and Yancey [6,7] proved a tight bound that every 4-critical graph on n vertices has at least $\frac{5n-2}{3}$ edges. By duality, their theorem shows the following result on 3-flow-critical planar graphs.

Theorem 1.1 (*Kostochka and Yancey [6,7]*)**.** *For any 3-flow-critical planar graph G on n vertices,*

$$|E(G)| \le \frac{5}{2}n - 4.$$

Moreover, the equality holds if and only if G is the dual of a planar 4-Ore graph.

A natural question is to ask what is the corresponding lower and upper bounds for nonplanar graphs. It is easy to see that the upper bound $\frac{5}{2}n - 4$ for planar graphs does not hold for general graphs. One may verify that (see Proposition 2.6) the graph $K^+_{3,n-3}$ (where $n \ge 6$) in Fig. 1 is 3-flow-critical with $3n - 8$ edges, where $K^+_{3,n-3}$ denotes the graph obtained from complete bipartite graph $K_{3,n-3}$ by adding a new edge between two vertices of degree $n - 3$.

In this paper, we provide linear lower and upper bounds on the size of any 3-flow-critical graph on n vertices.

Theorem 1.2. *Let G be a 3-flow-critical graph on n vertices. Then*

$$\frac{8n-2}{5} \le |E(G)| \le 4n - 10,$$

and each equality holds if and only if $G \cong K_4$. Moreover, we have $\frac{8n+2}{5} \le |E(G)| \le 4n - 11$ if $G \ncong K_4$.

We suspect that the bounds in Theorem 1.2 are not optimal in general. The dual of a construction of Yao and Zhou [13] on 4-critical planar graphs shows that there exist 3-flow-critical planar graphs on n vertices with $\frac{7n-1}{4}$ edges (see Theorem 4.1). However, determining the best possible lower bound on the size of 3-flow-critical planar graphs, or equivalently the highest density of 4-critical planar graphs, is a long-standing open problem (see [13]). It seems much more difficult for the best lower bound on the size of general nonplanar 3-flow-critical graphs, and we are even unclear about the candidate value. On the other hand, there are many rich families of 3-flow-critical graphs that we can construct by developing a 2-sum operation in Section 4. Specifically, from some known results, we are able to construct 3-flow-critical graphs on n vertices with size roughly rn for $\frac{7}{4} < r < 3$. Any 3-flow-critical graphs that we can construct seem to be sparser than the graph $K^+_{3,n-3}$. Thus we suggest the following conjecture concerning the tight upper bound.

Conjecture 1.3. *For any 3-flow-critical graph G on $n \geq 7$ vertices,*

$$|E(G)| \leq 3n - 8.$$

Perhaps $K^+_{3,n-3}$ is the only extreme graph to attain this bound when n is large. At least, it is true if $n_3(G) \geq n - 3$, as shown in Proposition 2.7 in Section 2.

Tutte's 3-flow conjecture (see Unsolved Problems #97 in [1]) asserts that every 4-edge-connected graph admits a nowhere-zero 3-flow. The density argument, even if Conjecture 1.3 was proved, cannot derive the 3-flow conjecture. We propose a stronger conjecture below, which, if true, implies the 3-flow conjecture.

Conjecture 1.4. *For any 3-flow-critical graph G on n vertices,*

$$|E(G)| < \frac{5}{2}n + n_3.$$

Note that $K^+_{3,n-3}$ satisfies Conjecture 1.4 since it has many 3-vertices. There is another family of 3-flow-critical graphs on $2k + 2$ vertices, constructed from 2-sum of K_4's (this 2-sum operation is defined in Definition 4.2), which contains four 3-vertices and $2k - 2$ 5-vertices, approaching the bound in Conjecture 1.4. To support Conjecture 1.4, we provide the following result.

Theorem 1.5. *For any 3-flow-critical graph G on n vertices,*

$$|E(G)| < \frac{5}{2}n + 9n_{\leq 8}.$$

The rest of the paper is organized as follows. In Section 2, we introduce a few basic notation and terminology, and then investigate structures of 3-flow-critical graphs to prove the lower bound in Theorem 1.2. In Section 3, we complete the proof of the upper bound in Theorem 1.2 as well as the proof of Theorem 1.5. Finally, we develop some operations to construct 3-flow-critical graphs with density between $\frac{7}{4}$ and 3 in Section 4.

2. Properties of 3-flow-critical graphs

For vertex subsets $U, W \subseteq V(G)$, let $[U, W]_G = \{uw \in E(G) | u \in U, w \in W\}$. When $U = \{u\}$ or $W = \{w\}$, we use $[u, W]_G$ or $[U, w]_G$ for $[U, W]_G$, respectively. The subgraph of G induced by U is denoted by $G[U]$. For any subset $S \subseteq V(G)$, we denote $S^c = V(G) \setminus S$ and set $d_G(S) = |[S, S^c]_G|$. An edge cut $[S, S^c]_G$ is called *essential* if there are at least two nontrivial components in $G - [S, S^c]_G$. A graph is called essentially k-edge-connected if it contains no essential edge cut with less than k edges. When there is no scope for ambiguity, the subscript G may be omitted. Contracting an edge of a graph means to identify its two endpoints and then delete the resulting loops. For an edge $e \in E(G)$ and a subgraph H of G, we write G/e to denote the graph obtained from G by contracting e, and denote by G/H the graph obtained from G by successively contracting the edges of $E(H)$.

Let $d^+_D(v) = |E^+_D(v)|$ and $d^-_D(v) = |E^-_D(v)|$ denote the out-degree and the in-degree of v under the orientation D, respectively. Let $\mathbb{Z}_n$ be the set of integers modulo n. A function $\beta\colon V(G) \to \mathbb{Z}_3$ is a

$\mathbb{Z}_3$-boundary if $\sum_{v \in V(G)} \beta(v) \equiv 0 \pmod 3$. For a given $\mathbb{Z}_3$-boundary β, a *β-orientation* is an orientation D of G such that $d_D^+(v) - d_D^-(v) \equiv \beta(v) \pmod 3$ for each $v \in V(G)$. Especially, a *modulo 3-orientation* of G is a β-orientation with $\beta(v) \equiv 0 \pmod 3$ for each $v \in V(G)$. We call a graph G *$\mathbb{Z}_3$-connected* if for any $\mathbb{Z}_3$-boundary β of G, there exists a β-orientation of G. A graph is called *$\mathbb{Z}_3$-irreducible* if it does not contain any nontrivial $\mathbb{Z}_3$-connected subgraphs. It is well-known that a graph admits a nowhere-zero 3-flow if and only if it admits a modulo 3-orientation (see [14]). Therefore, in the rest of this paper we will study nowhere-zero 3-flows in terms of modulo 3-orientations.

A useful method to prove $\mathbb{Z}_3$-connectedness is the following lemma.

Lemma 2.1 (*Lai [8]*). *Let G be a graph, and let $H \subseteq G$ be a subgraph of G.*
(i) If H is $\mathbb{Z}_3$-connected and G/H has a modulo 3-orientation, then G has a modulo 3-orientation.
(ii) If both H and G/H are $\mathbb{Z}_3$-connected, then G is also $\mathbb{Z}_3$-connected.
(iii) The graph $2K_2$ is $\mathbb{Z}_3$-connected, where $2K_2$ consists of two vertices and two parallel edges.

A wheel graph W_k is constructed by adding a new center vertex connecting to each vertex of a k-cycle, where $k \geq 3$. A wheel W_k is odd if k is odd, and even otherwise.

Lemma 2.2 (*DeVos, Xu, Yu [2]*). *A wheel W_k is $\mathbb{Z}_3$-connected if and only if k is even. Furthermore, each odd wheel does not admit a nowhere-zero 3-flow.*

As an example, it is an easy exercise to verify that each odd wheel is 3-flow-critical by Lemmas 2.1 and 2.2. The following observation about modulo 3-orientations will be useful in later proofs.

Observation 2.3. *Let G be a graph with a modulo 3-orientation D. Assume $V_3(G) \neq \emptyset$, and let $P = x_1x_2 \dots x_t$ be a path of $G[V_3]$. Then each of the following holds.*
(i) The number t is odd if and only if $d_D^+(x_1) = d_D^+(x_t) \in \{0, 3\}$.
(ii) The number t is even if and only if $d_D^+(x_1) = d_D^-(x_t) \in \{0, 3\}$.

Our first result of this section is the following fundamental structural properties of 3-flow-critical graphs.

Theorem 2.4. *Let G be a 3-flow-critical graph. Then each of the following holds.*
(i) For any $e \in E(G)$, $G - e$ admits a nowhere-zero 3-flow.
(ii) G is 3-edge-connected and essentially 4-edge-connected.
(iii) G is $\mathbb{Z}_3$-irreducible.
(iv) $G[V_3]$ contains no cycle, unless G is an odd wheel.

Proof. (i) Let $e = uv \in E(G)$, and let D be a modulo 3-orientation of G/e. Let D^* be the restriction of D on $G - e$. By arbitrarily orienting each edge in $E(G-e) \setminus E(G/e)$ (if any), we obtain an orientation D' of $G - e$. If D' is not a modulo 3-orientation of $G - e$, then either $d_{D'}^+(u) - d_{D'}^-(u) \equiv d_{D'}^-(v) - d_{D'}^+(v) \equiv 1 \pmod 3$ or $d_{D'}^+(u) - d_{D'}^-(u) \equiv d_{D'}^-(v) - d_{D'}^+(v) \equiv -1 \pmod 3$. So D' can be extended to a modulo 3-orientation of G by letting $v \to u$ or $u \to v$, a contradiction. Hence D' is a modulo 3-orientation of $G - e$.

(ii) By (i), we have $\delta(G) \geq 3$. Suppose to the contrary that G contains an edge cut $[S, S^c]_G$ such that $2 \leq d(S) \leq 3$, $|E(G[S])| \geq 1$ and $|E(G[S^c])| \geq 1$. Assume $e_1 \in E(G[S])$ and $e_2 \in E(G[S^c])$. By definition, G/e_1 admits a modulo 3-orientation D'. Then the restriction of D' to $G/G[S]$, say D_1, is a modulo 3-orientation. Similarly, $G/G[S^c]$ has a modulo 3-orientation D_2. Then either D_1 and D_2 agree along $[S, S^c]_G$ directly, or they agree after reversing all edge directions in D_2. Thus, their union provides a modulo 3-orientation of G, a contradiction. Hence G is 3-edge-connected and essentially 4-edge-connected.

(iii) Suppose that H is a nontrivial $\mathbb{Z}_3$-connected subgraph of G. Let $u_1v_1 \in E(H)$. By (i), $G - u_1v_1$ admits a modulo 3-orientation D_1. Thus the restriction D' of D_1 to G/H is also a modulo 3-orientation. By Lemma 2.1, G has a modulo 3-orientation, a contradiction. So G is $\mathbb{Z}_3$-irreducible.

(iv) Suppose, by contradiction, that G is not an odd wheel and $G[V_3]$ contains a cycle. Assume $C = v_1v_2\dots v_tv_1$ is a cycle with the minimum length in $G[V_3]$. Note that C is an induced subgraph of G. Let u_i be the neighbor of v_i which is not on C and let $e_i = u_iv_i$.

First, suppose t is even. By (i), $G-e_1$ admits a modulo 3-orientation D'. It implies that $d^+_{D'}(v_i) = 3$ or $d^-_{D'}(v_i) = 3$ for each $i \in \{2, 3, \dots, t\}$. Since t is even, by Observation 2.3(i), we have $d^+_{D'}(v_2) = d^+_{D'}(v_t) = 3$ or $d^-_{D'}(v_2) = d^-_{D'}(v_t) = 3$, which implies that $d^-_{D'}(v_1) = 2$ or $d^+_{D'}(v_1) = 2$. So v_1 is not balanced in D'. This leads to a contradiction.

Next, suppose t is odd. If there exists an edge e that is not incident to any vertex on C, then by (i), $G - e$ admits a modulo 3-orientation D'. It implies that $d^+_{D'}(v_i) = 3$ or $d^-_{D'}(v_i) = 3$ for each $i \in \{1, 2, \dots, t\}$. Since t is odd, by Observation 2.3(ii), we have either $d^+_{D'}(v_2) = d^-_{D'}(v_t) = 3$ or $d^-_{D'}(v_2) = d^+_{D'}(v_t) = 3$, which implies that v_1 is not balanced in D', a contradiction. Hence we suppose $E(G) = E(C) \cup \{e_1, e_2, \dots, e_t\}$. Since G is not an odd wheel, there exists an index $j \in \{1, 2, \dots, t-1\}$ such that $u_j \neq u_{j+1}$. By (i), $G-e_j$ admits a modulo 3-orientation D_j and $G-e_{j+1}$ admits a modulo 3-orientation D_{j+1}, respectively. Without loss of generality, assume $v_{j-1} \to v_j$ in D_j. Then we have $v_j \to v_{j+1}$ and $u_{j+1} \to v_{j+1}$ in D_j. Similarly, WLOG, assume $v_{j-1} \to v_j$ in D_{j+1}. Then we get $v_{j+1} \to v_j$ and $u_j \to v_j$ in D_{j+1}. Besides, we have $d^+_{D_j}(v_{j-1}) = d^+_{D_{j+1}}(v_{j-1}) = 3$ and so, by Observation 2.3(i)(ii), $d^+_{D_j}(v) = d^+_{D_{j+1}}(v)$ and $d^-_{D_j}(v) = d^-_{D_{j+1}}(v)$ for each $v \in V(C) \setminus \{v_j, v_{j+1}\}$. This implies that the direction of e in D_{j+1} is the same as that in D_j for each $e \in E(G) \setminus \{e_j, e_{j+1}, v_jv_{j+1}\}$. Thus we have $d^+_{D_j}(u_j) = d^+_{D_{j+1}}(u_j) - 1$ and $d^-_{D_j}(u_j) = d^-_{D_{j+1}}(u_j)$, which implies that u_j is not balanced in D_{j+1} since it is balanced in D_j, a contradiction again. ■

Kochol [4,5] obtained two equivalent statements of Tutte's 3-flow conjecture as follows: (i) every 5-edge-connected graph admits a nowhere-zero 3-flow, (ii) every bridgeless graph with at most three edge cuts of size three admits a nowhere-zero 3-flow. By Theorem 2.4, the results of Kochol [4,5] can be restated as certain properties of 3-flow-critical graphs.

Theorem 2.5 (*Kochol [4,5]*). *Tutte's 3-flow conjecture is equivalent to each of the following statements.*
(a) Every 3-flow-critical graph contains a vertex of degree 3.
(b) Every 3-flow-critical graph contains a vertex of degree at most 4.
(c) $|V_3(G)| \geq 4$ *for every 3-flow-critical graph G.*

It is proved in [3] that every $\mathbb{Z}_3$-irreducible graph has a vertex of degree at most 5, and so, combining Theorem 2.4(iii), it implies that every 3-flow-critical graph contains a vertex of degree at most 5.

Theorem 2.5 may suggest that some better structure properties of 3-flow-critical graphs could bring new ideas in solving Tutte's 3-flow conjecture. In particular, Theorem 2.5(b) shows that Conjecture 1.4 implies Tutte's 3-flow conjecture.

Next, we show in detail that $K^+_{3,n-3}$ is a 3-flow-critical graph and that Conjecture 1.3 holds for any 3-flow-critical graph G on n vertices with $n_3 \geq n-3 \geq 6$.

Proposition 2.6. *For any* $n \geq 6$, *the graph* $K^+_{3,n-3}$ *is a 3-flow-critical graph with* $3n-8$ *edges.*

Proof. It is easy to check that $K^+_{3,n-3}$ has $3n-8$ edges. So it remains to show that $K^+_{3,n-3}$ is 3-flow-critical. We use the notation in Fig. 1 to label the vertices of $K^+_{3,n-3}$, and let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, \dots, y_{n-3}\}$. To the contrary, suppose $K^+_{3,n-3}$ admits a modulo 3-orientation D. Since all vertices in Y are 3-vertices, we have $d^+_D(y_i) = 3$ or $d^-_D(y_i) = 3$ for each $y_i \in Y$. It is easy to check that $d^+_D(x_1) - d^-_D(x_1) \not\equiv 0 \pmod 3$ if $d^+_D(x_3) - d^-_D(x_3) \equiv 0 \pmod 3$, since x_1 has an extra neighbor x_2. Hence $K^+_{3,n-3}$ does not admit a modulo 3-orientation. For any $e \in E(K^+_{3,n-3})$, in order to show that $G' = K^+_{3,n-3}/e$ has a modulo 3-orientation, it is sufficient to prove that $G'' = K^+_{3,n-3} - e$ has a modulo 3-orientation.

We firstly give a special orientation of the complete bipartite graph $K_{3,t-3}$ with $t \geq 5$. Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, \dots, y_{t-3}\}$ be the two parts of $K_{3,t-3}$. Assign to each edge incident to x_1 a direction such that $d^+(x_1) - d^-(x_1) \equiv k \pmod 3$. Assign directions to the remain edges such that $d^+(v) - d^-(v) \equiv 0 \pmod 3$ for each $v \in Y$. Then we obtain an orientation $D(k)$ of $K_{3,t-3}$ such

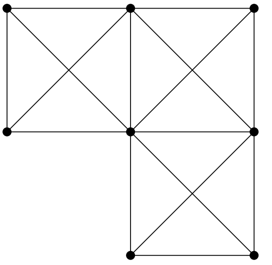

Fig. 2. A 3-flow-critical graph H on 8 vertices with 16 edges.

that $d^+_{D(k)}(u) - d^-_{D(k)}(u) \equiv k \pmod 3$ for each $u \in X$, and $d^+_{D(k)}(v) - d^-_{D(k)}(v) \equiv 0 \pmod 3$ for each $v \in Y$.

Now, by symmetry, it suffices to consider three cases $e = x_1x_2$, $e = x_1y_1$, and $e = x_3y_1$. If $e = x_1x_2$, then $G'' \cong K_{3,n-3}$. So G'' has a modulo 3-orientation $D(k)$ with $k = 0$. If $e = x_1y_1$, then $G_1 = G'' - y_1 - \{x_1x_2\}$ is isomorphic to $K_{3,n-4}$. So G_1 has an orientation $D(k)$ with $k = 1$. With the restriction of $D(1)$ on G'', we obtain a modulo 3-orientation of G'' by assigning $x_2 \to x_1, x_2 \to y_1$ and $y_1 \to x_3$. If $e = x_3y_1$, then $G_1 = G'' - y_1 - \{x_1x_2\}$ is isomorphic to $K_{3,n-4}$. So G_1 has an orientation $D(k)$ with $k = 0$. With the restriction of $D(0)$ on G'', we obtain a modulo 3-orientation of G'' by assigning $x_1 \to x_2, x_2 \to y_1$ and $y_1 \to x_1$.

Thus, for all cases above, we can obtain a modulo 3-orientation of G''. Hence we conclude that $K^+_{3,n-3}$ is 3-flow-critical. ■

Proposition 2.7. *Let G be a 3-flow-critical graph on $n \geq 9$ vertices. If $n_3 \geq n-3$, then*

$$|E(G)| \leq 3n - 8.$$

Moreover, the equality holds if and only if $G \cong K^+_{3,n-3}$.

Proof. By Lemma 2.1 and Theorem 2.4(iii), G contains no parallel edges. Let t denote the number of components of $G[V_3]$. We consider three cases in the following. Firstly, suppose $n_3 \geq n-1$. By Theorem 2.4(iv), the graph G is an odd wheel and $|E(G)| \leq 2n-2$, which is less than $3n-8$ when $n \geq 9$. Then suppose $n_3 = n-2$. By Theorem 2.4(iv), we know $G[V_3]$ is a forest, and hence $|E(G)| = |E(G[V_3])| + |[V_3, V_{\geq 4}]| + |E(G[V_{\geq 4}])| \leq (n-2-t) + (3(n-2) - 2(n-2-t)) + 1 = 2n+t-3$. Since G has no parallel edges and $G[V_3]$ has no isolated vertex, we obtain $t \leq \lfloor \frac{n-2}{2} \rfloor$, which implies $|E(G)| < 3n-8$ by $n \geq 9$.

Finally, suppose $n_3 = n-3$. Let $i = |E(G[V_{\geq 4}])|$ and $V_{\geq 4} = \{u_1, u_2, u_3\}$. Then $t \leq n-3$ and $0 \leq i \leq 3$. So we have $|E(G)| \leq (n-3-t) + (3(n-3) - 2(n-3-t)) + i = 2n+t+i-6$. If $t+i \leq n-3$, then $|E(G)| \leq 3n-9$. Now we consider the case $t+i \geq n-2$, whereas $i \geq 1$. If $i = 1$, then $t = n-3$ and $G = K^+_{3,n-3}$. If $2 \leq i \leq 3$, then $t \geq n-5$ and we assume $\{u_1u_2, u_2u_3\} \subseteq E(G[V_{\geq 4}])$ by symmetry. Let k be the number of isolated vertices of $G[V_3]$. We have $k + 2(t-k) \leq n_3 = n-3$ and then $n \leq 7+k$ since $t \geq n-5$. Hence we obtain $k \geq 2$ since $n \geq 9$. Now assume that v_1 and v_2 are two isolated vertices of $G[V_3]$. We use H to denote the graph induced by $\{v_1, v_2, u_1, u_2, u_3\}$. Let $H' = H$ if $u_1u_3 \notin E(G)$ and $H' = H - u_1u_3$ if $u_1u_3 \in E(G)$. So H' is a wheel W_4 and is $\mathbb{Z}_3$-connected by Lemma 2.2, which contradicts Theorem 2.4(iii). Hence $K^+_{3,n-3}$ is the only extreme graph to attain the bound. ■

Note that the condition $|V(G)| \geq 9$ in Proposition 2.7 is necessary, as there is another 3-flow-critical graph H on 8 vertices with $|E(H)| = 3|V(H)| - 8 = 16$, which is shown in Fig. 2.

Next we apply Theorem 2.4 and a counting argument to obtain the lower bound in Theorem 1.2. Since for an odd wheel W_{n-1} we have $|E(W_{n-1})| = 2n-2 \geq \frac{8n+2}{5}$ if $n \geq 6$, it suffices to prove the following proposition.

Proposition 2.8. *For any 3-flow-critical graph G on n vertices other than an odd wheel,*

$$|E(G)| \geq \frac{8n+2}{5}.$$

Proof. We double-count the number of edges in $[V_3, V_3^c]$.

On one hand, by Theorem 2.4(iv), $G[V_3]$ is acyclic, hence $|E(G[V_3])| \leq n_3 - 1$. Thus,

$$d(V_3) = 3n_3 - 2|E(G[V_3])| \geq 3n_3 - 2(n_3 - 1) = n_3 + 2, \tag{1}$$

with equality only if $G[V_3]$ is a tree.

On the other hand, counting the edges with respect to their endpoints in V_3^c, we have that

$$d(V_3) = \sum_{k\geq 4} kn_k - 2|E(G[V_{\geq 4}])| \leq \sum_{k\geq 4} kn_k = \sum_{k\geq 3} kn_k - 3n_3 = 2|E(G)| - 3n_3, \tag{2}$$

with equality only if $V_{\geq 4}$ is an independent set.

From (1) and (2) we conclude that

$$|E(G)| \geq 2n_3 + 1, \tag{3}$$

with equality only if $G[V_3]$ is a tree and $V_{\geq 4}$ is an independent set. Moreover, we have

$$\sum_{k\geq 4} kn_k \geq 4 \sum_{k\geq 4} n_k, \tag{4}$$

with equality only if $n_{\geq 5} = 0$.

Thus, we have

$$5|E(G)| = 4|E(G)| + |E(G)| \geq 2\sum_{k\geq 3} kn_k + 2n_3 + 1 = 8n_3 + 2\sum_{k\geq 4} kn_k + 1 \geq 8n + 1, \tag{5}$$

with equality only if $G[V_3]$ is a tree and $V_{\geq 4} = V_4$ is an independent set.

To obtain the bound $\frac{8n+2}{5}$ in the theorem, we shall show that $|E(G)| \neq \frac{8n+1}{5}$ below. Suppose to the contrary that $|E(G)| = \frac{8n+1}{5}$. From (5) we have that $G[V_3]$ is a tree and $V_{\geq 4} = V_4$ is an independent set. Let x_1 be a leaf vertex of the tree $G[V_3]$, and let y be a neighbor of x_1 with degree 4. Suppose the neighbors of y are x_1, x_2, x_3, x_4, where $x_i \in V_3$ for each $i \in \{1, 2, 3, 4\}$. Since $G[V_3]$ is a tree, there is a unique path, say P_{ij}, connecting the vertices x_i and x_j in $G[V_3]$. Then by symmetry, we consider two cases as follows.

Case 1. $x_2 \in V(P_{13})$ but $x_4 \notin V(P_{13})$.

Let $G' = G - yx_4$. Since G is 3-flow-critical, by Theorem 2.4(i), we have that G' admits a modulo 3-orientation D'. This implies that $d^+_{D'}(y) = 3$ or $d^-_{D'}(y) = 3$. Thus $|V(P_{13})|$ is odd by Observation 2.3(i). Let $G'' = G - yx_2$. By Theorem 2.4(i), G'' has a modulo 3-orientation D'', and then we have $d^+_{D''}(y) = 3$ or $d^-_{D''}(y) = 3$. However, the edges yx_1 and yx_3 must have opposite directions in D'' since $|V(P_{13})|$ is odd and $d_{G''}(x_2) = 2$, i.e., $y \to x_1$ if $x_3 \to y$ and $y \to x_3$ if $x_1 \to y$. This is a contradiction.

Case 2. $x_i \notin V(P_{1j})$ for any $\{i, j\} \subseteq \{2, 3, 4\}$.

By Observation 2.3(i), similar as Case 1, we know that $|V(P_{1j})|$ is an odd number for each $j \in \{2, 3, 4\}$. Since x_1 is a leaf of the tree $G[V_3]$, there is a neighbor z of x_1 such that $z \neq y$ and $z \in V_4$. Let $G' = G - zx_1$. Since G is 3-flow-critical, G' admits a modulo 3-orientation D'. Since $|V(P_{1j})|$ is odd for each $j \in \{2, 3, 4\}$, we have that the edges yx_2, yx_3 and yx_4 are all leaving or all entering y in D'. It implies that $d^+_{D'}(y) \geq 3$ or $d^-_{D'}(y) \geq 3$. Then we obtain $d^+_{D'}(y) - d^-_{D'}(y) \not\equiv 0 \pmod 3$ since $d_{G'}(y) = 4$, a contradiction again. ■

3. Upper bounds and $\mathbb{Z}_3$-irreducible graphs

In this section, we develop a method to prove an upper bound on the number of edges of 3-flow-critical graphs, which is tight for K_4. We start with a definition on the weight of a partition of the vertex-set of a graph.

Definition 3.1. Let $\mathcal{P} = \{X_1, X_2, \ldots, X_t\}$ be a partition of $V(G)$. Define

$$\rho_G(\mathcal{P}) = \sum_{i=1}^{t} d_G(X_i) - 8t + 20$$

and

$$\rho(G) = \min\{\rho_G(\mathcal{P}) : \mathcal{P} \text{ is a partition of } V(G)\}.$$

For a graph G with few vertices, it is easy to determine $\rho(G)$. For example, $\rho(K_2) = 6$, $\rho(2K_1) = 4$, $\rho(K_3) = 2$, $\rho(P_3) = 0$, and $\rho(K_4) = 0$, where $2K_1$ is an empty graph on 2 vertices. Note that for these graphs, $\rho(G)$ is attained only by the trivial partition, which is a partition with exact one vertex in each part.

For a partition $\mathcal{P} = \{X_1, X_2, \ldots, X_t\}$ of $V(G)$, let $G/\mathcal{P}$ be the graph obtained by identifying all vertices in each X_i to form a new vertex x_i. We say a graph G is $\mathbb{Z}_3$-*reduced* to a graph H if H is obtained from G by contracting all its $\mathbb{Z}_3$-connected subgraphs consecutively. In other words, there exists a partition $\mathcal{P} = \{X_1, X_2, \ldots, X_t\}$ of $V(G)$ such that $G/\mathcal{P} = H$ and $G[X_i]$ is $\mathbb{Z}_3$-connected for each $i \leq t$ (possibly $G[X_i] = K_1$).

Proposition 3.2. *Let $\mathcal{P} = \{X_1, X_2, \ldots, X_t\}$ be a partition of $V(G)$ with $|X_1| \geq 2$. Let $H = G[X_1]$ and let $\mathcal{Q}$ be a partition of X_1. Then we have*

$$\rho_H(\mathcal{Q}) = \rho_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{X_1\})) - \rho_G(\mathcal{P}) + 12.$$

Proof. Denote $\mathcal{Q} = \{Y_1, Y_2, \ldots, Y_s\}$ in $H = G[X_1]$. Then we have

$$\begin{aligned}\rho_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{X_1\})) &= \sum_{j=1}^{s} d_G(Y_j) + \sum_{i=2}^{t} d_G(X_i) - 8(s + t - 1) + 20 \\ &= [\sum_{j=1}^{s} d_G(Y_j) - d_G(X_1) - 8s + 20] + [\sum_{i=1}^{t} d_G(X_i) - 8(t - 1)] \\ &= \rho_H(\mathcal{Q}) + \rho_G(\mathcal{P}) - 12.\end{aligned}$$

Hence $\rho_H(\mathcal{Q}) = \rho_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{X_1\})) - \rho_G(\mathcal{P}) + 12$. ■

Indeed, Proposition 3.2 has a very important consequence to be used below.

Corollary 3.3. *Let $\mathcal{P} = \{X_1, X_2, \ldots, X_t\}$ be a partition of $V(G)$ with $|X_1| \geq 2$ such that $\rho(G) = \rho_G(\mathcal{P})$. Denote $H = G[X_1]$. Then, $\rho(H) \geq 12$.*

Proof. Let $\mathcal{Q}$ be a partition of $H = G[X_1]$. Then, by Proposition 3.2 we have

$$\rho(G) \leq \rho_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{X_1\})) = \rho_H(\mathcal{Q}) + \rho_G(\mathcal{P}) - 12 = \rho_H(\mathcal{Q}) + \rho(G) - 12,$$

and so $\rho_H(\mathcal{Q}) \geq 12$. This is true for each partition $\mathcal{Q}$ of H, and thus $\rho(H) \geq 12$. ■

The main result of this section is the following theorem.

Theorem 3.4. *Let $\mathcal{G} = \{K_2, K_3, P_3, K_4\}$. Let G be a connected graph with $\rho(G) \geq 0$. Then either*
(i) G is $\mathbb{Z}_3$-connected, or
(ii) G can be $\mathbb{Z}_3$-reduced to a graph in $\mathcal{G}$.

Proof. Assume, by way of contradiction, the result is false and study a minimal counterexample G with respect to $|V(G)| + |E(G)|$. That is, G is not $\mathbb{Z}_3$-connected and G cannot be $\mathbb{Z}_3$-reduced to a graph in $\mathcal{G}$. We first present some preliminary reductions on G.

Claim 1. *G is $\mathbb{Z}_3$-irreducible and $|V(G)| \geq 7$. In particular, G contains no parallel edges.*

Proof. Suppose to the contrary that there exists a subgraph H of G such that H is $\mathbb{Z}_3$-connected, where $|V(H)| > 1$. Clearly, G/H is connected and $\rho(G/H) \geq \rho(G) \geq 0$. Since G is a minimal counterexample, we consider two cases as follows. If G/H is $\mathbb{Z}_3$-connected, then by Lemma 2.1, G is $\mathbb{Z}_3$-connected, a contradiction. If G/H can be $\mathbb{Z}_3$-reduced to a graph in $\mathcal{G}$, then by definition G is $\mathbb{Z}_3$-reduced to a graph in $\mathcal{G}$. Each case leads to a contradiction. Hence G is $\mathbb{Z}_3$-irreducible and contains no nontrivial $\mathbb{Z}_3$-connected subgraph. Since $2K_2$ is $\mathbb{Z}_3$-connected, G contains no parallel edges.

Clearly, we have $|V(G)| \geq 3$. It is routine to verify that $|V(G)| \geq 7$ by some case analysis, but we shall apply a basic fact in [9] to accomplish this work. By Lemma 2.10 in [9], when $n = 3, 4, 5, 6$, any $\mathbb{Z}_3$-irreducible graph on n vertices contain at most 3, 6, 8, 11 edges, respectively. As $\rho(G) \geq 0$, G contains at least 2, 6, 10, 14 edges when $|V(G)| = 3, 4, 5, 6$, respectively. Thus either $G \in \{K_3, P_3, K_4\}$ or G is not $\mathbb{Z}_3$-irreducible, a contradiction. This shows $|V(G)| \geq 7$. ■

Claim 2. *Let H be a proper subgraph of G with $|V(H)| > 1$. Assume that $\rho_H(\mathcal{Q}) \geq 7$ for any nontrivial partition $\mathcal{Q}$ of H. Let $\mathcal{Q}_0$ denote the trivial partition of H. Then each of the following holds.*
(i) The trivial partition $\mathcal{Q}_0$ of H satisfies $\rho_H(\mathcal{Q}_0) \leq 6$.
(ii) If $\rho_H(\mathcal{Q}_0) \geq 1$, then $H \in \{2K_1, K_2, K_3\}$.

Proof. Since G is a minimal counterexample to Theorem 3.4, the theorem is applied for its proper subgraph H. Assume that $|V(H)| \geq 3$ and the trivial partition $\mathcal{Q}_0$ of H satisfies $\rho_H(\mathcal{Q}_0) \geq 0$. If H is not connected, then there exists a nontrivial partition $\mathcal{Q}'$ such that $\rho_H(\mathcal{Q}') = 0 - 8 \cdot 2 + 20 = 4$, a contradiction. Hence H is connected. Then Theorem 3.4 implies that either H is $\mathbb{Z}_3$-connected, or H can be $\mathbb{Z}_3$-reduced to a graph in $\mathcal{G}$. As G is $\mathbb{Z}_3$-irreducible, H and any nontrivial subgraph of H are not $\mathbb{Z}_3$-connected. Hence, the $\mathbb{Z}_3$-reduction of H is itself. So Theorem 3.4 implies that $H \in \mathcal{G}$. Note that $H \in \{K_2, 2K_1\}$ if $|V(H)| = 2$.

(i) Suppose to the contrary that $\rho_H(\mathcal{Q}_0) \geq 7$ for the trivial partition $\mathcal{Q}_0$ of H. Then we have $\rho(H) \geq 7$. It implies $H \notin \mathcal{G} \cup \{2K_1\}$, a contradiction.

(ii) We have that $\rho_H(\mathcal{Q}_0) \geq 1$ implies $H \notin \{P_3, K_4\}$, and so $H \in \{2K_1, K_2, K_3\}$. ■

For a partition $\mathcal{P}$ of $V(G)$, we set

$$r(\mathcal{P}) = |\{X \in \mathcal{P} : |X| \geq 2\}|,$$

and let

$$r_0(\mathcal{P}) = 1 \text{ if } \max\{|X| : X \in \mathcal{P}\} \geq 4, \text{ and } r_0(\mathcal{P}) = 0 \text{ otherwise.}$$

Claim 3. *Let $\mathcal{P}$ be a nontrivial partition of $V(G)$. Then we have*
(i) $\rho_G(\mathcal{P}) \geq 6$, and
(ii) $\rho_G(\mathcal{P}) \geq 12$ if $r(\mathcal{P}) + r_0(\mathcal{P}) \geq 2$.

Proof. Let $\mathcal{P} = \{X_1, X_2, \ldots, X_t\}$. If $t = 1$, then it is easy to verify $\rho_G(\mathcal{P}) = 12$. So we assume $t \geq 2$ and $|X_1| > 1$. Let $H = G[X_1]$.

(i) Suppose to the contrary that $\rho_G(\mathcal{P}) \leq 5$. Then for any partition $\mathcal{Q}$ of H, we have $\rho_H(\mathcal{Q}) = \rho_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{X_1\})) - \rho_G(\mathcal{P}) + 12 \geq 7$ by Proposition 3.2, and since $\rho(G) \geq 0$ by assumption, contradicting to Claim 2(i).

(ii) We first show that $\rho_G(\mathcal{P}) \geq 12$ if $\mathcal{P}$ is a partition with $|X_1| > 1$ and $|X_2| > 1$. Suppose to the contrary that $\rho_G(\mathcal{P}) \leq 11$. Since $|X_2| > 1$, for every partition $\mathcal{Q}$ of H, the partition $\mathcal{Q} \cup (\mathcal{P} \setminus \{X_1\})$ is a nontrivial partition of G. So $\rho_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{X_1\})) \geq 6$ by (i). Then we have

$$\rho_H(\mathcal{Q}) = \rho_G(\mathcal{Q} \cup (\mathcal{P} \setminus \{X_1\})) - \rho_G(\mathcal{P}) + 12 \geq 6 - 11 + 12 = 7$$

for any partition $\mathcal{Q}$ of H by Proposition 3.2, contradicting to Claim 2(i).

Now, as $r(\mathcal{P}) + r_0(\mathcal{P}) \geq 2$, it suffices to prove that $\rho_G(\mathcal{P}) \geq 12$ when $|X_1| \geq 4$ and $|X_i| = 1$ for each $i \in \{2, 3, \ldots, t\}$. Suppose to the contrary that $\rho_G(\mathcal{P}) \leq 11$. By Proposition 3.2 and by (i), we have $\rho_H(\mathcal{Q}) \geq 0 - 11 + 12 = 1$ for any partition $\mathcal{Q}$ of H, and additionally, $\rho_H(\mathcal{Q}) \geq 6 - 11 + 12 = 7$ for any nontrivial partition $\mathcal{Q}$ of H. Thus $H \in \{2K_1, K_2, K_3\}$ by Claim 2(ii), a contradiction. ■

Claim 4. *For any nonempty vertex subset $S \subsetneq V(G)$,*
(i) we have $d(S) \geq 4$. That is, G is 4-edge-connected.
(ii) If neither S nor S^c is trivial, then $d(S) \geq 7$. That is, G is essentially 7-edge-connected.

Proof. It is obvious that $\mathcal{P} = \{S, S^c\}$ is a partition of $V(G)$.

(i) Since $|V(G)| \geq 7$, $r(\mathcal{P}) \geq 1$ and $r_0(\mathcal{P}) = 1$. By Claim 3(ii), we have that $12 \leq \rho_G(\mathcal{P}) = 2d(S) - 16 + 20$, which yields $d(S) \geq 4$. This implies that G is 4-edge-connected.

(ii) It is sufficient to prove that if neither S nor S^c is trivial, then $\rho_G(\mathcal{P}) \geq 18$. It is clear that if $\rho_G(\mathcal{P}) \geq 18$, then we have $d(S) \geq 7$ by $\rho_G(\mathcal{P}) = 2d(S) - 16 + 20$. Now let us prove $\rho_G(\mathcal{P}) \geq 18$. By contradiction, suppose $\rho_G(\mathcal{P}) \leq 17$. Since $|V(G)| \geq 7$, by symmetry, we assume $|S^c| \geq 4$. Let $H = G[S]$. For any partition $\mathcal{Q}$ of H, we denote $\mathcal{P}' = Q \cup (\mathcal{P} \setminus \{S\})$. Then we have $r(\mathcal{P}') \geq 1$ and $r_0(\mathcal{P}') = 1$. Thus, by Claim 3(ii), $\rho_G(\mathcal{P}') \geq 12$. By Proposition 3.2, we have $\rho_H(\mathcal{Q}) = \rho_G(\mathcal{P}') - \rho_G(\mathcal{P}) + 12 \geq 12 - 17 + 12 = 7$ for any partition $\mathcal{Q}$ of H, a contradiction to Claim 2(i). This proves (ii). ■

Next we introduce a few more tools in order to complete the proof of Theorem 3.4. We will make use of a splitting operation as described in the following lemma, which preserves $\mathbb{Z}_3$-connectivity of the graph.

Lemma 3.5 (*Lemma 4.1 of [3]*). *Let G be a graph and let z be a vertex of G with degree at least 4 and $zv_1, zv_2 \in E_G(z)$. If $G' = G - z + v_1v_2$ is $\mathbb{Z}_3$-connected, then G is $\mathbb{Z}_3$-connected.*

Another key result is the following theorem due to Lovász, Thomassen, Wu and Zhang [10].

Theorem 3.6 (*Lovász et al. [10]*). *Every 6-edge-connected graph is $\mathbb{Z}_3$-connected.*

Now we are ready to finish the proof. By Claim 4(ii), each nontrivial edge cut of G has size at least 7. But G is not 6-edge-connected by Theorem 3.6. Hence the minimal degree of G is at most 5. Let z be a vertex in G of minimum degree. Then by Claim 4(i) we have

$$4 \leq d_G(z) \leq 5.$$

Our main strategy below is to show that by Claim 4 it is always possible to select $zv_1, zv_2 \in E_G(z)$ such that the modified graph $G' = G - z + v_1v_2$ still satisfies the condition of Theorem 3.4. Then the minimality of G and Theorem 3.4 would imply that G' is $\mathbb{Z}_3$-connected. Hence, G is $\mathbb{Z}_3$-connected by Lemma 3.5, a contradiction to Claim 1.

Claim 5. *Let $zv_1, zv_2 \in E_G(z)$ and let $G' = G - z + v_1v_2$. Then G' is 4-edge-connected.*

Proof. Let S be a nonempty proper subset of $V(G')$. We shall prove that $d_{G'}(S) \geq 4$. By Claim 1, G has no parallel edges and so $|N_G(z)| = d_G(z)$. As $|N_G(z)| \leq 5$, we may adjust notation, by interchanging S with S^c if necessary, so that $|S \cap N_G(z)| \leq 2$. Then, $d_{G'}(S) \geq d_G(S) - |S \cap N_G(z)|$. If $d_G(S) \geq 7$, then $d_{G'}(S) \geq 5$. We may thus assume that $d_G(S) < 7$. By Claim 4(ii), one of S and S^c is trivial. As $|S^c \cap N_G(z)| = |N_G(z)| - |S \cap N_G(z)| \geq 2$, we deduce that $|S| = 1$. Let v be the vertex of S, i.e. $S = \{v\}$. If $v \notin N_G(z)$, then $d_{G'}(v) = d_G(v) \geq 4$. Hence assume $v \in N_G(z)$. Now let us prove that $d_G(v) \geq 5$. This fact is clear when $\delta(G) = 5$. We may thus assume that $\delta(G) = 4$ and so $d_G(z) = 4$. Let $Y = \{v, z\}$. By Claim 4(ii), it follows that $7 \leq d_G(Y) = d_G(z) + d_G(v) - 2 = 2 + d_G(v)$, and so $d_G(v) \geq 5$. In both cases above, we deduce that $d_G(v) \geq 5$, which implies $d_{G'}(v) \geq d_G(v) - 1 \geq 4$.

We conclude that $d_{G'}(S) \geq 4$. This conclusion holds for every nonempty proper subset S of $V(G')$, and hence G' is 4-edge-connected. ■

Claim 6. *We have $\rho(G') \geq 0$.*

Proof. Let $\mathcal{Q}$ be a partition of $V(G')$, we shall prove that $\rho_{G'}(\mathcal{Q}) \geq 0$. To this end, we let $\mathcal{P} = \mathcal{Q} \cup \{\{z\}\}$, and let

$$s = \begin{cases} 0 & \text{if there exists a part } Y \text{ of } \mathcal{Q} \text{ such that } \{v_1, v_2\} \subseteq Y; \\ 2 & \text{otherwise.} \end{cases}$$

Clearly, $\sum_{X\in\mathcal{Q}} d_{G'}(X) \geq \sum_{X\in\mathcal{P}} d_G(X) - 2d_G(z) + s$. For convenience, we use $|\mathcal{Q}|$ to denote the number of parts of $\mathcal{Q}$. Then we have $|\mathcal{P}| = |\mathcal{Q}| + 1$. Thus,

$$\begin{aligned}\rho_{G'}(\mathcal{Q}) &= \sum_{X\in\mathcal{Q}} d_{G'}(X) - 8|\mathcal{Q}| + 20\\ &\geq \sum_{X\in\mathcal{P}} d_G(X) - 2d_G(z) + s - 8|\mathcal{P}| + 8 + 20\\ &= \rho_G(\mathcal{P}) - 2d_G(z) + 8 + s.\end{aligned}$$

If $s = 2$, then $\rho_{G'}(\mathcal{Q}) \geq \rho_G(\mathcal{P}) \geq \rho(G) \geq 0$ since $4 \leq d_G(z) \leq 5$. We may thus assume that $s = 0$. In this case, $\mathcal{Q}$ contains a set Y such that $\{v_1, v_2\} \subseteq Y$. Clearly, $Y \in \mathcal{P}$, hence $\mathcal{P}$ is nontrivial. By Claim 3(i), we have $\rho_G(\mathcal{P}) \geq 6$. Thus, $\rho_{G'}(\mathcal{Q}) \geq \rho_G(\mathcal{P}) - 2 > 0$.

In both cases above, we have $\rho_{G'}(\mathcal{Q}) \geq 0$. This conclusion holds for each partition $\mathcal{Q}$ of $V(G')$, and hence $\rho(G') \geq 0$. ■

Now the minimality of G implies that Theorem 3.4 is applicable to G'. Thus either G' is $\mathbb{Z}_3$-connected, or there is a partition $\mathcal{Q}$ of G' such that $G'/\mathcal{Q} \in \mathcal{G}$. But the latter case cannot happen since G' is 4-edge-connected. Hence G' is $\mathbb{Z}_3$-connected, and so G is $\mathbb{Z}_3$-connected by Lemma 3.5, a contradiction. ■

Corollary 3.7. *(i) Every graph G satisfying $\rho(G) \geq 8$ is $\mathbb{Z}_3$-connected.*

(ii) [3] Every graph with four edge-disjoint spanning trees is $\mathbb{Z}_3$-connected.

Proof. (i) The statement holds vacuously for $|V(G)| = 1, 2$, and so we assume $|V(G)| \geq 3$. If G is not connected, then we have $\rho(G) \leq 4$ by Definition 3.1, a contradiction to $\rho(G) \geq 8$. Thus, G is connected. By Theorem 3.4, either G is $\mathbb{Z}_3$-connected, or there is a partition $\mathcal{P}$ of G such that $G/\mathcal{P} \in \mathcal{G}$. Since any partition of $G/\mathcal{P}$ can be obtained from a partition of G by collapsing vertex sets in $\mathcal{P}$ to become vertices, we have $\rho(G/\mathcal{P}) \geq \rho(G) \geq 8$. Thus, $G/\mathcal{P} \notin \mathcal{G}$ and so G is $\mathbb{Z}_3$-connected.

(ii) If a graph G contains 4 edge-disjoint spanning trees, then $\rho(G) \geq 12$, and so G is $\mathbb{Z}_3$-connected by (i). This reproves the main result in [3]. Actually, Theorem 3.4 is an improvement of the result in [3]. ■

To complete the proof of the upper bound in Theorem 1.2, we need the following corollary.

Corollary 3.8. *Let G be a $\mathbb{Z}_3$-irreducible graph. Then for every nontrivial partition $\mathcal{P}$ of $V(G)$, $\rho_G(\mathcal{P}) > \rho(G)$. Consequently, $\rho(G) = 2|E(G)| - 8|V(G)| + 20$.*

Proof. Let $Z \in \mathcal{P}$ with $|Z| \geq 2$ and let $H = G[Z]$. If $\rho(H) \geq 12$, then H is $\mathbb{Z}_3$-connected by Corollary 3.7(i). This contradicts the fact that G is $\mathbb{Z}_3$-irreducible. Thus $\rho(H) \leq 11$. Hence by Corollary 3.3, we have $\rho_G(\mathcal{P}) > \rho(G)$. ■

Proof of the upper bound in Theorem 1.2 using Theorem 3.4. Let G be a 3-flow-critical graph on n vertices. By Theorem 2.4(iii) and Corollary 3.8, we have that G is $\mathbb{Z}_3$-irreducible and $\rho(G) = 2|E(G)| - 8n + 20$. If $\rho(G) < 0$, then $|E(G)| < 4n - 10$ holds. We may thus assume that $\rho(G) \geq 0$. Since G and any nontrivial subgraph of G are not $\mathbb{Z}_3$-connected, we obtain $G \in \mathcal{G}$ by Theorem 3.4. Since K_2, K_3 and P_3 are not 3-flow-critical, we have $G = K_4$, and so $|E(K_4)| = 4|V(K_4)| - 10$ in this case. ■

Proof of Theorem 1.5. By way of contradiction, we suppose $|E(G)| \geq \frac{5n}{2} + 9n_{\leq 8}(G)$. If $n_{\leq 8}(G) \geq \frac{n}{6}$, then $|E(G)| \geq \frac{5n}{2} + \frac{9n}{6} = 4n$, which contradicts to Theorem 1.2. So we assume $n_{\leq 8}(G) < \frac{n}{6}$. Since $\delta(G) \geq 3$, we have $2|E(G)| = \sum_{v\in V(G)} d(v) \geq 3n_{\leq 8}(G) + 9(n - n_{\leq 8}(G)) = 9n - 6n_{\leq 8}(G) > 8n$, still a contradiction to Theorem 1.2. This proves Theorem 1.5. ■

4. Construction of 3-flow-critical graphs

Yao and Zhou [13] proved that for each positive integer k, there exists a 4-critical planar graph with $6k+7$ vertices and $14k+12$ edges. By duality, their theorem shows the following result on 3-flow-critical planar graphs.

Theorem 4.1 (*Yao and Zhou [13]*). *For each positive integer k, there exists a* 3*-flow-critical planar graph with* $8k+7$ *vertices and* $14k+12$ *edges.*

Definition 4.2. Let G_1 and G_2 be two graphs. Let $G_1 \oplus G_2$ be a graph which is obtained as the 2-sum of G_1 and G_2, that is, a graph obtained from the disjoint union of $G_1 - e_1$ and $G_2 - e_2$ by identifying u_1 and u_2 to form a vertex u, identifying v_1 and v_2 to form a vertex v, and adding a new edge uv, where $e_1 = u_1v_1 \in E(G_1)$ and $e_2 = u_2v_2 \in E(G_2)$.

Lemma 4.3. *If* G_1 *and* G_2 *are both* 3*-flow-critical graphs, then* $G_1 \oplus G_2$ *is* 3*-flow-critical.*

Proof. Assume $e_1 = u_1v_1 \in E(G_1)$ and $e_2 = u_2v_2 \in E(G_2)$, and assume that $G_1 \oplus G_2$ is constructed as shown in Definition 4.2. First, we show that $G_1 \oplus G_2$ has no modulo 3-orientation. To the contrary, we suppose $G_1 \oplus G_2$ has a modulo 3-orientation D with $v \to u$. Let D_i be the restriction of D on G_i for each $i \in \{1, 2\}$. Denote $d^+_{D_i}(u_i) - d^-_{D_i}(u_i) \equiv a_i \pmod 3$ and $d^+_{D_i}(v_i) - d^-_{D_i}(v_i) \equiv b_i \pmod 3$. Then we have $a_1 + a_2 + 1 \equiv 0 \pmod 3$ since u is balanced in D, and $a_i + b_i \equiv 0 \pmod 3$ since every vertex, except perhaps u_i and v_i, is balanced in D_i. If $a_1 = 0$, then $b_1 = 0$ and D_1 is a modulo 3-orientation of G_1, a contradiction. If $a_1 = 1$, then $b_1 = 2$. We can obtain a modulo 3-orientation of G_1 by reversing the direction of the arc v_1u_1 in D_1, a contradiction. If $a_1 = 2$, then $a_2 = 0$ and $b_2 = 0$, and so D_2 is a modulo 3-orientation of G_2, a contradiction again.

Then it suffices to show that $G_1 \oplus G_2 - e$ has a modulo 3-orientation for each edge e in $G_1 \oplus G_2$. Recall that $G_i - e'$ has a modulo 3-orientation for each $e' \in E(G_i)$ by Theorem 2.4(i). If $e = uv$, then the union of the modulo 3-orientations of $G_i - u_iv_i$ is a modulo 3-orientation of $G_1 \oplus G_2 - e$. If $e \in E(G_1)$ and $e \neq u_1v_1$, then the union of the modulo 3-orientations of $G_1 - e$ and $G_2 - u_2v_2$ is a modulo 3-orientation of $G_1 \oplus G_2 - e$. If $e \in E(G_2)$ and $e \neq u_2v_2$, then we can also find a modulo 3-orientation of $G_1 \oplus G_2 - e$ by a symmetric argument. This proves that $G_1 \oplus G_2$ is a 3-flow-critical graph. ■

Finally we apply Theorem 4.1 and Lemma 4.3 to construct 3-flow-critical graphs with density from $\frac{7}{4}$ up to 3.

Theorem 4.4. *For any positive integer N and any rational number r with* $\frac{7}{4} < r < 3$*, there exists a* 3*-flow-critical graph G on* $n \geq N$ *vertices with*

$$rn - \frac{5}{8} \leq |E(G)| \leq rn + \frac{5}{8}.$$

Proof. Assume $r = \frac{q}{p}$, where p, q are two positive integers. Note that Lemma 4.3 provides a way to construct 3-flow-critical graphs from smaller graphs. Now let $s \geq \frac{6(3p-q)}{8q-14p} + N$ and let G_1 be a 3-flow-critical planar graph with $8s+7$ vertices and $14s+12$ edges as described in Theorem 4.1. Let

$$a = \frac{1}{3p-q}((8q - 14p)s + 5q - 3p - \frac{5p}{8})$$

and

$$b = \frac{1}{3p-q}((8q - 14p)s + 5q - 3p + \frac{5p}{8}).$$

Since $\frac{7}{4} < \frac{q}{p} < 3$, we have $3p-q > 0$, $8q-14p > 0$ and $5q-3p-\frac{5p}{8} > 0$. So $s > N$ and $a > 6$. Since $b - a = \frac{5p}{4(3p-q)} = \frac{5}{4(3-\frac{q}{p})} > 1$, there exists a positive integer t satisfying $a \leq t \leq b$. Let $G_2 = K^+_{3,t-3}$

and let $G = G_1 \oplus G_2$. Then G is 3-flow-critical by Lemma 4.3. By the construction of G, the graph G has $8s + 7 + t - 2 = 8s + t + 5$ vertices and $14s + 12 + 3t - 8 - 1 = 14s + 3t + 3$ edges. So $|V(G)| > N$. It is routine to compute that $rn - \frac{5}{8} \leq |E(G)| \leq rn + \frac{5}{8}$. In fact, with a straightforward calculation, it follows from $a \leq t \leq b$ that

$$rn + \frac{5}{8} - |E(G)| = \frac{q}{p}(8s + t + 5) + \frac{5}{8} - (14s + 3t + 3) = \frac{3p - q}{p}(b - t) \geq 0$$

and

$$|E(G)| - (rn - \frac{5}{8}) = (14s + 3t + 3) - \frac{q}{p}(8s + t + 5) + \frac{5}{8} = \frac{3p - q}{p}(t - a) \geq 0.$$

This completes the proof. ■

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